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This paper deals with regressive functions and regressive isols. It was proven by J. C. E. Dekker in [2] that the collection Λ_R of all regressive isols is not closed under addition. In the first note of this paper we shall given another proof of this fact by considering a new relation, denoted by $\overset{*}{\vee}$, between infinite regressive isols. Let A and B denote infinite regressive isols. The main results established in the first note are:

(1) $A \leqq^* B \Longrightarrow A \overset{*}{\vee} B$, yet not conversely.

(2) $A + B \in \Lambda_R \Longrightarrow A \overset{*}{\vee} B$, yet not conversely.

(3) There exist infinite regressive isols which are not $\overset{*}{\vee}$ related.

(4) Λ_R is not closed under addition.

In addition, the following result is stated.

(5) $A + B \in \Lambda_R \Longrightarrow \min(A, B) \leqq A + B$, yet not conversely.

In the second note we consider the $\leqq^*$ relation between regressive isols. A natural question concerning this relation is whether $A \leqq^* B$, where A and B are regressive isols, is a necessary or a sufficient condition for the sum $A + B$ to be regressive. In the second note we show that this condition is neither necessary nor sufficient.

We shall assume that the reader is familiar with the notations, terminology and main results of [1] and [2].

Preliminaries. Let $\varepsilon = \{0, 1, 2, 3, \cdots\}$ be the set of nonnegative integers (*numbers*). A one-to-one function t_n from ε into ε is *regressive* if there is a partial recursive function $p(x)$ such that $\rho t \subseteqq \delta p$ and $p(t_0) = t_0$, $(\forall n)[p(t_{n+1}) = t_n]$. The function p is a *regressing function* of t_n if p has the following additional properties: $\rho p \subseteqq \delta p$ and $(\forall x)[x \in \delta p \rightarrow (\exists n)[p^{n+1}(x) = p^n(x)]]$. It is known (cf. [1]) that every regressive function has a regressing function. A set is *regressive* if it is finite or the range of a regressive function. A set is *retraceable* if it is finite or the range of a strictly increasing regressive function. Let p be a regressing function of t_n, then the function p^* is defined by: $\delta p^* = \delta p$ and $p^*(x) = (\mu n)[p^{n+1}(x) = p^n(x)]$. It follows that p^* is a partial recursive function and $(\forall n)[p^*(t_n) = n]$.

Let s_n and t_n be two one-to-one functions from ε into ε. Then $s_n \leqq^* t_n$, if there is a partial recursive function f such that

$$(1) \qquad \rho s \subseteqq \delta f \quad \text{and} \quad (\forall n)[f(s_n) = t_n]\,.$$

Also, s_n and t_n are said to be *recursively equivalent* (denoted $s_n \simeq t_n$)

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if there is a one-to-one partial recursive function f such that (1) holds. Let σ and τ be two sets. Then $\sigma \leqq^* \tau$, if either σ is finite and card. $\sigma \leqq$ card. τ, or σ is infinite and there is a partial recursive function f such that $\sigma \subseteqq \delta f$, f is one-to-one on σ and $f(\sigma) = \tau$. Let S and T be two isols. Then $S \leqq^* T$, if there are sets $\sigma \in S$ and $\tau \in T$ such that $\sigma \leqq^* \tau$. The following propositions will be useful:

(a) Retraceable sets are either recursive or immune.

(b) Every function recursively equivalent to a regressive function is regressive.

(c) Every set recursively equivalent to a regressive set is regressive.

(d) Let $\sigma = \rho s_n$ and $\tau = \rho t_n$ where s_n and t_n are one-to-one regressive functions. Then $\sigma \leqq^* \tau$ if and only if $s_n \leqq^* t_n$, and $\sigma \simeq \tau$ if and only if $s_n \simeq t_n$.

(e) Let s_n and t_n be one-to-one functions from ε into ε. Then $s_n \simeq t_n$ if and only if $s_n \leqq^* t_n$ and $t_n \leqq^* s_n$.

