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PACIFIC JOURNAL OF MATHEMATICS
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STABILITY OF LINEAR DIFFERENTIAL EQUATIONS WITH PERIODIC COEFFICIENTS IN HILBERT SPACE

GERT ALMKVIST

In this paper we study the stability of the solutions of the differential equation

$$u'(t) = A(t) \cdot u(t) \tag{1}$$

for $t \geqq 0$ in a separable Hilbert space. It is assumed that $A(t)$ is periodic with period one and satisfies the following symmetry condition: There exists a continuous constant invertible operator Q such that

$$A(t)^* = -Q \cdot A(t) \cdot Q^{-1} \quad \text{for all} \quad t \geqq 0 .$$

We use a perturbation technique. Let $A(t) = A_0(t)+B(t)$ where $A_0(t)$ is compact and antihermitian for all t. We denote by $U_0(t)$ the solution operator of $u'(t) = A_0(t)u(t)$. It is shown that (1) is stable if $B(t)$ satisfies a certain smallness condition involving the distribution of the eigenvalues of $U_0(1)$ and the action of $B(t)$ on the eigenvectors of $U_0(1)$. The results can be applied to the second order equation

$$y'' + C(t)y = 0$$

where $C(t)$ is selfadjoint for all t.

Throughout this paper we consider the differential equation (1) where u is a function from the positive reals, $\mathbf{R}^+$, into a separable Hilbert space X with norm $\|x\| = (x, x)^{1/2}$. A is a function from $\mathbf{R}^+$ into $B(X)$, the algebra of continuous linear operators on X. We assume that $A(t)$ is Bochner integrable on every finite subinterval of $\mathbf{R}^+$. Then for a given initial value $u(0)$, there exists a unique solution of (1) (see [4, p. 521]).

Further we always assume that $A(t)$ is periodic. It is no restriction to assume that the period is one, that is $A(t+1) = A(t)$ for all $t \in \mathbf{R}^+$.

The equation (1) is said to be *stable* if for every initial value $u(0)$, there exists a constant M, such that $\|u(t)\| \leqq M$ for all $t \in \mathbf{R}^+$. It is convenient to study the equation

$$U(t)' = A(t)U(t) , \qquad U(0) = I \tag{2}$$

in $B(X)$. Using the principle of uniform boundedness it is easily seen that (1) is stable if and only if the solution of (2) is bounded.

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Let $$\Phi(A) = \lim_{\alpha\to+0} \alpha^{-1}(\| I + \alpha A \| - 1)$$

denote the Gateau differential of A. When X is a Hilbert space $\Phi(A)$ can be calculated by the formula $\Phi(A) = \sup_{\|x\|=1} Re(Ax, x)$

PROPOSITION 1. If $\int_0^1 \Phi(A(t))dt \leqq 0$, then (1) is stable.

Proof. Let n be the greatest integer $\leqq t$. Then using [1, Th. 4] we get

$$\| U(t) \| \leqq \exp \int_0^t \Phi(A(s))ds \leqq \exp \left(n\int_0^1 \phi(A(s))ds\right) \cdot \exp \int_0^{t-n} \phi(A(s))ds$$
$$\leqq \exp \int_0^1 | \Phi(A(s)) | \, ds$$

which ends the proof.

From now on we assume that $A(t)$ satisfies the following symmetry condition:

There exists a constant continuous operator Q such that Q^{-1} is continuous and

(S) $$A(t)^* = - QA(t)Q^{-1} \qquad \text{for all} \quad t \geqq 0 .$$

Here A^* denotes the adjoint of A.

PROPOSITION 2. Condition (S) is equivalent to

$$U(t)^* = QU(t)^{-1}Q^{-1} \qquad \text{for all} \quad t \geqq 0 .$$

Proof. We have $U^*(0)QU(0) = Q$ because $U(0) = I$. But

$$\frac{d}{dt}(U(t)^*QU(t)) = U(t)^*A^*(t)QU(t) + U(t)^*QA(t)U(t) = 0$$

if and only if

$$A^*(t)Q + QA(t) = 0 .$$

Let $\sigma(U)$ be the spectrum of U. From Proposition 2 it follows that $\sigma(U^*(t)) = \sigma(QU^{-1}(t)Q^{-1}) = \sigma(U^{-1}(t)$ that is $\lambda \in \sigma(U(t))$ implies $\bar{\lambda}^{-1} \in \sigma(U(t))$.

PROPOSITION 3. If Q is positive definite, then (1) is stable.

Proof. Q has a positive definite square root S, that is $Q = S^2$. Moreover S^{-1} exists and is continuous. From Proposition 2 we get

$$U^* = S^2 U^{-1} S^{-2}$$

and after some calculations $(SUS^{-1})^* = (SUS^{-1})^{-1}$, that is SUS^{-1} is unitary and hence $\| U(t) \| \leqq \| S \| \cdot \| S^{-1} \|$ for all $t \geqq 0$.

The uniqueness of the solution of (2) implies that

$$U(n+t) = U(t)U(1)^n \qquad \text{for } n = 1, 2, \cdots$$

Hence (1) is stable if and only if there exists a constant M such that

$$\| U(1)^n \| \leqq M \qquad \text{for } n = 1, 2, \cdots$$

Since $\| U(1)^n \| \geqq (\nu(U(1)))^n$, where ν is the spectral radius, it follows that $\sigma(U(1)) \subset \{\lambda;\ |\lambda| \leqq 1\}$ is necessary for the stability of (1). When (S) is satisfied $\sigma(U(1))$is symmetric about the unit circle and hence $\sigma(U(1)) \subset \{\lambda;\ |\lambda| = 1\}$ is necessary.

Now we study the stability of (1) with a perturbation method, due to G. Borg [3] in the finite dimensional case. In order to state the next theorem we introduce some notations. Let the equation be

$$u'(t) = (A_0(t) + B(t))u(t) \tag{3}$$

We assume that

(a) $A_0(t)$ and $B(t)$ are periodic with period one.
(b) $A_0(t)$ is compact and antihermitian $(A_0(t)^* = -A_0(t))$ for all t.

Let further $U_0(t)$ be the unique solution of $U_0'(t) = A_0(t)U_0(t)$, $U_0(0) = I$. Suppose that

(c) $U_0(1)$ has only simple eigenvalues, λ_n, all $\neq 1$.
(d) $A_0(t) + B(t)$ satisfies condition (S).

Let further e_n be the eigenvector with norm one of $U_0(1)$ corresponding to the eigenvalue λ_n. Put

$$b_n^2 = \int_0^1 \| B(t) U_0(t) e_n \|^2 \, dt$$
$$K = \int_0^1 \exp\left[2\int_t^1 \Phi(B(s))ds \right] dt$$
$$r_n = 2^{-1} \inf_{k \neq n} |\lambda_n - \lambda_k| .$$

THEOREM. *If* (a), (b), (c), (d) *and*

$$K \cdot \sup_k \sum_{n=1}^{\infty} b_n^2 (|\lambda_k - \lambda_n| - r_k)^{-2} < 1 \tag{e}$$

and

(f) $$\sum_{n=1}^{\infty} b_n^2 r_n^{-2} < \infty$$

are satisfied, then (3) *is stable.*

REMARK 1. The theorem is true if K and b_n are replaced by

$$K' = \exp\left\{2 \max_{0 \leq t \leq 1} \int_t^1 \Phi(B(s))ds\right\}, \qquad b_n' = \int_0^1 \|B(t)U_0(t)e_n\|\,dt\,.$$

It is easily seen that $K \leqq K'$ but $b_n' \leqq b_n$.

REMARK 2. If X is finite dimensional, then condition (f) is automatically fulfilled.

REMARK 3. $K \cdot \sum_1^\infty b_n^2 r_n^{-2} < 1$ implies both (e) and (f).

Proof of the theorem. The rather lengthy proof is divided in eight parts.

(i) *$U_0(t)$ is unitary for all t.*

A calculation shows that $U_0(t)^{-1} = V(t)^*$ where V is the unique solution of $V' = -A_0^*(t)V$, $V(0)=I$. But since $-A_0^* = A_0$ it follows that $U_0(t)^{-1} = U_0(t)^*$.

(ii) *$U_0(1) - I$ is compact.*

We have $U_0(1) - I = \int_0^1 A_0(t)U_0(t)dt$. The integral is compact because it is the limit of compact operators of the form $\sum_{i=1}^n A_0(t_i)U_0(t_i)\Delta t_i$.

