A THEOREM OF LITTLEWOOD AND LACUNARY SERIES
FOR COMPACT GROUPS

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Let \( G \) be a compact group and \( f \in L^2(G) \). We prove that given \( p < \infty \) there exists a unitary transformation \( U \) of \( L^2(G) \) into \( L^p(G) \), which commutes with left translations and such that \( Uf \in L^p \). The proof is based on techniques developed by S. Helgason for a similar question. The result stated above, which is an extension of a theorem of Littlewood for the unit circle is then applied to the study of lacunary Fourier series.

The following two results concerning Fourier series of functions defined on the unit circle were proved by Littlewood [5]:

I. Suppose that for any choice of complex numbers \( \alpha_n \), with \( |\alpha_n| = 1 \), \( \sum \alpha_n e^{inz} \) is the Fourier series of an integrable function (or a Fourier-Stieltjes series) then \( \sum |a_n|^2 < \infty \).

II. Let \( \sum |a_n|^2 < \infty \). Then given \( p < \infty \) there exist complex numbers \( \alpha_n \), with \( |\alpha_n| = 1 \), such that \( \sum \alpha_n e^{inz} \) is the Fourier series of a function in \( L^p \).

Helgason [3] has generalized I to Fourier series on compact groups. Let \( G \) be a compact group with normalized Haar measure \( dx \). If \( f \in L^1(G) \) then \( f \) is uniquely represented by a Fourier series

\[
\sum_{\gamma \in \Gamma} d_{\gamma} Tr(A_{\gamma} D_{\gamma}(x))
\]

where \( Tr \) denotes the usual trace, \( \Gamma \) is the set of equivalence classes of irreducible unitary representations of \( G \), \( D_{\gamma} \) is a representative of the class \( \gamma \), \( d_{\gamma} \) is the degree of \( \gamma \), and \( A_{\gamma} \) is the linear transformation given by

\[
A_{\gamma} = \int_g f(x) D_{\gamma}(x^{-1}) dx.
\]

Helgason has proved

II'. Suppose that, for any choice of unitary transformations \( U_{\gamma} \) on the Hilbert space of dimension \( d_{\gamma} \), \( \sum_{\gamma \in \Gamma} d_{\gamma} Tr(U_{\gamma} A_{\gamma} D_{\gamma}(x)) \) is the Fourier series of an integrable function (or a Fourier-Stieltjes series) then

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In view of the Schur-Peter-Weyl formula
\[
\int_{\mathbb{R}} |f(x)|^2 dx = \sum_{\gamma \in \mathcal{G}} d_\gamma Tr(A_\gamma^* A_\gamma) ,
\]
Helgason's result is an extension of I.

In this paper, using Helgason's techniques, we propose to extend II to compact groups in the same sense. That is we prove

II'. Let \( \sum d_\gamma Tr(A_\gamma A_\gamma^*) < \infty \). Given \( p < \infty \) there exist unitary transformations \( U_\gamma \) such that \( \sum d_\gamma Tr(U_\gamma A_\gamma D_\gamma(x)) \) is the Fourier series of a function in \( L^p \).

This is accomplished as in [3] by proving and exploiting the "lacunarity" of a certain subset of the space of irreducible unitary representations of the product group \( \prod_{i \in S} U(d_i) \) where \( U(d_i) \) is the group of unitary transformations of the Hilbert space of dimension \( d_i \) and \( S \) is an arbitrary index set. In the last section we discuss in general lacunary properties of subsets of the space of irreducible representations of a compact group.

2. The main result. For a positive integer \( n \) let \( U(n) \) be the group of unitary transformations of the Hilbert space of dimension \( n \). The normalized Haar measure on \( U(n) \) will be denote by \( dV \).

**Lemma 1.** Let \( A \) be an \( n \times n \) matrix. Then for \( s = 1, 2, 3, \cdots \)

(1) \[ \int_{U(n)} |Tr(AV)|^{2s} dV \leq \frac{B(s)}{n^s} [Tr(AA^*)]^s \]

where \( B(s) \) is a constant depending only on \( s \).