Proposition (a) is proven in [3]. Propositions (b) and (c), and the second part of (d) are proven [1]. Both (e) and the first part of (d) are given in [2].

Two sets α and β are said to be *separated* (denoted $\alpha \mid \beta$) if there are disjoint r.e. sets α^* and β^* such that $\alpha \subseteqq \alpha^*$ and $\beta \subseteqq \beta^*$. Two functions a_n and b_n are said to be *separated* (denoted $a_n \mid b_n$) if their ranges are separated sets. We will use the familiar primitive recursive functions j, k and l defined by

$$j(x, y) = x + (x + y)(x + y + 1)/2 ,$$
$$j(k(n), l(n)) = n .$$

The function j maps ε^2 one-to-one onto ε.

Note 1. The $\overset{*}{\vee}$ relation.

DEFINITION 1. Let a_n and b_n be any two one-to-one functions from ε into ε. Then $a_n \overset{*}{\vee} b_n$ if there is a partial recursive function $p(x)$ such that

$$(\forall n)[a_n \in \delta p \text{ and } p(a_n) = b_n) \vee (b_n \in \delta p \text{ and } p(b_n) = a_n)] .$$

The following proposition can be readily proven using the definitions of the concepts involved. Its proof will be omitted.

PROPOSITION 1.1. Let a_n and b_n be any two one-to-one functions from ε into ε. Then

(a) $a_n \overset{*}{\vee} b_n \Longrightarrow b_n \overset{*}{\vee} a_n$,

(b) $a_n \leqq^* b_n \Longrightarrow a_n \overset{*}{\vee} b_n$,

$$\text{(c)}\quad \left.\begin{array}{l} a_n \overset{*}{\vee} b_n, \\ a_n \simeq a'_n,\ b_n \simeq b'_n, \\ a_n \mid b_n,\ a'_n \mid b'_n, \end{array}\right\} \Longrightarrow a'_n \overset{*}{\vee} b'_n .$$

DEFINITION 2. Let A and B be any two infinite regressive isols. Then $A \overset{*}{\vee} B$ if there are regressive functions a_n and b_n such that

$$\rho a_n \in A,\quad \rho b_n \in \beta,\quad a_n \mid b_n \quad \text{and} \quad a_n \overset{*}{\vee} b_n .$$

RERARK. In view of Proposition (d) and part (c) of Proposition 1.1, we see that if A and B are infinite regressive isols, then $A \overset{*}{\vee} B$ means that $a_n \overset{*}{\vee} b_n$ for every pair a_n and b_n of separated, regressive functions ranging over sets in A and B respectively.

THEOREM 1.1. *Let A and B be infinite regressive isols. Then*

$$A \leqq^* B \Longrightarrow A \overset{*}{\vee} B .$$

Proof. Let a_n and b_n be any two (one-to-one) regressive functions ranging over sets in A and B respectively and such that $a_n \leqq^* b_n$. Set $a'_n = 2a_n$ and $b'_n = 2b_n + 1$. Then $a_n \simeq a'_n$, $b_n \simeq b'_n$ and $a'_n \mid b'_n$. Taking into account Propositions (b), (c) and (d) it follows that a'_n and b' are separated, regressive functions which range over sets in A and B respectively. In addition, $a_n \leqq^* b_n$ implies $a'_n \leqq^* b'_n$. By Proposition 1.1 (b) this means $a'_n \overset{*}{\vee} b'_n$, and therefore $A \overset{*}{\vee} B$.