From (i) and (ii) we conclude that $\{e_n\}_1^\infty$ is an orthonormal set and indeed a basis because $U_0(1) - I$ is compact and 1 is not an eigenvalue of $U_0(1)$. Further $\lim_{n\to\infty} \lambda_n = 1$. Since $U_0(t)$ is unitary

$$\|U_0(t)\| = \|U_0(t)^{-1}\| = 1 \quad \text{for all } t \text{ and } \quad |\lambda_n| = 1\,.$$

Put $W(t) = U(t) - U_0(t)$. Further it is convenient to write $U(1) = U$, $U_0(1) = U_0$ and $W(1) = W$. Let C_k be the circumference of a circle with center λ_k and radius r_k.

(iii) $R_\lambda = (\lambda I - U)^{-1}$ *exists if* $\lambda \in \bigcup_1^\infty C_k$.

Put $R_\lambda^0 = (\lambda I - U_0)^{-1}$. For a λ such that R_λ^0 and $(I - WR_\lambda^0)^{-1}$

exist, we have

$$R_\lambda = R_\lambda^0(I - WR_\lambda^0)^{-1}$$

It is clear that R_λ^0 exists whenever $\lambda \in \bigcup_1^\infty C_k$ and if $\| WR_\lambda^0 \| < 1$ it follows that R_λ exists. Since $\{e_n\}_1^\infty$ is an orthonormal basis it follows that

$$\| WR_\lambda^0 \|^2 \leqq \sum_1^\infty \| WR_\lambda^0 e_n \|^2 .$$

But

$$\| WR_\lambda^0 e_n \| = | \lambda - \lambda_n |^{-1} \cdot \| We_n \|$$

since

$$R_\lambda^0 e_n = (\lambda - \lambda_n)^{-1} e_n .$$

One verifies that $W(t)$ satisfies the equation

$$W'(t) = (A_0(t) + B(t))W(t) + B(t)U_0(t)$$

which has the solution

$$W = W(1) = \int_0^1 U(1)U(s)^{-1}B(s)U_0(s)ds$$

Then we get

$$\| We_n \| \leqq \int_0^1 \| U(1)U(s)^{-1} \| \cdot \| B(s)U_0(s)e_n \| ds.$$

From Theorem 4 in [1] we find

$$\| U(1)U(s)^{-1} \| \leqq \exp \int_s^1 \Phi(A_0(t) + B(t))dt .$$

But $\Phi(A_0(t) + B(t)) = \Phi(B(t))$ since $A_0(t)$ is antihermitian. We finally get

$$\begin{aligned} \| We_n \|^2 &\leqq \left\{ \int_0^1 \exp\left[\int_s^1 \Phi(B(t))dt \right] \| B(s)U_0(s)e_n \| ds \right\}^2 \\ &\leqq \int_0^1 \exp\left(2\int_s^1 \Phi(B(t))dt \right) ds \cdot \int_0^1 \| B(s)U_0(s)e_n \|^2 ds = K \cdot b_n^2 . \end{aligned}$$

From condition (e) we conclude that

$$\begin{aligned} \sum_1^\infty \| WR_\lambda^0 e_n \|^2 &\leqq K \cdot \sum_1^\infty b_n^2 | \lambda - \lambda_n |^{-2} \\ &\leqq K \cdot \sup_k \sum_{n=1}^\infty b_n^2 (| \lambda_k - \lambda_n | - r_k)^{-2} < 1 \end{aligned}$$

and hence $\| WR_\lambda^0 \| < 1$ for all $\lambda \in \bigcup_1^\infty C_k$. Thus we have shown that R_λ exists if $\lambda \in \bigcup_1^\infty C_k$.

(iv) *$U - I$ is compact.*

From (iii) it follows that $\sum_1^\infty \| We_n \|^2 \leqq K \sum_1^\infty b_n^2 < \infty$ since (e) implies that $\sum_1^\infty b_n^2 < \infty$. Hence W belongs to the Schmidt class, cf. [5], and is compact. Further $U - I = (U_0 - I) + W$ is compact since $U_0 - I$ is compact (ii).

Put $D_n = \{\lambda;\ |\lambda - \lambda_n| < r_n\}$.

(v) *U has exactly one eigenvalue, α_n,* in *D_n and α_n is simple.*

Since $U - I$ is compact and $1 \notin D_n$ it follows that there is only a finite number of eigenvalues of U in D_n.

Now it is convenient to introduce a parameter μ in the equation. Thus we study $U' = (A_0(t) + \mu B(t))U$, $U(0) = I$ where $0 \leqq \mu \leqq 1$. A simple calculation shows that $R_\lambda(\mu)$ is a continuous function of μ. Hence the projection

$$E_n(\mu) = (2\pi i)^{-1} \int_{\sigma_n} R_\lambda(\mu) d\lambda$$

is also continuous in $[0, 1]$. Further we can find a partition

$$0 = \mu_1 < \mu_2 < \cdots < \mu_k = 1$$

such that

$$\| E_n(\mu_{\nu+1}) - E_n(\mu_\nu) \| < (2M)^{-1} \qquad \text{for } \nu = 1, 2, \cdots, k,$$

where $M = \max_{0 \leqq \mu \leqq 1} \| E_n(\mu) \|$. According to a well known lemma (see [6, p. 424]) it follows that $\dim E_n(\mu_{\nu+1})X = \dim E_n(\mu_\nu)X$ if both sides are finite. This is the case here because $U(\mu) - I$ is compact for $0 \leqq \mu \leqq 1$ and D_n contains only a finite number of eigenvalues. Now $\dim E_n(0)X = 1$ and hence, $\dim E_n(1)X = 1$ by induction. Thus there is exactly one point $\alpha_n \in \sigma(U)$ in D_n and this α_n must be simple.

(vi) $|\alpha_n| = 1$.

Assume that $|\alpha_n| > 1$. Then it follows that $\bar{\alpha}_n^{-1} \in D_n$. But due to (S) we find that $\bar{\alpha}_n^{-1} \in \sigma(U)$ and there will be two points belonging to $\sigma(U)$ in D_n. This is impossible.

Assume now that $|\alpha_n| < 1$. If $\bar{\alpha}_n^{-1} \in D_u$ we can apply the same argument as above. If $\bar{\alpha}_n^{-1} \notin D_n$ it is easily seen that $\bar{\alpha}_n^{-1} \notin \sigma(U)$. In fact we show that if $\lambda \notin \bigcup_1^\infty D_k$ and $\lambda \neq 1$ it follows that $\lambda \notin \sigma(U)$. We need only consider λ with $|\lambda| > 1$. Let D_k be the circle closest to λ. Then it is clear that $\{\lambda - \lambda_n| \geqq ||\lambda_n - \lambda_k| - r_k|$ for all n and we get

$$K\sum_1^\infty \| WR_\lambda^0 e_n \|^2 \leqq K\sum_1^\infty b_n^2 |\lambda - \lambda_n|^{-2} \leqq K\sum_{n=1}^\infty b_n^2(|\lambda_n - \lambda_k| - r_k)^{-2} < 1$$

due to (e). Hence R_λ exists.

Now we have proved that $\sigma(U)$ consists of simple eigenvalues on the unit circle with limit point 1. In the finite dimensional case it follows immediately that (3) is stable (see Boman [2]). In the infinite dimensional case we have to use condition (f).

Put $E_n(0) = E_n$ and $E_n(1) = F_n$. If $F_n e_n \neq 0$ we put $\varphi_n = F_n e_n$ and if $F_n e_n = 0$ we choose φ_n as an arbitrary eigenvector of U corresponding to α_n. We have $E_n e_n = e_n$ and $U\varphi_n = \alpha_n \varphi_n$.

(vii) $\sum_1^\infty \| \varphi_n - e_n \|^2 < \infty$,

$$(F_n - E_n)e_n = (2\pi i)^{-1}\int_{\sigma_n} (R_\lambda - R_\lambda^0)e_n d\lambda \,.$$

A calculation shows that

$$R_\lambda - R_\lambda^0 = R_\lambda^0(I - WR_\lambda^0)^{-1} WR_\lambda^0 \,.$$

Thus

$$\begin{aligned} \| (F_n - E_n)e_n \| &\leqq (2\pi)^{-1}\int_{\sigma_n} \| R_\lambda^0 \| \cdot \| (I - WR_\lambda^0)^{-1} \| \cdot \| WR_\lambda^0 e_n \| \cdot |d\lambda| \\ &\leqq (2\pi)^{-1} r_n^{-1} \sup_{\lambda \in \sigma_n} (1 - \| WR_\lambda^0 \|)^{-1} \cdot K^{1/2} b_n r_n^{-1} 2\pi r_n \\ &= \text{const} \cdot b_n r_n^{-1} \,. \end{aligned}$$

Here we used the fact that $\| R_\lambda^0 \| = r_n^{-1}$ for all $\lambda \in c_n$. Then

$$\sum_1^\infty \| (F_n - E_n)e_n \|^2 \leqq \text{const.} \sum_1^\infty b_n^2 r_n^{-2} < \infty \qquad \text{due to (f)}\,.$$

It follows that $F_n e_n = 0$ only for a finite number of n and hence

$$\sum_1^\infty \| \varphi_n - e_n \|^2 < \infty \,.$$

We define a linear operator P by the relation $Px = \sum_1^\infty c_\nu \varphi_\nu$ where $x = \sum_1^\infty c_\nu e_\nu$ and $\sum_1^\infty |c_\nu|^2 < \infty$. We recall that an operator T is called injective if $Tx = 0$ implies $x = 0$.