**Proof.** Since \( dV \) is left and right invariant it is sufficient to prove the lemma when \( A \) is diagonal. Letting \( e_1, e_2, \cdots, e_n \) be a basis for the Hilbert space on which \( A \) and \( V \) act and \( a_i = \langle Ae_i, e_i \rangle, v_i = \langle Ve_i, e_i \rangle \) we have

(2) \[ \int_{U(n)} |Tr(AV)|^{2s} dV = \sum a_{i_1} \bar{a}_{i_2} a_{i_3} \bar{a}_{i_4} \cdots a_{i_{2s-1}} \bar{a}_{i_{2s}} \]

\[ \int_{U(n)} v_{i_1} \bar{v}_{i_2} \cdots v_{i_{2s-1}} \bar{v}_{i_{2s}} dV , \]

where the sum extends over all \( i_1, i_2, \cdots, i_{2s} \) such that \( 1 \leq i_j \leq n \). Each integral in the sum is of the form
Now each such integral is zero unless \( j_1 = k_1, \ldots, j_n = k_n \). For let \( W \) be a diagonal unitary matrix with elements \( \alpha_i \) of modulus one on the main diagonal. Then by the invariance of \( dV \), (3) becomes

\[
\int_{U(n)} \prod_{i=1}^{n} \langle Ve_i, e_i \rangle^{i-j} \langle e_i, Ve_i \rangle^{i \cdot dV} = \int_{U(n)} \prod_{i=1}^{n} \langle WVe_i, e_i \rangle^{i-j} \langle e_i, WVe_i \rangle^{i \cdot dV} = \prod_{i=1}^{n} \alpha_i^{j-i} \int_{U(n)} \prod_{i=1}^{n} \langle Ve_i, e_i \rangle^{i-j} \langle e_i, Ve_i \rangle^{i \cdot dV}.
\]

Thus if the integral is not zero, \( \prod_{i=1}^{n} \alpha_i^{j-i} = 1 \), for all choices of the \( \alpha_i \). Clearly this is possible only if \( j_1 = k_1, \ldots, j_n = k_n \). Thus the sum (2) is equal to

\[
\sum |a_{j_1}|^2 |a_{j_2}|^2 \cdots |a_{j_n}|^2 \int_{U(n)} |v_{i_1}|^2 \cdots |v_{i_2}|^2 dV.
\]

We shall see that for each integer \( s \)

\[
\int_{U(n)} |v_i|^{2s} dV \leq \frac{B(s)}{n^s} \quad (s = 1, 2, \ldots)
\]

where \( B(s) \) depends only on \( s \). It then follows from Hölder’s inequality that the integrals in (4) are bounded by \( B(s)/n^s \) so that (4) is majorized by

\[
\frac{B(s)}{n^s} \sum |a_{j_1}|^2 \cdots |a_{j_n}|^2 = \frac{B(s)}{n^s} [\text{Tr}(AA^*)]^s,
\]

and the lemma will be proved.

It is sufficient to calculate (5) for \( i = 1 \). Let \( U_i(n-1) \) be the subgroup \( \{ T \in U(n) : Te_i = e_i \} \). The space \( U(n)/U_i(n-1) \) of left cosets \( \{ \tilde{V} = VU_i(n-1) : V \in U(n) \} \) can be identified with the unit sphere \( \Sigma_n \) in a complex \( n \)-dimensional Hilbert space. Since \( v_i \) is constant on these cosets

\[
\int_{U(n)} |v_i|^{2s} dV = \int_{x_n} \langle Ve_i, e_i \rangle^{2s} d \tilde{V} = \int_{|w_1|^2 + \cdots + |w_n|^2 = 1} |w_i|^{2s} d \tilde{W}
\]

where \( d \tilde{W} \) is the unique normalized measure on \( \Sigma_n \) invariant with respect to \( U(n) \) and

\[
Ve_i = w_1 e_i + \cdots + w_n e_n.
\]