THEOREM 1.2. *For all infinite regressive isols A and B,*

$$A + B \in \Lambda_R \Longrightarrow A \overset{*}{\vee} B .$$

Proof. Let A and B denote two infinite regressive isols whose sum is also regressive. Let a_n and b_n be regressive functions with $\alpha = \rho a_n \in A$, $\beta = \rho b_n \in B$ and $\alpha \mid \beta$. Then $\alpha + \beta \in A + B$ and $\alpha + \beta$ is a regressive set. Let c_n be a regressive function ranging over the set $\alpha + \beta$ and let $p(x)$ be a regressing function of c_n. Set

$$\delta = \{x \mid (x = a_n \text{ and } p^*(b_n) < p^*(a_n)) \vee (x = b_n \text{ and } p^*(a_n) < p^*(b_n))\} .$$

We note that $\delta \subseteqq \alpha + \beta$ and that for each number n, exactly one of the numbers a_n and b_n belongs to δ. Let the function f with domain δ be defined by

$$f(x) = \begin{cases} b_n, & \text{if} \quad x = a_n , \\ a_n, & \text{if} \quad x = b_n . \end{cases}$$

It is easily seen that if f has a partial recursive extension then $a_n \overset{*}{\vee} b_n$.

Since a_n and b_n are separated functions this fact would also imply that $A \overset{*}{\vee} B$. Hence to complete the proof it suffices to show that f has a partial recursive extension. For this purpose, assume that $x \in \delta$. Since α and β are separated sets we can determine whether $x \in \alpha$ or $x \in \beta$. First suppose that $x \in \alpha$. Taking into account that a_n and c_n are regressive functions, we can find the numbers u and v such that $x = a_u = c_v$. The number a_u belongs to δ and therefore

$$b_u \in (c_0, c_1, \cdots, c_{v \dot{-} 1}) = \{p^r(x) \mid 1 \leqq r \leqq v\} .$$

The members of the set on the right side can be effectively obtained from x, since p is a partial recursive function. In addition, using once again the separability of the sets α and β, and the regressiveness of the function b_n, it follows that we can find the number b_u. This gives the value of $f(x)$. In a similar fashion one can determine the value of $f(x)$ in the event $x \in \beta$. From these remarks we can conclude that f will have a partial recursive extension. This completes the proof.

REMARK. We shall state without proof, two additional facts which can be established in the proof of Theorem 1.2. These are

(a) $$\delta \in \min(A, B) ,$$

(b) $$\delta \mid (\alpha + \beta) - \delta .$$

Since $\alpha + \beta \in A + B$, these facts imply that

(*) $$\min(A, B) \leqq A + B .$$

In the proof of Theorem 1.2, A and B were assumed to be infinite regressive isols. However, it is easily seen that the relation denoted by (*) is also true in the event either A or B is finite, for in this case $\min(A, B)$ assumes one of the values (A, B). From these remarks one has the following

THEOREM. *For all regressive isols A and B,*

$$A + B \in \Lambda_R \Longrightarrow \min(A, B) \leqq A + B .$$

The statement obtained by reversing the implication in the above theorem is false, for in the second note it is shown that there are two infinite regressive isols which are comparable relative to the $\leqq^*$ relation, hence their minimum assumes one of these two values, and yet whose sum is not regressive. According to Theorem 1.1, this also means that reversing the implication in Theorem 1.2 yields a false statement as well.

THEOREM 1.3. *There exist infinite regressive isols A and B which* *are not* $\overset{*}{\vee}$ *related.*

Proof. Let $\{p_i\}$ be an enumeration of partial recursive functions of one variable such that:

(a) every partial recursive function of one variable occurs at least once in $\{p_i\}$,

(b) $p_0(1) \neq 3$ and $p_0(3) \neq 1$.

We shall define two functions a_n and b_n such that the recursive equivalence types, $A = \text{Req}\, \rho a_n$ and $B = \text{Req}\, \rho b_n$ satisfy the conditions of the Theorem.