(viii) *$I - P$ is compact and P is injective. Hence P^{-1} is continuous.*

$$\sum_1^\infty \| (I-P)e_n \|^2 = \sum_1^\infty \| e_n - \varphi_n \|^2 < \infty \qquad \text{due to (vii)} .$$

Thus $I - P$ belongs to the Schmidt class and is compact (see [5]). Assume now that $Px = \sum_1^\infty c_\nu \varphi_\nu = 0$. We apply the projection F_k and get

$$F_k \sum_1^\infty c_\nu \varphi_\nu = c_k F_k \varphi_k = c_k \varphi_k = 0$$

and $c_k = 0$ for every k. Hence $x = 0$ and P is injective.

Now we end the proof of the theorem. We have to estimate $\| U^n x \|$ for an arbitrary $x \in X$. Put $y = P^{-1}x$ and assume that $y = \sum_1^\infty a_\nu e_\nu$. We get $x = Py = \sum_1^\infty a_\nu \varphi_\nu$ and

$$U^n x = U^n P y = \sum_1^\infty a_\nu U^n \varphi_\nu = \sum_1^\infty a_\nu \alpha_\nu^n \varphi_\nu = P \sum_1^\infty a_\nu \alpha_\nu^n e_\nu .$$

Further

$$\| U^n x \| \leqq \| P \| \cdot \{ \sum_1^\infty | a_\nu \alpha_\nu^n |^2 \}^{1/2} = \| P \| \cdot \{ \sum_1^\infty | a_\nu |^2 \}^{1/2}$$
$$= \| P \| \cdot \| y \| \leqq \| P \| \cdot \| P^{-1} \| \cdot \| x \| ,$$

which implies that $\| U^n \| \leqq \| P \| \, \| P^{-1} \|$ for every n and the proof is finished.

REMARK 4. If $C = (K \cdot \sum_1^\infty b_n^2 r_n^{-2})^{1/2} < 2^{-1}$, then $\| U^n \| < (1 - 2C)^{-1}$.

Proof. From the proof of (iii) it follows that $\| W R_\lambda^0 \| \leqq C$ for all $\lambda \in \bigcup_1^\infty C_k$. Further we get

$$\| (F_n - E_n) e_n \| \leqq (1 - C)^{-1} K^{1/2} b_n r_n^{-1} < 1$$

for all n since

$$(1 - C)^{-2} K \sum_1^\infty b_n^2 r_n^{-2} = C^2 (1 - C)^{-2} < 1 .$$

Hence $F_n e_n \neq 0$ and $\varphi_n = F_n e_n$ for all n. Then

$$\| I - P \|^2 \leqq \sum_1^\infty \| \varphi_\nu - e_\nu \|^2 \leqq C^2 (1 - C)^{-2}$$

and

$$\| P \| \leqq 1 + C(1 - C)^{-1} = (1 - C)^{-1} .$$

Further

$$\| P^{-1} \| = \| (I - (I - P))^{-1} \| \leqq (1 - \| I - P \|)^{-1} \leqq (1 - C)(1-2C)^{-1} .$$

Finally

$$\| U^n \| \leqq \| P \| \cdot \| P^{-1} \| \leqq (1 - 2C)^{-1} .$$

An interesting application of the theorem is the second order equation

$$y'' + C(t)y = 0$$

in a Hilbert space Y, where $C(t)$ is selfadjoint. Put $X = Y \oplus Y$ and $u = \begin{pmatrix} y \\ y' \end{pmatrix}$. Then we get

$$u' = \begin{pmatrix} 0 & I \\ - C(t) & 0 \end{pmatrix} u .$$

This equation satisfies the symmetry condition (S) with $Q - \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$.

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A TRANSPLANTATION THEOREM FOR ULTRASPHERICAL COEFFICIENTS

RICHARD ASKEY AND STEPHEN WAINGER

Let $f(\theta)$ be integrable on $(0, \pi)$ and define

$$a_n = \int_0^\pi f(\theta) \cos n\theta \, d\theta \,, \qquad b_n = n^{1/2} \int_0^\pi f(\theta) P_n(\cos \theta)(\sin \theta)^{1/2} d\theta$$

where $P_n(x)$ is the Legendre polynomial of degree n. Then

$$(1) \qquad c \leqq \sum_{n=0}^{\infty} | a_n |^p (n+1)^\alpha \Big/ \sum_{n=0}^{\infty} | b_n |^p (n+1)^\alpha \leqq C$$

for $1 < p < \infty$, $-1 < \alpha < p - 1$, where C and c depend on p and α but not on f. From this we obtain a form of the Marcinkiewicz multiplier theorem for Legendre coefficients. Also an analogue of the Hardy-Littlewood theorem on Fourier coefficients of monotone coefficients is obtained. In fact, any norm theorem for Fourier functions can be transplanted by (1) to a corresponding theorem for Legendre coefficients.

Actually, the main theorem of this paper deals with ultraspherical coefficients and (1) is just a typical special case, which is stated as above for simplicity.

Let $P_n^\lambda(x)$ be defined by $(1 - 2rx + r^2)^{-\lambda} = \sum_{n=0}^{\infty} P_n^\lambda(x) r^n$ for $\lambda > 0$. The functions $P_n^\lambda(\cos \theta)$ are orthogonal on $(0, \pi)$ with respect to the measure $(\sin \theta)^{2\lambda} d\theta$ and

$$(1) \qquad \int_0^\pi [P_n^\lambda(\cos \theta)]^2 (\sin \theta)^{2\lambda} d\theta = \frac{\Gamma(n + 2\lambda)\Gamma(1/2)\Gamma(\lambda + 1/2)}{n!(n+\lambda)\Gamma(\lambda)\Gamma(2\lambda)} = [t_n^\lambda]^{-2} \,.$$

Observe that $t_n^\lambda = An^{1-\lambda} + O(n^{-\lambda})$ where A will denote a constant whose numerical value is of no interest to us. For simplicity we set $\varphi_n^\lambda(\theta) = t_n^\lambda P_n^\lambda(\cos \theta)(\sin \theta)^\lambda$. The functions $\{\varphi_n^\lambda(\theta)\}_{n=0}^{\infty}$ form a complete orthonormal sequence of functions on $(0, \pi)$ which for $\lambda = 1$ reduce to $\{A \sin (n+1)\theta\}_0^\infty$. Also $\lim_{\lambda \to 0} \varphi_n^\lambda(\theta) = A \cos n\theta$ so the functions $\varphi_n^\lambda(\theta)$ are generalizations of the trigonometric functions which are used in classical Fourier series. For uniformity we define $\varphi_n^0(\theta) = (2/\pi)^{1/2} \cos n\theta$. Later we shall state an asymptotic formula for $\varphi_n^\lambda(\theta)$ which shows another close connection with trigonometric functions. In essence it says that $\varphi_n^\lambda(\theta)$ looks like $\cos [(n + \lambda)\theta - \pi(\lambda/2)]$. All of the facts about φ_n^λ that are quoted without reference are in [15]. Since $\varphi_n^\lambda(\theta)$ are a bounded orthonormal sequence we may consider their Fourier coefficients. Let $f \in L^1(0, \pi)$ and define

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$$a_n^\lambda = \int_0^\pi f(\theta)\varphi_n^\lambda(\theta)d\theta \ .$$

Let $\|a_n\|_p = [\sum_{n=0}^\infty |a_n|^p]^{1/p}$. Then using M. Riesz's inequality [12] for $b_n = \sum_{k\neq n} a_k/(n-k)$, i.e. $\|b_n\|_p \leqq A_p \|a_n\|_p$, $1 < p < \infty$, and Hilbert's inequality, i.e. if $c_n = \Sigma a_k/(n+k)$ then $\|c_n\|_p \leqq A_p \|a_n\|_p$, $1 < p < \infty$, it is easy to show that $\|a_n^1\|_p \leqq A_p \|a_n^0\|$ and conversely $\|a_n^0\|_p \leqq A_p \|a_n^1\|_p$, $1 < p < \infty$. It is this inequality that we generalize to all $\lambda > 0$. For some of the applications we actually want a slight generalization of the above. Instead of considering the l^p norm we work in a weighted l^p norm,

$$\|a_n\|_{p,\alpha} = \left[\sum_{n=0}^\infty |a_n|^p (n+1)^\alpha\right]^{1/p} . \tag{2}$$

These applications will be given in the last section.