If we identify \( \Sigma_n \) with the real \((2n-1)\)-dimensional sphere \( S^{2n-1} \) in real \( 2n \)-dimensional space and \( d \tilde{V} \) with \( dw \), the normalized invariant
measure on $S^{2n-1}$, then

\[(6) \quad \int_{U(n)} |v_1|^2dV = \int_{x_1^2+\ldots+x_{2n}^2=1} (x_1^2 + x_2^2)d\mu.
\]

By Minkowski’s inequality and the invariance of $d\mu$ (6) is bounded by

\[2^s \int_{S^{2n-1}} x_1^s dx = 2^s \frac{\Omega(S^{2n-1})}{\Omega(S^{m-1})} \int_{-1}^1 x_1^s(1 - x_1^2)^{n-1}(1 - x_1^2)^{-1/2}dx_1
\]

where $\Omega(S^m) = \frac{2\pi^{(m+1)/2}}{\Gamma\left(\frac{m+1}{2}\right)}$ is the Euclidean surface area of the real $m$-dimensional unit sphere. Thus the integral in (6) is bounded by

\[\frac{2^s \cdot 2\pi^{(2n-1)/2}}{\Gamma\left(\frac{n-1}{2}\right)} \cdot \frac{\Gamma\left(s + \frac{1}{2}\right)\Gamma\left(n - \frac{1}{2}\right)}{\Gamma(n + s)} \leq B(s)\frac{\Gamma(n)}{n^s}
\]

which proves (5).

**Corollary 2.** Let $J$ be the canonical representation $U \rightarrow U$ of $U(n)$ and $J_{s,t}$ be the tensor product of $J$, $s$ times and $\bar{J}$, the conjugate representation, $t$ times. $J_{s,t}$ decomposes into at most $B(s + t)$ irreducible components. If $s \neq t$ then none of the components is the identity representation.

**Proof.** If $\chi_T$ is the character of the representation $T$, then $\chi_{J_{s,t}}(V) = (\chi_J(V))^s(\chi_{\bar{J}}(V))^t = (Tr(V))^s(Tr(V))^t$. Thus by the lemma

\[\int_{U(n)} |\chi_{J_{s,t}}(V)|^2dV \leq B(s + t),
\]

which proves the first statement.

The number of times the identity representation occurs in $J_{s,t}$ is

\[\int_{U(n)} \chi_{J_{s,t}}(V)dV = \int_{U(n)} (Tr(V))^s(Tr(V))^tdV = 0
\]

if $s \neq t$ by the statement following (3).

**Lemma 3.** Let $G = \prod_{i \in S} U(d_i)$ be a product of unitary groups $U(d_i)$. Let $F(V)$ be a function on $G$ of the form

\[F(V) = \sum_{i \in S} d_i Tr(A_i V_i)
\]

where $A_i$ is a $d_i \times d_i$ matrix and $V_i$ is the projection of $V$ on $U(d_i)$. Then
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\[ \int_G |F(V)|^2 dV \leq B(s) \left( \int |F(V)|^dV \right)^s \]

where \( dV \) is the normalized Haar measure on \( G \).

Proof. It suffices to prove the lemma when

\[ F(V) = \sum_{i=1}^N d_i Tr(A_i V_i). \]

Then

(7) \[ \int_G |F(V)|^2 dV \]

\[ = \sum_{U(d_1) \times \cdots \times U(d_n)} d_i d_i Tr(A_i V_i) \cdots d_i d_i Tr(A_i V_i) dV \]

where the sum extends over all \( i_1, \ldots, i_n \) such that \( 1 \leq i_1 \leq N \). By the corollary the only terms in the sum which do not vanish are those of the form

(8) \[ \int_{U(d_1) \times \cdots \times U(d_n)} d_i^2 \ldots d_i^2 \left[ \int_{U(d_1)} |Tr(A_i V_i)|^2 dV_i \right]^{i/s} \ldots \left[ \int_{U(d_n)} |Tr(A_i V_i)|^2 dV_i \right]^{i/s}. \]