Put $a_0 = 1$ and $b_0 = 3$. We note that (b) implies

$$(1) \qquad p_0(a_0) \neq b_0 \quad \text{and} \quad p_0(b_0) \neq a_0 .$$

Let $t \geqq 1$ and suppose that $a_0, \cdots, a_{t-1}$ and $b_0, \cdots, b_{t-1}$ have already been defined. We define a_t and b_t by setting

$$a_t = j(a_{t-1}, u_t) ,$$
$$b_t = j(b_{t-1}, v_t) ,$$

where the numbers u_t and v_t will be defined in such a manner that

$$(2) \qquad p_t(a_t) \neq b_t \quad \text{and} \quad p_t(b_t) \neq a_t .$$

The definition of u_t and v_t. Set

$$\eta = \{u \mid j(a_{t-1}, u) \in \delta p_t\} ,$$
$$\zeta = \{v \mid j(b_{t-1}, v) \in \delta p_t\} .$$

We consider three cases:

Case I. $\eta' \neq \phi$. Let u be the smallest number belonging to η'. Then $p_t j(a_{t-1}, u)$ is undefined.

Subcase I.1. There exists a number v such that

$$p_t j(b_{t-1}, v) \neq j(a_{t-1}, u) .$$

Set

$$u_t = u ,$$
$$v_t = (\mu v)[p_t j(b_{t-1}, v) \neq j(a_{t-1}, u)] .$$

Subcase I.2. For all numbers v,

$$p_t j(b_{t-1}, v) = j(a_{t-1}, u) .$$

Consider the number $j(a_{t-1}, u + 1)$. Since j maps ε^2 one-to-one onto ε, it follows that $j(a_{t-1}, u + 1) \neq j(a_{t-1}, u)$. Hence for all numbers v,

$$p_t j(b_{t-1}, v) \neq j(a_{t-1}, u + 1) .$$

Clearly there exist numbers v' such that $j(b_{t-1}, v') \neq p_t(a_{t-1}, u + 1)$. Set

$$u_t = u + 1 ,$$
$$v_t = (\mu v')[j(b_{t-1}, v') \neq p_t j(a_{t-1}, u + 1)] .$$

Case II. $\zeta' \neq \phi$. Here we proceed in a fashion similar to Case I. The details are omitted.

Case III. $\eta' = \zeta' = \phi$, i.e., $\eta = \zeta = \varepsilon$, i.e.,

$$(\forall u)[j(a_{t-1}, u) \in \delta] \quad \text{and} \quad (\forall v)[j(b_{t-1}, v) \in \delta] ,$$

where $\delta = \delta p_t$. The numbers in the following four lists:

L1. $j(a_{t-1}, 0),\ j(a_{t-1}, 1), \cdots$

L2. $p_t j(b_{t-1}, 0),\ p_t j(b_{t-1}, 1), \cdots$

L3. $j(b_{t-1}, 0),\ j(b_{t-1}, 1), \cdots$

L4. $p_t j(a_{t-1}, 0),\ p_t j(a_{t-1}, 1), \cdots$

are therefore all defined. Since the function $j(x, y)$ is one-to-one, all numbers in L1 are distinct and all numbers in L3 are distinct.

*Subcase III.*1. L1 contains a number which does not occur in L2. Set

$$u_t = (\mu u)(\forall v)[j(a_{t-1}, u) \neq p_t j(b_{-1}, v)] .$$

Since all of the numbers in L3 are distinct, it follows that

$$(\exists v)[j(b_{t-1}, v) \neq p_t j(a_{t-1}, u_t)] .$$

Set

$$v_t = (\mu v)[j(b_{t-1}, v) \neq p_t j(a_{t-1}, u_t)] .$$

*Subcase III.*2. Every number of L1 occurs at least once in L2. Since L1 contains infinitely many numbers this implies that L2 contains infinitely many numbers. Hence, not only

$$(\forall u)(\exists v)[j(a_{t-1}, u) \neq p_t j(b_{t-1}, v)] ,$$

but also

$$(\forall u)(\exists \text{ infinitely many } v)[j(a_{t-1}, u) \neq p_t j(b_{t-1}, v)] .$$