Our main theorem is as follows.

THEOREM 1. *Let $f \in L^1(0, \pi)$ and define a_n^λ as above. Then if $\|a_n\|_{p,\alpha}$ is defined by* (2) *we have*

$$A \leqq \|a_n^\lambda\|_{p,\alpha} / \|a_n^\mu\|_{p,\alpha} \leqq A \tag{3}$$

for all $\lambda, \mu \geqq 0$ and $1 < p < \infty$, $-1 < \alpha < p - 1$.

It will be sufficient to prove the inequalities (3) when $\mu < \lambda < \mu + 1$. We first give in detail the proof when $\mu = 0$ and $0 < \lambda < 1$. The formulas that we use in this case are all in the literature and are reasonably well known. Also this proof is easier to follow than the proof of the general case. Then we will sketch the proof for general λ, μ, $\mu < \lambda < \mu + 1$. For simplicity we set $a_n^0 = a_n$ and use $\cos n\theta$ instead of $\varphi_n^0(\theta)$.

Let $f_r(\theta) = \sum_{n=0}^\infty a_n r^n \cos n\theta$. Since $f_r(\theta) \to f(\theta)$ almost everywhere and boundedly in L^1 we have

$$\begin{aligned} a_n^\lambda &= \lim_{r\to 1} \int_0^\pi f_r(\theta)\varphi_n^\lambda(\theta)d\theta = \lim_{r\to 1} t_n^\lambda \int_0^\pi f_r(\theta) P_n^\lambda(\cos\theta)(\sin\theta)^\lambda d\theta \\ &= \lim_{r\to 1} \sum_{k=0}^\infty a_k r^k t_n^\lambda \int_0^\pi P_n^\lambda(\cos\theta)\cos k\theta\,(\sin\theta)^\lambda d\theta \ . \end{aligned}$$

We break the sum up into three parts, $0 \leqq k \leqq [n/2]$, $[n/2] < k < 2n$ and $2n \leqq k$. What we need in each of these intervals is a good estimate for $t_n^\lambda \int_0^\pi P_n^\lambda(\cos\theta)\cos k\theta\,(\sin\theta)^\lambda d\theta = G(k, n)$.

Consider first the case $k \geqq 2n$. We use the following well-known representation for $P_n^\lambda(\cos\theta)$ in terms of $\cos j\theta$.

$$P_n^\lambda(\cos\theta) = \sum_{j=0}^n \alpha_j \alpha_{n-j} \cos(n-2j)\theta \tag{4}$$

where $\alpha_j = (j)_\lambda/j! = Aj^{\lambda-1} + O(j^{\lambda-2})$. Then

$$G(k, n) = \sum_{j=0}^{n} \alpha_j \alpha_{n-j} t_n^\lambda \int_0^\pi (\sin\theta)^\lambda \times [\cos(k-n+2j)\theta + \cos(k+n-2j)\theta] d\theta .$$

Since $\int_0^\pi (\sin\theta)^\lambda \cos r\theta d\theta = O(r^{-1-\lambda})$ and $k \geqq 2n$ we see that

$$|G(k, n)| \leqq A \sum_{j=1}^{n-1} j^{\lambda-1}(n-j)^{\lambda-1} k^{-1-\lambda} n^{1-\lambda} = O((n/k)^\lambda k^{-1}) = O(k^{-1}) .$$

For the theorem that we want the last estimate $O(k^{-1})$ is sufficient. Observe however that we actually have a better estimate. Because of this it is possible to change Theorem 1 to get similar theorems where the Fourier coefficients are defined by $\int_0^\pi f(\theta) P_n^\lambda(\cos\theta)(\sin\theta)^\lambda n^\alpha d\theta$ for various values of α. A possible transplantation then goes to $\int_0^\pi f(\theta) P_n^{\lambda+\beta}(\cos\theta)(\sin\theta)^{\lambda+\beta} n^{\alpha-\beta} d\theta$. Or the $(\sin\theta)^\lambda$ can be omitted from both of these integrals. We mention these facts only because in the dual case different transplantation theorems have been considered by Muckenhoupt and Stein [11] and by the authors [3]. The reason that both types of theorems are true is best seen in the proof of the present theorem, which is essentially easier than either of the theorems in [11] or [3].

Next consider $G(k, n)$ for $k \leqq [n/2]$. This time we need a formula of Szegö. For $0 < \lambda < 1$

$$(5) \qquad (\sin\theta)^{2\lambda-1} P_n^\lambda(\cos\theta) = \frac{2^{2-2\lambda}}{\Gamma(\lambda)} \frac{\Gamma(n+2\lambda)}{\Gamma(n+\lambda+1)} \sum_{j=0}^{\infty} f_{j,n}^\lambda \sin(n+2j+1)\theta$$

where $f_{0,n}^\lambda = 1$ and

$$f_{j,n}^\lambda = \frac{(1-\lambda)(2-\lambda)\cdots(j-\lambda)}{j!} \frac{(n+1)\cdots(n+j)}{(n+\lambda+1)\cdots(n+\lambda+j)} .$$

See [15, p. 96]. A simple estimate shows that

$$f_{j,n}^\lambda = O(j^{-\lambda}(n+j)^{-\lambda} n^\lambda) .$$

Then

$$G(k, n) = O\Big[\sum_{j=0}^{\infty} n^{1-\lambda} f_{j,n}^\lambda n^{\lambda-1} \int_0^\pi (\sin\theta)^{1-\lambda} \times [\sin(n+2j-k+1)\theta + \sin(n+2j+k+1)\theta] d\theta\Big]$$

$$= O\Big[\sum_{j=0}^{\infty} f_{j,n}^\lambda (n+j)^{-2+\lambda}\Big]$$

$$= O(n^{-2+\lambda}) + O\Big(\sum_{j=1}^{\infty} j^{-\lambda}(n+j)^{-2} n^\lambda\Big) = O(n^{-1}) .$$

As usual in results of this nature the region where k and n overlap is harder to handle. This is because a Hilbert transform of some sort always seems to arise. This time we not only have the usual Hilbert transform but we also get a strange variant of it. The transformation we encounter is

$$b_n = \frac{1}{n} \sum_{k=[n/2]}^{2n} a_k \log \frac{n}{|n - k + \lambda + 1|} \,.$$

In § 2 we prove the following lemma, which we will use in the following argument.

LEMMA 1. *If* $\{a_n\} \in l^{p,\alpha}$, $1 < p < \infty$, $-1 < \alpha < p - 1$, *and*

$$b_n = \frac{1}{n} \sum_{k=[n/2]}^{2n} a_k \log \frac{n}{|n - k + \lambda + 1|}$$

then $\|b_n\|_{p,\alpha} \leqq A_p \|a_n\|_{p,\alpha}$.

For reference we state a form of the asymptotic formula for $P_n^\lambda(\cos\theta)$ which we will use, [15, p. 195].

For $0 < \lambda < 1$, $1/n \leqq \theta \leqq \pi/2$

$$(6)\quad \begin{aligned} P_n^\lambda(\cos\theta) = \frac{\Gamma(n+2\lambda)}{\Gamma(n+\lambda+1)} \Bigg[& \frac{A\cos\left\{(n+\lambda)\theta - \frac{\lambda\pi}{2}\right\}}{(\sin\theta)^\lambda} \\ & + \frac{B\cos\left\{(n+\lambda+1)\theta - (\lambda+1)\frac{\pi}{2}\right\}}{(n+\lambda+1)(\sin\theta)^{\lambda+1}} + O(n^{-2}(\sin\theta)^{-\lambda-2}) \Bigg] . \end{aligned}$$

where A and B depend upon λ but not on n.