By Hölder's inequality (8) is majorized by

\[ d_i^2 \ldots d_i^2 \left[ \int_{U(d_1)} |Tr(A_i V_i)|^2 dV_i \right]^{i/s} \ldots \left[ \int_{U(d_n)} |Tr(A_i V_i)|^2 dV_i \right]^{i/s}. \]

which by Lemma 1 is majorized by

\[ d_i^2 \ldots d_i^2 B(s) \frac{Tr(A_i A_i^\gamma)}{d_i} \ldots \frac{Tr(A_i A_i^\gamma)}{d_i}. \]

Hence the left side of (7) is bounded by

\[ B(s) \left[ \sum_{i=1}^N d_i Tr(A_i A_i^\gamma) \right]^s = B(s) \left[ \int_G |f(V)|^2 dV \right]^s, \]

where the equality follows from the Peter-Weyl formula.

Now let \( G \) be an arbitrary compact group and \( \Gamma \) be the set of equivalence classes of irreducible representations of \( G \). Let \( d_\gamma \) be the degree of the class \( \gamma \). Then \( G = \prod_{\gamma \in \Gamma} U(d_\gamma) \) is a compact group which can be thought of as the group of unitary transformations of \( L^2(G) \) into \( L^2(G) \) which commute with left translations. That is, if \( V \) is such a transformation then \( V \) corresponds to the element \( \{V_\gamma \} \in G \) such that

\[ Vf(x) \sim \sum_{\gamma \in \Gamma} d_\gamma Tr(V_\gamma A_\gamma D_\gamma(x)) \]

whenever \( f(x) \sim \sum_{\gamma \in \Gamma} Tr(A_\gamma D_\gamma(x)) \in L^2(G) \).
THEOREM 4. Let $f \in L^p(G)$ and $p < \infty$, then for almost every $V \in G$, $Vf \in L^p(G)$.

Proof. Let $Vf(x) = f(V, x) = \sum \partial V \text{Tr}(V_\gamma A_\gamma(x))$. Then $f(V, x)$ can be considered as a function on $G \times G$. For fixed $x \in G$ we have by Lemma 2 that

$$\int_G |f(V, x)|^2 dV \leq B(s) \left( \int_G |f(V, x)|^2 dV \right)^* \leq B(s) \left( \sum_{\gamma \in \Gamma} \partial V \text{Tr}(A_\gamma A_\gamma^*) \right)^* = B(s) \left( \int_G |f(x)|^2 dx \right)^*,$$

so that

$$\int_G \int_G |f(V, x)|^2 dxdV = \int_G \int_G |f(V, x)|^2 dVdx \leq B(s) \left( \int_G |f(x)|^2 dx \right)^*.$$

Therefore if $f \in L^2(G)$, then for almost every $V \in G$, $\int_G |Vf(x)|^2 dx < \infty$.

Letting $s > p/2$ we obtain the theorem.

We remark for later use that for some $V$

$$||Vf||_p \leq ||Vf||_2 \leq 2B(s) ||f||_2.$$

Indeed the set of $V$ for which

$$\int \int |Vf(x)|^2 dx > 2B(s) \left( \int |f(x)|^2 dx \right)^*$$

cannot be of measure one.

We will also use the following

REMARK 5. Let $f \in C(G)$ be a continuous function such that for all self adjoint $V \in G$, $Vf \in C(G)$, then $f(x) \sim \sum \partial V \text{Tr}(A_\gamma D_\gamma(x))$ with $\sum \partial V \text{Tr}(|A_\gamma|) < \infty$ ($|A_\gamma|$ is the absolute value of the matrix $A_\gamma$). Indeed letting $\tilde{f}(x) = \tilde{f}(x^{-1})$ we can write $f = \frac{f + \tilde{f}}{2} + i \frac{f - \tilde{f}}{2i} = f_x + if_\gamma$. If $f_x(x) \sim \sum \partial V \text{Tr}(A_{\gamma,i} D_\gamma(x))$ ($i = 1, 2$) then $A_{\gamma,i} = A_{\gamma,i}$. Therefore there exists a self adjoint $V = \{V_\gamma\} \in G$ such that $A_{\gamma,i} = |A_{\gamma,i}|$. Thus $\sum \partial V \text{Tr}(A_{\gamma,i} D_\gamma(x))$ is continuous so that applying a method of summation as in [4, 8.3] we obtain that the partial sums of $\sum \partial V \text{Tr}(|A_{\gamma,i}|) = \sum \partial V \text{Tr}(A_{\gamma,i} D_\gamma(x))$ are bounded. Thus $\sum \partial V \text{Tr}(|A_\gamma|) \leq \sum \partial V \text{Tr}(|A_{\gamma,i}|) + \sum \partial V \text{Tr}(A_{\gamma,i}) < \infty$.