This must be true in particular for $u = 0$. Thus there exists an infinite sequence $v_0, v_1, v_2, \cdots$ of distinct numbers such that

$$(\forall n)[j(a_{t-1}, 0) \neq p_t j(b_{t-1}, v_n)] .$$

Let

$$n^* = (\mu n)[j(b_{t-1}, v_n) \neq p_t j(a_{t-1}, 0)] .$$

Define

$$u_t = 0 ,$$
$$v_t = v_{n^*} .$$

This completes the definition of the numbers u_t and v_t, and hence also of the functions a_n and b_n. It is readily verified that the numbers a_t and b_t have been so defined as to satisfy (2), that is

$$p_t(a_t) \neq b_t \quad \text{and} \quad p_t(b_t) \neq a_t .$$

Combining this fact with (1) gives

$$(3) \qquad (\forall n)[p_n(a_n) \neq b_n \quad \text{and} \quad p_n(b_n) \neq a_n] .$$

Let

$$\alpha = \rho a_n \quad \text{and} \quad \beta = \rho b_n .$$

We claim:

(a) a_n and b_n are strictly increasing regressive functions and α and β are retraceable sets,

(b) $\alpha \mid \beta$,

(c) a_n and b_n are not $\overset{*}{\vee}$ related,

(d) α and β are immune sets.

Re (a): It follows from the definition of the function $j(x, y)$ that $x < j(x, y)$ for $x > 0$. Moreover, we have

$$a_0 > 0 \quad \text{and} \quad (\forall n)(\exists u)[a_{n+1} = j(a_n, u)] ,$$
$$b_0 > 0 \quad \text{and} \quad (\forall n)(\exists v)[b_{n+1} = j(b_n, v)] .$$

Hence

$$a_0 < a_1 < a_2 < \cdots \quad \text{and} \quad b_0 < b_1 < b_2 < \cdots ,$$

and therefore a_n and b_n are strictly increasing functions. Set

$$q(x) = \begin{cases} a_0, & \text{if} \quad x = a_0 , \\ k(x), & \text{if} \quad x \neq a_0 . \end{cases}$$

Clearly $q(x)$ is a recursive function and it can be readily shown that $q(x)$ is a regressing function of a_n. By replacing a_0 by b_0 in the definition of $q(x)$ yields a regressing function of b_n. Hence, a_n and b_n are each strictly increasing regressive functions and therefore α and β are retraceable sets.

Re (*b*): As a consequence of the definition of the functions a_n and b_n, we have

$$\alpha \subset \{z \mid z = 1 \vee (\exists n)[k^n(z) = 1]\} ,$$
$$\beta \subset \{z \mid z = 3 \vee (\exists n)[k^n(z) = 3]\} .$$

The sets appearing on the right sides are clearly r.e. Also, since $k(3) = 0$, $k(1) = 0$ and $k(0) = 0$, they are disjoint. Hence $\alpha \mid \beta$.

Re (*c*): Suppose that statement (c) were false; this would then mean $a_n \overset{*}{\vee} b_n$. Hence there would be a partial recursive function $p(x)$ such that

$$(4) \qquad (\forall n)[p(a_n) = b_n) \vee (p(b_n) = a_n)] .$$

Assume that the index of p in our enumeration is i, i.e., $p(x) = p_i(x)$. In view of (4), we would have

$$p_i(a_i) = b_i \cdot \text{ or } \quad p_i(b_i) = a_i .$$

However, according to (3) this statement must be false. This contradiction establishes the desired conclusion that a_n and b_n are not $\overset{*}{\vee}$ related.

Re (*d*): By part (a), each of the sets α and β is retraceable and hence is either recursive or immune. If one of these sets is recursive then the strictly increasing function ranging over the set would be a recursive function. Thus, if α were a recursive set then a_n would be a recursive function. In this event, we would have that

$$b_n \leqq^* n, \text{ since } b_n \text{ is a regressive function,}$$
$$n \leqq^* a_n, \text{ since } a_n \text{ is a regressive function,}$$

and, by the transitivity of the $\leqq^*$ relation, also that $b_n \leqq^* a_n$. By Proposition 1.1 (b), this means that $a_n \overset{*}{\vee} b_n$, which is not possible according to part (c). Therefore α must be an immune set. In a similar way it can be shown that β is also an immune set. This verifies (d).