From this we have

$$(7)\quad \begin{aligned} t_n^\lambda P_n^\lambda(\cos\theta)(\sin\theta)^\lambda = {} & A\cos\left\{(n+\lambda)\theta - \frac{\lambda\pi}{2}\right\} \\ & + \frac{B\cos\left\{(n+\lambda+1)\theta\,(\lambda+1)\frac{\pi}{2}\right\}}{n\sin\theta} + O((n\theta)^{-2}) + O(n^{-1}) , \end{aligned}$$

where $1/n \leqq \theta \leqq \pi/2$ and the O terms are uniform in n and θ. Also we shall use the fact that $t_n^\lambda P_n^\lambda(\cos\theta)(\sin\theta)^\lambda$ are uniformly bounded functions, [15, 7. 33, 6]. Instead of considering

$$t_n^\lambda \int_0^\pi P_n^\lambda(\cos\theta)(\sin\theta)^\lambda \cos k\theta d\theta$$

we may consider $\int_0^{\pi/2}$ since the integrand is either even or odd with respect to $\theta = \pi/2$. Using (7) we get

$$\begin{aligned} t_n^\lambda \int_0^{\pi/2} & P_n^\lambda(\cos\theta)(\sin\theta)^\lambda \cos k\theta d\theta \\ &= t_n^\lambda \int_{1/n}^{\pi/2} P_n^\lambda(\cos\theta)(\sin\theta)^\lambda \cos k\theta d\theta + O\Big(\frac{1}{n}\Big) \\ &= A\int_{1/n}^{\pi/2} \cos\Big\{(n+\lambda)\theta - \frac{\lambda}{2}\Big\} \cos k\theta d\theta \\ &\quad + B\int_{1/n}^{\pi/2} \frac{\cos\Big\{(n+\lambda+1)\theta - (\lambda+1)\dfrac{\pi}{2}\Big\}\cos k\theta d\theta}{n\sin\theta} \\ &\quad + O\Big[\int_{1/n}^{\pi/2}\frac{d\theta}{n^2\theta^2}\Big] + O\Big(\frac{1}{n}\Big) . \end{aligned}$$

The last two terms are $O(1/n)$ and the first is $A'/(n-k+\lambda) + O(1/n)$. We need to consider the second term. Using the addition theorem for $\cos\theta$ we get $B/n \int_{1/n}^{\pi/2} [\{\cos(n-k+1)\theta\}/\sin\theta] d\theta$ + three more terms which are similar but easier to handle. Since $1/\sin\theta - 1/\theta$ is a bounded function for $0 < \theta \leqq \pi/2$ we may instead consider

$$J = \frac{B}{n}\int_{1/n}^{\pi/2} \frac{\cos(n-k+\lambda+1)\theta}{\theta} d\theta .$$

Assume first that $k < n+1+\lambda$. Then changing variables by $(n-k+\lambda+1)\theta = y$, we find

$$J = \frac{B}{n}\int_{(n-k+\lambda+1)/n}^{1} \frac{\cos y dy}{y} + \frac{B}{n}\int_1^{n-k+\lambda+1} \frac{\cos y dy}{y} .$$

The second term is $O(1/n)$ by an integration by parts. The first term is

$$\begin{aligned} \frac{B}{n}\int_{(n-k+\lambda+1)/n}^{1} & \frac{dy}{y} + O\Big(\frac{1}{n}\int_{(n-k+\lambda+1)/n}^{1} y\, dy\Big) \\ &= \frac{B}{n}\log\frac{n}{n-k+\lambda+1} + O\Big(\frac{1}{n}\Big) . \end{aligned}$$

If $k > n+\lambda+1$ we get instead that

$$J = \frac{B}{n}\log\frac{n}{k-n-\lambda-1} + O\Big(\frac{1}{n}\Big) .$$

Using all of the estimates, we have

$$\begin{aligned} a_n^\lambda = O\Big[\frac{1}{n}\sum_{k=0}^{[n/2]} |a_k|\Big] &+ A\sum_{k=[n/2]}^{2n} \frac{a_k}{n-k+\lambda} \\ &+ \frac{B}{n}\sum_{k=[n/2]}^{2n} a_k \log\frac{n}{|n-k+\lambda+1|} \\ &+ O\Big[\frac{1}{n}\sum_{k=[n/2]}^{2n} |a_k|\Big] + \lim_{r\to 1}\sum_{k=2n}^{\infty} a_k r^k A(k,n) \end{aligned}$$

where $A(k, n) = O(k^{-1})$.

To show the $l^{p,\alpha}$ boundedness of these sums we need two forms of Hardy's inequality and M. Riesz's inequality for the discrete Hilbert transform as well as Lemma 1. The relevant forms of Hardy's inequality are in [6], p. 255, # 346 (a), (b), part (α). The continuous analogue of the $l^{p,\alpha}$ boundedness of the discrete Hilbert transform is in [5].

Using these inequalities we see that the first and fourth terms are bounded by Hardy's inequality. By dominated convergence we may let $r \to 1$ in the fifth term and it is bounded in $l^{p,\alpha}$ by Hardy's inequality. The second term is just the discrete Hilbert transform plus two terms like the first and last terms. Thus it is bounded in $l^{p,\alpha}$. The third term is handled by Lemma 1.

In actual fact the second and third terms given above are not exactly right since the terms in which k and n have opposite parity are zero. The notation to include this is too cumbersome to be worth including and this point causes no trouble.

To show that $\| a_n \|_{p,\alpha} \leqq A \| a_n^\lambda \|_{n,\alpha}$ observe that (formally)

$$a_k = \int_0^\pi f(\theta) \cos k\theta d\theta = \sum_{n=0}^\infty a_n^\lambda t_n^\lambda \int_0^\pi P_n^\lambda(\cos\theta) \cos k\theta (\sin\theta)^\lambda d\theta \,.$$

We have the same $G(k, n)$ that we analysed above and so no more work need be done on it. However there is the problem of Abel summability of ultraspherical expansions. Estimates for the Poisson kernel which allows us to prove the dominated L^1 convergence of the Abel means are in [11, § 4]. The argument that is needed to prove this is well known.

We now consider the general case of Theorem 1 with $\mu < \lambda < \mu + 1$. The proof proceeds along the same general lines but the formulas for P_n^λ that we need are considerably more complicated. To take the place of (4) we need the following result of Gegenbauer [4].

If $0 < \alpha < \beta$ then

$$P_n^\beta(\cos\theta) = \sum_{j=0}^{[n/2]} \alpha_j P_{n-2j}^\alpha(\cos\theta) \tag{8}$$

where

$$\alpha_j = \frac{\Gamma(\alpha)(n - 2j + \alpha)\Gamma(j + \beta - \alpha)\Gamma(n - j + \beta)}{\Gamma(\beta)\Gamma(\beta - \alpha)j!\Gamma(n - j + \alpha + 1)} \,.$$

Instead of (5) we need a result that follows from (8) and is given in [2]. If $(\beta - 1)/2 < \alpha < \beta$ then

$$(\sin\theta)^{2\alpha} P_n^\alpha(\cos\theta) = \sum_{j=0}^\infty \beta_j P_{n+2j}^\beta(\cos\theta)(\sin\theta)^{2\beta} \tag{9}$$

where

$$\beta_j = \frac{\Gamma(\beta)2^{2\beta-2\alpha}(n+2j)!(n+2j+\beta)\Gamma(n+2\alpha)\Gamma(n+j+\beta)\Gamma(j+\beta-\alpha)}{\Gamma(\beta-\alpha)\Gamma(\alpha)j!n!\Gamma(n+j+\alpha+1)\Gamma(n+2j+2\beta)} .$$

Observe that β_j is positive if $\alpha < \beta$. This result also holds for $\alpha > \beta$ but then the coefficients are no longer positive and changes must be made for $\alpha = \beta + 1, \beta + 2, \cdots$, since the right hand side is then a finite sum. A simple computation shows that

$$(10) \qquad \alpha_j \sim (n - 2j + \alpha) j^{\beta-\alpha-1} n^{\beta-\alpha-1}$$

and

$$(11) \qquad \beta_j \sim (n + 2j)^{2-2\beta} n^{2\alpha-1} j^{\beta-\alpha-1} (n + j)^{\beta-\alpha-1} \sim (n + j)^{1-\beta-\alpha} n^{2\alpha-1} j^{\beta-\alpha-1} ,$$

for $\alpha < \beta$. For $\alpha > \beta$ and $\alpha \neq \beta + 1, \beta + 2, \cdots$, we have

$$|\beta_j| \sim (n + j)^{1-\beta-\alpha} n^{2\alpha-1} j^{\beta-\alpha-1} .$$

By $a_j \sim b_j$ we mean $0 < c \leqq a_j/b_j \leqq C < \infty$.