We shall call a series $\sum \partial V \text{Tr}(A_{\gamma} D_\gamma(x))$ satisfying $\sum \partial V \text{Tr}(|A_{\gamma}|) < \infty$ an absolutely convergent series. The space of such functions will be denoted by $A(G)$. It is easy to see that $A(G)$ consists of functions of the type $f \ast g$ with $f, g \in L^2(G)$. The space $A(G) = L^2(G) \ast L^2(G)$ has been
studied in [1].

3. Lacunary Fourier series. Given a compact group \( G \) we shall say that a subset \( E \subseteq \Gamma \) of the set of irreducible unitary representation of \( G \) is a *Sidon set* if it satisfies the following property:

\[ \sum_{y \in E} \text{Tr}(A_y) < \infty \quad \text{whenever} \quad \sum_{y \in E} d_y \text{Tr}(A_y D_y(x)) \text{ is the Fourier series of a continuous function} \quad (\text{cf. [6, 5.7]}) .\]

A set \( E \subseteq \Gamma \) will be called a set of type \( \mathcal{A}(p) \) (or \( E \in \mathcal{A}(p) \)) for \( p > 1 \) if it satisfies

\[ \text{If} \quad \sum_{y \in E} d_y \text{Tr}(A_y D_y(x)) \text{ is the Fourier series of an integrable function then it is the Fourier series of a function in } L^p \quad \text{(cf. [8]).} \]

If \( B \) is a space of functions on \( G \) and \( E \subseteq \Gamma \) we will denote by \( B_E \) those functions in \( B \) with a series of the form \( \sum_{y \in E} d_y \text{Tr}(A_y D_y(x)) \).

It is seen as in [8, 1.4] that \( E \in \mathcal{A}(p) \) if, for some \( r < p \), \( L^r_E = L^r \).

Clearly \( \mathcal{A}(p_1) \subseteq \mathcal{A}(p_2) \) if \( p_1 \geq p_2 \).

If \( G = \bigsqcup_{i \in I} U(d_i) \) then \( S \) can be thought of as the set of irreducible representations of \( G \) consisting of the projections of \( G \) onto the \( U(d_i) \).

Lemma 2 shows that \( S \in \mathcal{A}(p) \) for every \( p < \infty \). It is a simple matter to prove that \( S \) is also a Sidon set. Indeed, if \( f(V) = \sum_{i \in S} d_i \text{Tr}(A_i V_i) \) is a continuous function belonging to \( C_s(G) \) and if \( U = \{ U_i \} \in G \) then

\[ U f(V) = \sum_{i \in S} d_i \text{Tr}(A_i U_i V_i) \text{ = left translation of } f \text{ by } U, \]

is also continuous. It suffices to pick the \( U_i \) so that \( A_i U_i = |A_i| \) to obtain that \( \sum_{i \in S} d_i \text{Tr}(|A_i|) < \infty \).

We shall now establish a characterization of sets of type \( \mathcal{A}(p) \) which will imply that every Sidon set is a \( \mathcal{A}(p) \) set for every \( p < \infty \). For a group \( G \) denote by \( \mathcal{B} = \mathcal{B}(G) \) the algebra of operators on \( L^2(G) \) generated in the weak operator topology by the operators \( \{ R_y : y \in G \} \) where \( R_y f(x) = f(xy) \). We shall use the fact [2, Th. 6] that \( \mathcal{B} \) is (isometric and isomorphic to) the dual space of a Banach space \( A^p \) of continuous functions on \( G \). \( A^p = A(G) \) the space of functions with absolutely convergent Fourier series [1].