To complete the proof, let

$$A = \operatorname{Req} \alpha \quad \text{and} \quad B = \operatorname{Req} \beta .$$

By statements (a) and (d) it follows that A and B are infinite regressive isols. In addition, combining statements (a) and (c) with the Remark following Definition 2 implies that A and B are not $\overset{*}{\vee}$ related. Hence A and B satisfy the requirements of the Theorem.

REMARK A. In [2, Theorem T2] it is shown that both the collection Λ_R of all regressive isols and the collection Λ_{CR} of all cosimple regressive isols are not closed under addition. We note that the first of these results can be obtained by combining Theorems 1.2 and 1.3.

REMARK B. It is readily seen from Definitions 1 and 2, that the $\overset{*}{\vee}$ relation for infinite regressive isols is both reflexive and symmetric. The following Corollary to Theorem 1.3 shows that $\overset{*}{\vee}$ is a not a transitive relation.

COROLLARY. *There exist infinite regressive isols A, B and W with $A \overset{*}{\vee} W$, $B \overset{*}{\vee} W$, while A and B are not $\overset{*}{\vee}$ related.*

Proof. Let A and B be any two infinite regressive isols which are not $\overset{*}{\vee}$ related. Set $W = \min(A, B)$. Then W is an infinite regressive isol with

$$W \leqq^* A \quad \text{and} \quad W \leqq^* B .$$

Hence, by Theorem 1.1

$$W \overset{*}{\vee} A \quad \text{and} \quad W \overset{*}{\vee} B .$$

According to our choice of A and B, the proof is complete.

Note 2. The main results of this note will establish the fact that $A \leqq^* B$ (where $A, B \in \Lambda_R$) represents neither a necessary condition nor a sufficient condition for the sum $A + B$ to belong to Λ_R. In the following discussion we will use the notion of the *degree of unsolvability of a regressive isol*. This concept is studied in [2]. If A is a regressive isol, then Δ_A will denote its degree of unsolvability.

THEOREM 2.1. *There exist regressive isols A and B with $A \leqq^* B$, yet whose sum $A + B$ is not regressive.*

Proof. Let P and Q denote two (infinite) regressive isols with different degrees of unsolvability, i.e., $\Delta_P \neq \Delta_Q$. Set

$$A = \min(P, Q) .$$

Then A is an infinite regressive isol such that

$$A \leqq^* P \quad \text{and} \quad A \leqq^* Q .$$

To complete the proof we need only show that at least one of the two isols $A+P$ and $A+Q$ is not regressive. To prove this fact, let us suppose otherwise, namely that both $A+P$ and $A+Q$ are regressive isols. Then according to [2, Proposition 17(d)], it follows that

$$\Delta_A = \Delta_P \quad \text{and} \quad \Delta_A = \Delta_Q ,$$

and therefore $\Delta_P = \Delta_Q$. This last equality contradicts our choice of P and Q. Hence, either $A+P$ or $A+Q$ is not regressive. If we define B to be P if $A+P \notin \Lambda_R$ and to be Q otherwise, then A and B will satisfy the requirements of the Theorem.

REMARK. It is proven in [2] that there are cosimple regressive isols with different degrees of unsolvability. Moreover, the minimum of two cosimple regressive isols is again a cosimple regressive isol. Thus, as a consequence of the previous proof, we see that the following result is also true.

THEOREM. *There exist cosimple regressive isols A and B with $A \leqq^* B$ yet whose sum $A+B$ is not regressive.*

THEOREM 2.2. *There exist regressive isols S and T which are incomparable relative to the $\leqq^*$ relation and whose sum is regressive.*

Proof. This shall be a constructive type of proof and we shall use a technique introduced in the proof of [4, Theorem 95]. The proof will progress in four steps.

Step I. In this step we shall define a particular function a_n from ε into ε, and show that it is strictly increasing and regressive.