If in (9) we let $n = 0$ and use (1) we have

$$(12) \qquad \left| \int_0^\pi (\sin\theta)^{2\alpha} P_{2j}^{\beta}(\cos\theta) d\theta \right| = O(j^{2\beta-2\alpha-2}) .$$

Next we need something to take the place of $\cos x \cos y = [\cos(x + y) + \cos(x - y)]/2$ and $\sin x \cos y = [\sin(x + y) + \sin(x - y)]/2$. For the first we use a formula of Dougall which is given in [9] and reduces to it for $\lambda \to 0$. If $\lambda > 0$ then

$$(13) \qquad \frac{P_n^\lambda(x)}{P_n^\lambda(1)} \frac{P_m^\lambda(x)}{P_m^\lambda(1)} = \sum_{k=|n-m|}^{n+m} c_\lambda(k, m, n) \frac{P_k^\lambda(x)}{P_k^\lambda(1)}$$

where $c_\lambda(k, m, n) \geqq 0$ and $\sum_k c_\lambda(k, m, n) = 1$. We define $c_\lambda(k, m, n) = 0$ if $k < |n - m|$ or $k > n + m$ and then we may sum on all non-negative k. The numbers c_λ are known [9], but we shall not need them in our argument.

For the second formula above we use the following substitute which again reduces to it for $\lambda \to 0$. If $\lambda > 0$ then

$$(14) \qquad \frac{P_n^{\lambda+1}(x)}{P_n^{\lambda+1}(1)} \frac{P_m^\lambda(x)}{P_m^\lambda(1)} = \sum_{k=|n-m|-2}^{n+m} d_\lambda(k, m, n) \frac{P_k^{\lambda+1}(x)}{P_k^{\lambda+1}(1)}$$

where $d_\lambda \geqq 0$ if $n \geqq m - 1$. This is found in [1]. From (14) it follows that $\sum_{k=0}^\infty d_\lambda(k, m, n) = 1$ where $d_\lambda(k, m, n) = 0$ if $k < |n - m| - 2$ or $k > n + m$. Finally recall that

$$(15) \qquad P_n^\lambda(1) \sim n^{2\lambda-1} .$$

These results are sufficient to allow us to estimate $\int_0^\pi \varphi_n^\lambda(\theta)\varphi_k^\mu(\theta)d\theta$ for $\mu < \lambda < \mu + 1$ and $k \leqq n/2$ or $n \leqq k/2$. To estimate this integral for $k/2 < n < 2k$ we use the following asymptotic formulas due to Szegö.

LEMMA 2. *Let $\mu > 0$, μ not an integer. Then*

$$P_n^\mu(\cos\theta) = \frac{2}{\pi}\sin\pi\mu\,\frac{\Gamma(n+2\mu)}{\Gamma(\mu)}$$
$$\times\left[\sum_{m=0}^{p-1}\frac{\Gamma(m+\mu)\Gamma(m-\mu+1)}{\Gamma(n+m+\mu+1)m!}\,\frac{\cos\left[(n+m+\mu)\theta-(m+\mu)\frac{\pi}{2}\right]}{(2\sin\theta)^{m+\mu}}\right]+R_p$$

where

$$|R_p| = O[(\sin\theta)^{-p-\mu}n^{\mu-p-1}]$$

and the O holds uniformly for $0 < \theta < \pi$.

For $\mu = 1, 2, 3, \cdots$ we have

LEMMA 3.

$$P_n^\mu(\cos\theta) = 2\sum_{m=0}^{\mu-1}(-1)^m\binom{m+\mu-1}{m}\binom{n+2\mu-1}{\mu-m+1}$$
$$\times\frac{\cos\left[(n+m+\mu)\theta-(m+\mu)\frac{\pi}{2}\right]}{[(2\sin\theta)(m+\mu)]}.$$

The same estimates hold for an error term in Lemma 3 as in Lemma 2 if one stops before $m = \mu - 1$. These two lemmas are in [14, p. 49 and p. 59]. In fact we do not need the full force of either of these Lemmas but they are relatively inaccessible and not as well known as they should be.

Now to complete the proof of Theorem 1. Let $\mu < \lambda < \mu + 1$ and $f_r(\theta) = \sum_{n=0}^\infty a_n^\mu r^n \varphi_n^\mu(\theta)$. Then by dominated convergence and the boundedness of the Abel means of an ultraspherical expansion we have

$$a_n^\lambda = \lim_{r\to 1-}\int_0^\pi f_r(\theta)\varphi_n^\lambda(\theta)d\theta = \lim_{r\to 1-}\sum_{k=0}^\infty a_k^\mu r^k\int_0^\pi \varphi_k^\mu(\theta)\varphi_n^\lambda(\theta)d\theta .$$

As above we need to estimate $\int_0^\pi \varphi_k^\lambda(\theta)\varphi_n^\mu(\theta)d\theta = G(k, n)$ for three cases, $k \leqq n/2$, $n/2 < k < 2n$, and $2n \leqq k$. Consider the third case first. Using (8) and (13) we have

$$\begin{aligned} G(k, n) &= t_n^\lambda t_k^\mu \int_0^\pi P_k^\mu(\cos\theta) P_n^\lambda(\cos\theta)(\sin\theta)^{\lambda+\mu} d\theta \\ &= \sum_{j=0}^{[n/2]} t_n^\lambda t_k^\mu \alpha_j \int_0^\pi P_k^\mu(\cos\theta) P_{n-2j}^\mu(\cos\theta)(\sin\theta)^{\lambda+\mu} d\theta \\ &= \sum_{j=0}^{[n/2]} t_n^\lambda t_k^\mu \alpha_j P_k^\mu(1) P_{n-2j}^\mu(1) \sum_{l=0}^{\infty} c_\mu(l, k, n-2j)[P_l^\mu(1)]^{-1} \\ &\quad \cdot \int_0^\pi P_l^\mu(\cos\theta)(\sin\theta)^{\lambda-\mu}(\sin\theta)^{2\mu} d\theta . \end{aligned}$$

Then using (10), (12), (15), and recalling that $c_\mu(l, k, n-2j) = 0$ unless $k - n + 2j \leqq l \leqq k + n - 2j$ and so $l \sim k$.

For simplification of printing we use n, k, j in the following arguments instead of $n+1, k+1, j+1$, etc. This leads to some infinite terms which clearly aren't infinite and they are to be interpreted in the obvious way.

$$\begin{aligned} |G(k, n)| &\leqq \sum_{j=0}^{[n/2]} (n)^{1-\lambda}(k)^{1-\mu}(n-2j+\mu)(j)^{\lambda-\mu-1}(n)^{\lambda-\mu-1}(n-2j)^{2\mu-1}(k)^{\mu-\lambda-2} \\ &\leqq A(n/k)^\lambda (k)^{-1} \leqq A(k)^{-1} . \end{aligned}$$

Next we consider $G(k, n)$ for $k \leqq n/2$. Using (9) and (14) we have

$$\begin{aligned} G(k, n) &= \sum_{j=0}^{\infty} t_n^\lambda t_k^\mu \beta_j \int_0^\pi P_k^\mu(\cos\theta) P_{n+2j}^{\mu+1}(\cos\theta)(\sin\theta)^{3\mu-\lambda+2} d\theta \\ &= \sum_{j=0}^{\infty} t_n^\lambda t_k^\mu \beta_j P_k^\mu(1) P_{n+2j}^{\mu+1}(1) \sum_{l=0}^{\infty} d_\mu(l, k, n+2j)[P_l^{\mu+1}(1)]^{-1} \\ &\quad \cdot \int_0^\pi P_l^{\mu+1}(\cos\theta)(\sin\theta)^{\mu-\lambda}(\sin\theta)^{2\mu+2} d\theta . \end{aligned}$$

This time $d_\lambda(l, k, n+2j) = 0$ unless $n/2 + j \leqq l \leqq 2n + 4j$ (actually it is zero for many values in this range also but that doesn't matter) and so $l \approx n + 2j$ and thus using (11), (12), and (15) we obtain

$$\begin{aligned} |G(k, n)| &\leqq \sum_{j=0}^{\infty} (n)^{1-\lambda}(k)^{1-\mu}(k)^{2\mu-1}\beta_j \int_0^\pi P_{n+2j}^{\mu+1}(\cos\theta)(\sin\theta)^{3\mu-\lambda+2} d\theta . \\ &\leqq \sum_{j=0}^{\infty} (n)^{1-\lambda}(k)^{\mu}(n)^{2\lambda-1}(n+j)^{-\lambda-\mu}(j)^{\mu-\lambda} \int_0^\pi P_{n+2j}^{\mu+1}(\cos\theta)(\sin\theta)^{3\mu-\lambda+2} d\theta \\ &\leqq (n)^\lambda (k)^\mu \sum_{j=0}^{\infty} (j)^{\mu-\lambda}(n+j)^{-\lambda-\mu}(n+j)^{-2+2(\mu+1)-2+\lambda-3\mu} \\ &\leqq (n)^\lambda (k)^\mu \left[\sum_{j=0}^{n} (j)^{\mu-\lambda} n^{-2-2\mu} + \sum_{j=n}^{\infty} (j)^{-2-\mu-\lambda} \right] \\ &\leqq (n)^\lambda (k)^\mu [(n)^{-1-\mu-\lambda} + (n)^{-1-\mu-\lambda}] \leqq [k/n]^\mu (n)^{-1} . \end{aligned}$$

For the terms with $k/2 \leqq n \leqq 2k$ we use Lemmas 2 and 3. As in the case $\mu = 0$, $0 < \lambda < 1$ we first reduce the integral to

$$\int_{1/n}^{\pi/2} \varphi_n^\lambda(\theta) \varphi_k^\mu(\theta) d\theta + O(n^{-1})$$

and then terms of the same type as previously appear. The proof is then finished by the same appeal to Hardy's inequality, M. Riesz's inequality, and Lemma 1.