The isomorphism between \( \mathcal{B} \) and the dual space of \( A^p \) is given by \( T \rightarrow \varphi_T \) where \( \varphi_T(f) = Tf(e) \). This correspondence is well defined because every \( T \in \mathcal{B} \) maps each element of \( A^p \) into a continuous function, indeed an element of \( A^p \). We also have that \( \mathcal{B} \) consists exactly of those bounded operators on \( L^p \) which commute with left translations.

Now if \( T \in \mathcal{B}, \ p > 2 \), then \( T \in \mathcal{B} \) and \( \| T \|_{\mathcal{B}} \leq M \| T \|_{L^p} \) where \( M \) is a constant depending only on \( p \). For if \( f \in L^p \) then by Theorem
there exists a unitary transformation $U$ commuting with right translations (and therefore with elements of $\mathcal{P}$) such that $Uf \in L^p$. We can also choose $U$ such that $\|Uf\|_p \leq 2B(s)\|f\|_p$ where $B(s)$ is the constant appearing in Lemma 1 and $s > p/2$ (cf. the remarks following the proof of Theorem 4).

We then have that $TUf \in L^p$ and $U^*TUf = TU^*Uf = Tf \in L^p$. Also $\|Tf\|_p = \|U^*TUf\|_p = \|TUf\|_p = \|T\|_{\mathcal{P}}\|Uf\|_p \leq \|T\|\|Uf\|_p = \|T\|_{\mathcal{P}}\|f\|_p$. Therefore $\|T\|_{\mathcal{P}} \leq M \|T\|_{\mathcal{P}}$. This implies that $A^2 \subseteq A^p$ and $\|A^p\|_p \leq M\|A\|_p$. It is now a simple matter to prove:

**Theorem 6.** Let $E \subseteq \Gamma$ be a set of irreducible unitary representations of $G$ and $p > 2$. The following are equivalent:

(a) $E$ is a set of type $A(p)$.
(b) If $T \in \mathcal{P}$ there exists $S \in \mathcal{P}$ such that $Tf = Sf$ for all $f \in L^2_E$.
(c) If $f \in A^2_E$ then $f \in A^2 = A(G)$.
(d) Every closed subspace of $L^2_E$ which is invariant under left translations is the range of a projection $P$ belonging to $\mathcal{P}$ which is self-adjoint in the sense that $P^* = P$. It is now a simple matter to prove:

**Proof.** Let $E \in A(p)$. Then $L^2_E = L^2_E$ so that by the open mapping theorem there exists $B$ such that $\|f\|_p \leq B\|f\|_p$ for $f \in L^2_E$. As $L^2_E$ is invariant under right and left translations there exists a projection $P_E$ of $L^2$ onto $L^2_E$ which commutes with right and left translations. If $T \in \mathcal{P}$ let $S = TP_E$, then $\|Sf\|_p \leq \|T\|\|P_Ef\|_p \leq \|T\|\|f\|_p \leq B\|T\|\|f\|_p$. Thus $S \in \mathcal{P}$ and (a) implies (b).

Now assume (b) holds. If $f \in L^2_E$ then by Theorem 3 there exists $U \in \mathcal{P}$ such that $Uf \in L^2_E$. Let $S \in \mathcal{P}$ be such that $Sg = U^*g$ for all $g \in L^2_E$; then $SUf = U^*Uf = f \in L^p_E$. Hence $L^2_E = L^2_E$ so that (b) implies (a).

We now show that (a) and (b) imply (d). Indeed if (a) holds the projection $P_E$ of $L^1$ onto $L^2_E$ is bounded in $L^p$. Suppose the $Y \subseteq L^2_E$ is invariant under left translations, let $P_Y$ be the projection (belonging to $\mathcal{P}$) of $L^1$ onto the left invariant subspace of $L^1$ generated by $Y$. By (b) there exists $S \in \mathcal{P}$ with $S = P_Y$ on $L^2_E$. Then $P_YS = P_Y$ so that $P_Y \in \mathcal{P}$.