Let $p_i(x)$ denote a function of the two variables i and x such that every one-to-one partial recursive function and no other function appears in the sequence $\{p_i\}$. For any numbers $t_0, \cdots, t_m, i$; $\max^* \{p_i(t_0), \cdots, p_i(t_m)\}$ is defined to be 0 if none of the $m+1$ numbers $p_i(t_0), \cdots, p_i(t_m)$ is defined; and is defined to be the maximum of those numbers $p_i(t_0), \cdots, p_i(t_m)$ which are defined; if at least one of them is defined.

The function a_n is defined by,

$$a_0 = 1 ,$$
$$a_{k+1} = j(a_k, u_{k+1}) , \quad \text{where}$$

$u_{k+1} = 0$, if either $k = 4n+1$ or $k = 4n+3$,

$u_{k+1} = (\mu y)[j(a_k, y) > \max^* \{p_n(a_0), \cdots, p_n(a_n)\}]$, if either $k = 4n$ or $k = 4n+2$.

It is readily seen that a_n is an everywhere defined function from ε into ε. Moreover, just as the function a_n in the proof of Theorem 1.3 was shown to be strictly increasing and regressive, it can be shown that

a_n is also strictly increasing and regressive.

Step II. Let the four sets δ_0, δ_1, δ_2 and δ_3 denote the ranges of the functions a_{4n}, a_{4n+1}, a_{4n+2} and a_{4n+3} respectively. Since each of the functions $4n$, $4n+1$, $4n+2$ and $4n+3$ is strictly increasing and recursive, it follows that each of the functions a_{4n}, a_{4n+1}, a_{n+2} and a_{4n+3} is regressive. Hence the four sets δ_0, δ_1, δ_2 and δ_3 are each regressive. We shall now prove:

(a) not $[\delta_0 \simeq \delta_1]$,

(b) not $[\delta_2 \simeq \delta_3]$,

(c) a_n ranges over an immune set.

Re (*a*): To prove statement (a), let us suppose that it is false. Then, by the enumeration in Step I, there would be a number i such that

$$\delta_0 \subset \delta p_i \quad \text{and} \quad p_i(\delta_0) = \delta_1 .$$

One consequence of this fact is

$$(1) \qquad (p_i(a_0), p_i(a_4), \cdots, p_i(a_{4i})) \subset \delta_1 .$$

By the definition of the function a_n, it follows that a_{4i+1} would exceed each of the numbers $p_i(a_0), p_i(a_4), \cdots, p_i(a_{4i})$. Since a_n is strictly increasing, the same would be true for a_{4j+1} with $j \geqq 1$. Hence from (1) we can conclude that

$$(p_i(a_0), p_i(a_4), \cdots, p_i(a_{4i})) \subset (a_1, a_5, \cdots, a_{4(i-1)+1}) .$$

However, the set on the left side has exactly $i+1$ members while the set on the right side has only i members. This contradicts the fact that p_i is a one-to-one function. This means that statement (a) must be true.

Re (*b*): We can prove statement (b) in a way similar to the one used to prove (a). Assuming that statement (b) is false implies that there is a number i such that

$$\delta_2 \subset \delta p_i, \quad \text{and} \quad p_i(\delta_2) = \delta_3 ,$$

and

$$(2) \qquad (p_i(a_2), p_i(a_6), \cdots, p_i(a_{4i+2})) \subset \delta_3 .$$

The definition of the function a_n implies that a_{4i+3} will exceed each of the numbers $p_i(a_2), p_i(a_6), \cdots, p_i(a_{4i+2})$, and since a_n is strictly increasing, the same will be true for a_{4j+3} with $j \geqq i$. Hence from (2) we can conclude that

$$(p_i(a_2), p_i(a_6), \cdots, p_i(a_{4i+2})) \subset (a_3, a_7, \cdots, a_{4(i-1)+3}) .$$

Yet the set on the left side has exactly $i + 1$ members while the set on the right side has exactly i members. This contradicts the fact that p_i is a one-to-one function. Therefore (b) must be a true statement.