Theorem 1 then follows by a repeated application of the inequalities just proven.

2. A lemma. We now give a proof of Lemma 1. Recall that

$$b_n = \frac{1}{n} \sum_{k=[n/2]}^{2n} a_k \log \frac{n}{|n+\lambda+1-k|}\,.$$

We define $A_k = \sum_{j=[n/2]}^{k} a_j$. Then

$$\begin{aligned} b_n &= \frac{1}{n} \sum_{k=[n/2]+1}^{2n} (A_k - A_{k-1}) \log \frac{n}{|n+\lambda+1-k|} + \frac{1}{n} a_{[n/2]} \log \frac{n}{\frac{n}{2}+\lambda+1} \\ &= \frac{1}{n} \sum_{k=[n/2]+1}^{2n-1} A_k \left[\log \frac{n}{|n+\lambda+1-k|} - \log \frac{n}{|n+\lambda-k|}\right] \\ &\quad + O\left(\frac{1}{n} a_{[n/2]}\right) + O\left(\frac{A_{2n}}{n^2}\right) \\ &= +\frac{1}{n} \sum_{k=[n/2]}^{2n} A_k \log \left|\frac{n+\lambda-k}{n+\lambda+1-k}\right| + R_n \end{aligned}$$

where R_n is a bounded sequence in $l^{p,\alpha}$ if $\{a_n\} \in l^{p,\alpha}$. But

$$\begin{aligned} \log \left|\frac{n+\lambda-k}{n+\lambda+1-k}\right| &= -\log \left|1 + \frac{1}{n+\lambda-k}\right| \\ &= +\frac{1}{k-n-\lambda} + O\frac{1}{(k-n-\lambda)^3}\,. \end{aligned}$$

So we have

$$b_n = -\frac{1}{n} \sum_{k=[n/2]}^{2n} \frac{A_k}{n+\lambda-k} + \frac{1}{n} \sum_{k=[n/2]}^{2n} \frac{A_k}{(k-n-\lambda)^2} + R_n\,.$$

The second term is a bounded sequence in $l^{p,\alpha}$ by [6, p. 198, # 274]. We write the first term as

$$-\frac{1}{n} \sum_{k=[n/2]}^{2n} \frac{A_k}{n+\lambda-k} = \sum_{k=[n/2]}^{2n} \frac{\frac{-A_k}{(k-\lambda)}}{n+\lambda-k} + \sum_{k=[n/2]}^{2n} \frac{A_k}{n(k-\lambda)}\,.$$

But $A_k/(k-\lambda)$ is in $l^{p,\alpha}$ and so we have that $\{b_n\}$ is an $l^{p,\alpha}$ sequence by Hardy's inequality and M. Riesz's inequality for the discrete Hilbert transform.

A similar proof also shows that

$$c_n = \sum_{k=[n/2]}^{2n} \frac{a_k}{k} \log \frac{n}{|n - k + \alpha|}$$

is a bounded operator for α not an integer. If α is an integer the transformations are bounded if the term when the logarithm is undefined is dropped.

A similar theorem is also true in the continuous case where an integration by parts takes the place of our summation by parts.

3. Applications. Our first application is an analogue of a theorem of Hardy and Littlewood concerning the Fourier coefficients of even functions, monotonically decreasing in $(0, \pi)$, [16, p. 130]. Their theorem is

THEOREM A. *If $f(\theta)$ is a decreasing integrable function on $(0, \pi)$ and if a_n are the Fourier cosine coefficients of f, then*

$$\left[\sum_{n=0}^{\infty} |a_n|^p (n+1)^\alpha\right]^{1/p}$$

is finite if and only if

$$\left[\int_0^\pi |f(\theta)|^p \theta^{p-2+\alpha} d\theta\right]^{1/p}$$

is finite, $1 < p < \infty$, $-1 < \alpha < p - 1$.

From this and Theorem 1 we obtain

THEOREM 2. *Let $f(\theta)$ be decreasing and integrable on $(0, \pi)$ and* $a_n = t_n^\lambda \int_0^\pi f(\theta) P_n^\lambda(\cos\theta)(\sin\theta)^\lambda d\theta$, $0 < \lambda$. *Then* $\left[\sum_{n=0}^{\infty} |a_n|^p (n+1)^\alpha\right]^{1/p}$ *is finite if and only if* $\left[\int_0^\pi |f(\theta)|^p \theta^{p-2-\alpha} d\theta\right]^{1/p}$ *is finite,* $1 < p < \infty$, $-1 < \alpha < p - 1$.

Another application is the analogue of the Marcinkiewicz Multiplier theorem. In the case of Fourier coefficients it is due to Sunouchi [13] for $\{a_n\} \in l^p$ and to Igari [10] for $\{a_n\} \in l^{p,\alpha}$.

THEOREM B. *Let* $f(\theta) \in L^1(0, \pi)$, $a_n = \int_0^\pi f(\theta) \cos n\theta d\theta$, $|t(\theta)| \leqq C$,

$$\int_{\pi 2^{-n-1}}^{\pi 2^{-n}} |dt(\theta)| \leqq C, \qquad n = 0, 1, \cdots.$$

Then if $b_n = \int_0^\pi t(\theta) f(\theta) \cos n\theta d\theta$ *and* $\{a_n\} \in l^{p,\alpha}$, $1<p<\infty$, $-1<\alpha<p-1$, *we have* $\{b_n\} \in l^{p,\alpha}$ *and* $\|b_n\|_{p,\alpha} \leqq A \|a_n\|_{p,\alpha}$.

From this we get a form of the Marcinkiewicz theorem for ultraspherical coefficients.

THEOREM 3. *Let* $f(\theta) \in L^1(0, \pi)$, $a_n = t_n^\lambda \int_0^\pi f(\theta) P_n^\lambda(\cos\theta)(\sin\theta)^\lambda d\theta$, $\lambda > 0$, $|t(\theta)| \leqq C$,

$$\int_{\pi 2^{-n-1}}^{\pi 2^{-n}} |dt(\theta)| \leqq C, \qquad n = 0, 1, \cdots .$$

Then if $b_n = t_n^\lambda \int_0^\pi t(\theta) f(\theta) P_n^\lambda(\cos\theta)(\sin\theta)^\lambda d\theta$ *and if* $\{a_n\} \in l^{p,\alpha}$, $1<p<\infty$, $-1 < \alpha < p - 1$ *then* $\{b_n\} \in l^{p,\alpha}$ *and* $\|b_n\|_{p,\alpha} \leqq A \|a_n\|_{p,\alpha}$.

For $p = 2$ Hirschman has already obtained a form of the Marcinkiewicz theorem. If we let

$$t_r(\theta) = \begin{cases} 1 & 0 \leqq \theta \leqq 1/r \\ 0 & 1/r < \theta \leqq \pi \end{cases}$$

then we get the projection theorem of Hirschman [8] but only for ultraspherical coefficients. Hirschman proves his result for Jacobi coefficients and presumably Theorem 1 is also true for Jacobi polynomials. However this is still open.

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TWO NOTES ON REGRESSIVE ISOLS

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This paper deals with regressive functions and regressive isols. It was proven by J. C. E. Dekker in [2] that the collection Λ_R of all regressive isols is not closed under addition. In the first note of this paper we shall given another proof of this fact by considering a new relation, denoted by $\overset{*}{\vee}$, between infinite regressive isols. Let A and B denote infinite regressive isols. The main results established in the first note are:

(1) $A \leqq^* B \Longrightarrow A \overset{*}{\vee} B$, yet not conversely.

(2) $A + B \in \Lambda_R \Longrightarrow A \overset{*}{\vee} B$, yet not conversely.

(3) There exist infinite regressive isols which are not $\overset{*}{\vee}$ related.

(4) Λ_R is not closed under addition.