Suppose (d) holds and let $U$ be a unitary self-adjoint element of $\mathcal{P}$. Then $U^2 = I$ so that $P = (U + I)/2$ is a projection which commutes with left translations. Let $Y$ be the subspace of $L^2_E$ generated by $PL^2_E \cap L^2_E$. Then $Y$ is invariant under left translations so that (d) there is a self-adjoint projection of $L^1$ onto $Y$ commuting with left translations. Clearly this projection is $PP_E$ so that $PP_E \in \mathcal{P}_p$. Hence $UP_E = (2P - I)P_E \in \mathcal{P}_p$. Therefore $UP_Ef$ is continuous for every $f \in A^p_E$. In particular if $f(x) \sim \sum_{\gamma \in \Gamma} d_\gamma Tr(A_\gamma D_\gamma(x)) \in A^2_E$ then $UP_Ef = Uf$ is
continuous. Therefore by Remark 4, \( A_p^0 \subseteq A(G) \).

Finally since \( A \subseteq A^p \) with \( \|f\|_{A^p} \leq M \|f\|_A \), (c) implies that \( A_e = A_e^p \), so that, by the closed graph theorem, \( \|f\|_A \leq B \|f\|_{A^p} \) for each \( f \in A_e^p \).

Each \( T \in \mathcal{B} \) defines therefore a continuous linear functional on \( A_e^p \) by \( Tf(e) \). The Hahn-Banach extension of this functional determines, in view of the duality between \( A^p \) and \( \mathcal{B} \), an element \( S \in \mathcal{B} \) such that \( Sf(x) = (SL_x f)(e) = (TL_x f)(e) = Tf(x) \) for \( f \in A_e^p \); therefore \( S = T \) on \( L_e^p \). Thus (c) implies (b) and the theorem is proved.

**Remark 7.** It suffices for condition (d) to be true that every closed left invariant subspace of \( L_e^p \) is the range of a projection. Indeed the argument used in [7, Th. 1] will show that such a projection can be chosen to be left invariant (and therefore belonging to \( \mathcal{B} \)).

**Theorem 8.** \( E \subseteq I' \) is a Sidon set if and only if for each \( T \in \mathcal{B} \) there exists a finite measure \( \mu \) on \( G \) such that \( Tf = f^* \mu \) for each \( f \in L_e^p \).

**Proof.** One applies the same duality argument used in the proof of Theorem 6 (cf. also [6, 5.7.3]). Assume first that \( E \) is a Sidon set. Then given \( T \in \mathcal{B} \), define a linear functional \( F \) on \( C_e \) by \( F(f) = Tf(e) \). Then \( F \) is well defined, since \( f \in C_e \Rightarrow f \in A \); by the closed graph theorem \( F \) is continuous and has a Hahn-Banach extension to all of \( C(G) \). That is, by the Riesz representation theorem there exists a bounded measure \( \mu \) satisfying

\[
F(f) = \int_{\sigma} f(x^*) d\mu(x) \quad \text{for all } f \in C_e.
\]

Since \( T \) commutes with left translations \( Tf = f^* \mu \) for all \( f \in C_e \). Conversely let \( E \) satisfy the hypothesis of the theorem, to prove that \( E \) is a Sidon set, let \( f \in C_e \) and let \( T \) be an unitary element of \( \mathcal{B} \).

By hypothesis there exists a measure \( \mu \) such that \( Tf = f^* \mu \). Hence \( Tf \in C(G) \) and by Remark 5, \( f \in A \).

**Corollary 9.** Every Sidon set is a \( \Lambda(p) \) set for every \( p \).

**Proof.** If \( \mu \) is a bounded measure and \( R_\mu f = f^* \mu \), then \( R_\mu \in \mathcal{B} \) for every \( p \). Therefore, by Theorem 8 if \( E \) is a Sidon set condition (b) of Theorem 6 holds.

**Remark 10.** In [4, 9.2] a sufficient condition for a set \( E \subseteq I' \) to be a Sidon set is given. This condition includes the requirements that the degrees of the representations of \( E \) be bounded. The fact that for \( \Pi_{\eta} \mathcal{U}(d_i) \), \( S \) is a Sidon set shows that this requirement is not necessary.
REFERENCES


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