Re (*c*): Since a_n is a strictly increasing regressive function it ranges over an infinite retraceable set. We know that this set will be either recursive or immune. But it is easily seen that if a_n ranges over an infinite recursive set then each of the sets δ_0 and δ_1 will also be infinite and recursive. According to statement (a), this is not possible. Hence a_n ranges over an immune set. This verifies (c) and also completes Step II.

Step III. Let

$$\sigma = \delta_0 + \delta_3 \quad \text{and} \quad \tau = \delta_1 + \delta_2 .$$

We shall now prove:
(d) σ and τ are infinite regressive sets,
(e) $\sigma \mid \tau$,
(f) not $[\sigma \leqq^* \tau]$,
(g) not $[\tau \leqq^* \sigma]$.
For this purpose, let

$$g(x) = \begin{cases} 4n , & \text{if} \quad x = 2n , \\ 4n + 3 , & \text{if} \quad x = 2n + 1 , \end{cases}$$
$$h(x) = \begin{cases} 4n + 1 , & \text{if} \quad x = 2n , \\ 4n + 2 , & \text{if} \quad x = 2n + 1 . \end{cases}$$

Then

$$(3) \qquad \rho a_{g(n)} = \sigma \quad \text{and} \quad \rho a_{h(n)} = \tau .$$

We also note that the functions g and h are each recursive and strictly increasing. In addition, their ranges are disjoint and the union of their ranges is ε.

Re (*d*): Since both g and h are strictly increasing, recursive functions and a_n is a regressive function it readily follows that both $a_{g(n)}$ and $a_{h(n)}$ are regressive function. By (3), this means that σ and τ are infinite regressive sets.

Re (*e*): From the two facts, a_n is a regressive function, and the ranges of the recursive functions g and h are disjoint, one can easily show that the two functions $a_{g(n)}$ and $a_{h(n)}$ are separated. This means that σ and τ are separated sets.

Re (*f*): Suppose that statement (f) were false, namely assume that $\sigma \leqq^* \tau$. According to Proposition (d), this implies that $a_{g(n)} \leqq^* a_{h(n)}$. Comparing the definitions of $g(x)$ and $h(x)$, we can conclude from this fact that

$$a_{4n} \leqq^* a_{4n+1} .$$

Clearly,

$$a_{4n+1} \leqq^* a_{4n} ,$$

and hence by Proposition (e),

$$a_{4n} \simeq a_{4n+1} .$$

According to Proposition (d), this implies that $\delta_0 \simeq \delta_1$ which is not possible in view of part (a). Therefore statement (f) is true.

Re (*g*): To verify (g) we can proceed as in the previous case. Suppose that statement (g) is false. This will imply that $a_{h(n)} \leqq^* a_{g(n)}$, and this fact gives

$$a_{4n+2} \leqq^* a_{4n+3} .$$

Clearly,

$$a_{4n+3} \leqq^* a_{4n+2} ,$$

and hence

$$a_{4n+2} \simeq a_{4n+3} .$$

This means that $\delta_2 \simeq \delta_3$ which is not possible in view of part (b). This contradiction establishes (g) and also completes Step III.

Step IV. Let

$$S = \text{Req}\,\sigma \quad \text{and} \quad T = \text{Req}\,\tau .$$

Both σ and τ are infinite subsets of the immune set ρa_n, and therefore are themselves immune sets. Also, by part (d), σ and τ are regressive. Hence

(i) S and T are infinite regressive isols.

Combining [2, Proposition P 10] and statement (f) and (g), implies that

(ii) S and T are incomparable relative to the $\leqq^*$ relation.

In view of (i) and (ii), in order to complete the proof it remains only to show that

(iii) $S + T \in \Lambda_R$.

Since σ and τ are separated sets, it follows that $\sigma + \tau \in S + T$. Moreover, $\sigma + \tau$ is a regressive set since $\sigma + \tau = \rho a_n$. Hence $S + T$ is a regressive isol. This verifies (iii) and completes the proof.

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