In addition, the following result is stated.

(5) $A + B \in \Lambda_R \Longrightarrow \min(A, B) \leqq A + B$, yet not conversely.

In the second note we consider the $\leqq^*$ relation between regressive isols. A natural question concerning this relation is whether A $\leqq^* B$, where A and B are regressive isols, is a necessary or a sufficient condition for the sum $A + B$ to be regressive. In the second note we show that this condition is neither necessary nor sufficient.

We shall assume that the reader is familiar with the notations, terminology and main results of [1] and [2].

Preliminaries. Let $\varepsilon = \{0, 1, 2, 3, \cdots\}$ be the set of nonnegative integers (*numbers*). A one-to-one function t_n from ε into ε is *regressive* if there is a partial recursive function $p(x)$ such that $\rho t \subseteqq \delta p$ and $p(t_0) = t_0$, $(\forall n)[p(t_{n+1}) = t_n]$. The function p is a *regressing function* of t_n if p has the following additional properties: $\rho p \subseteqq \delta p$ and $(\forall x)[x \in \delta p \rightarrow (\exists n)[p^{n+1}(x) = p^n(x)]]$. It is known (cf. [1]) that every regressive function has a regressing function. A set is *regressive* if it is finite or the range of a regressive function. A set is *retraceable* if it is finite or the range of a strictly increasing regressive function. Let p be a regressing function of t_n, then the function p^* is defined by: $\delta p^* = \delta p$ and $p^*(x) = (\mu n)[p^{n+1}(x) = p^n(x)]$. It follows that p^* is a partial recursive function and $(\forall n)[p^*(t_n) = n]$.

Let s_n and t_n be two one-to-one functions from ε into ε. Then $s_n \leqq^* t_n$, if there is a partial recursive function f such that

$$(1) \qquad \rho s \subseteqq \delta f \quad \text{and} \quad (\forall n)[f(s_n) = t_n] .$$

Also, s_n and t_n are said to be *recursively equivalent* (denoted $s_n \simeq t_n$)

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if there is a one-to-one partial recursive function f such that (1) holds. Let σ and τ be two sets. Then $\sigma \leqq^* \tau$, if either σ is finite and card. $\sigma \leqq$ card. τ, or σ is infinite and there is a partial recursive function f such that $\sigma \subseteqq \delta f$, f is one-to-one on σ and $f(\sigma) = \tau$. Let S and T be two isols. Then $S \leqq^* T$, if there are sets $\sigma \in S$ and $\tau \in T$ such that $\sigma \leqq^* \tau$. The following propositions will be useful:

(a) Retraceable sets are either recursive or immune.

(b) Every function recursively equivalent to a regressive function is regressive.

(c) Every set recursively equivalent to a regressive set is regressive.

(d) Let $\sigma = \rho s_n$ and $\tau = \rho t_n$ where s_n and t_n are one-to-one regressive functions. Then $\sigma \leqq^* \tau$ if and only if $s_n \leqq^* t_n$, and $\sigma \simeq \tau$ if and only if $s_n \simeq t_n$.

(e) Let s_n and t_n be one-to-one functions from ε into ε. Then $s_n \simeq t_n$ if and only if $s_n \leqq^* t_n$ and $t_n \leqq^* s_n$.

Proposition (a) is proven in [3]. Propositions (b) and (c), and the second part of (d) are proven [1]. Both (e) and the first part of (d) are given in [2].

Two sets α and β are said to be *separated* (denoted $\alpha \mid \beta$) if there are disjoint r.e. sets α^* and β^* such that $\alpha \subseteqq \alpha^*$ and $\beta \subseteqq \beta^*$. Two functions a_n and b_n are said to be *separated* (denoted $a_n \mid b_n$) if their ranges are separated sets. We will use the familiar primitive recursive functions j, k and l defined by

$$j(x, y) = x + (x + y)(x + y + 1)/2 ,$$
$$j(k(n), l(n)) = n .$$

The function j maps ε^2 one-to-one onto ε.

Note 1. The $\overset{*}{\vee}$ relation.

DEFINITION 1. Let a_n and b_n be any two one-to-one functions from ε into ε. Then $a_n \overset{*}{\vee} b_n$ if there is a partial recursive function $p(x)$ such that

$$(\forall n)[a_n \in \delta p \text{ and } p(a_n) = b_n) \vee (b_n \in \delta p \text{ and } p(b_n) = a_n)] .$$

The following proposition can be readily proven using the definitions of the concepts involved. Its proof will be omitted.

PROPOSITION 1.1. Let a_n and b_n be any two one-to-one functions from ε into ε. Then

(a) $a_n \overset{*}{\vee} b_n \Longrightarrow b_n \overset{*}{\vee} a_n$,

(b) $a_n \leqq^* b_n \Longrightarrow a_n \overset{*}{\vee} b_n$,

$$\text{(c)}\quad \left.\begin{array}{l} a_n \overset{*}{\vee} b_n, \\ a_n \simeq a'_n,\ b_n \simeq b'_n\ , \\ a_n \mid b_n,\ a'_n \mid b'_n\ , \end{array}\right\} \Longrightarrow a'_n \overset{*}{\vee} b'_n\ .$$

DEFINITION 2. Let A and B be any two infinite regressive isols. Then $A \overset{*}{\vee} B$ if there are regressive functions a_n and b_n such that

$$\rho a_n \in A,\quad \rho b_n \in \beta,\quad a_n \mid b_n \quad \text{and} \quad a_n \overset{*}{\vee} b_n\ .$$

RERARK. In view of Proposition (d) and part (c) of Proposition 1.1, we see that if A and B are infinite regressive isols, then $A \overset{*}{\vee} B$ means that $a_n \overset{*}{\vee} b_n$ for every pair a_n and b_n of separated, regressive functions ranging over sets in A and B respectively.

THEOREM 1.1. *Let A and B be infinite regressive isols. Then*

$$A \leqq^* B \Longrightarrow A \overset{*}{\vee} B\ .$$

Proof. Let a_n and b_n be any two (one-to-one) regressive functions ranging over sets in A and B respectively and such that $a_n \leqq^* b_n$. Set $a'_n = 2a_n$ and $b'_n = 2b_n + 1$. Then $a_n \simeq a'_n$, $b_n \simeq b'_n$ and $a'_n \mid b'_n$. Taking into account Propositions (b), (c) and (d) it follows that a'_n and b' are separated, regressive functions which range over sets in A and B respectively. In addition, $a_n \leqq^* b_n$ implies $a'_n \leqq^* b'_n$. By Proposition 1.1 (b) this means $a'_n \overset{*}{\vee} b'_n$, and therefore $A \overset{*}{\vee} B$.

THEOREM 1.2. *For all infinite regressive isols A and B,*

$$A + B \in \Lambda_R \Longrightarrow A \overset{*}{\vee} B\ .$$

Proof. Let A and B denote two infinite regressive isols whose sum is also regressive. Let a_n and b_n be regressive functions with $\alpha = \rho a_n \in A$, $\beta = \rho b_n \in B$ and $\alpha \mid \beta$. Then $\alpha + \beta \in A + B$ and $\alpha + \beta$ is a regressive set. Let c_n be a regressive function ranging over the set $\alpha + \beta$ and let $p(x)$ be a regressing function of c_n. Set

$$\delta = \{x \mid (x = a_n \text{ and } p^*(b_n) < p^*(a_n)) \vee (x = b_n \text{ and } p^*(a_n) < p^*(b_n))\}\ .$$

We note that $\delta \subseteqq \alpha + \beta$ and that for each number n, exactly one of the numbers a_n and b_n belongs to δ. Let the function f with domain δ be defined by

$$f(x) = \begin{cases} b_n, & \text{if} \quad x = a_n\ , \\ a_n, & \text{if} \quad x = b_n\ . \end{cases}$$

It is easily seen that if f has a partial recursive extension then $a_n \overset{*}{\vee} b_n$.

Since a_n and b_n are separated functions this fact would also imply that $A \overset{*}{\vee} B$. Hence to complete the proof it suffices to show that f has a partial recursive extension. For this purpose, assume that $x \in \delta$. Since α and β are separated sets we can determine whether $x \in \alpha$ or $x \in \beta$. First suppose that $x \in \alpha$. Taking into account that a_n and c_n are regressive functions, we can find the numbers u and v such that $x = a_u = c_v$. The number a_u belongs to δ and therefore

$$b_u \in (c_0, c_1, \cdots, c_{v \dot{-} 1}) = \{p^r(x) \mid 1 \leqq r \leqq v\} .$$

The members of the set on the right side can be effectively obtained from x, since p is a partial recursive function. In addition, using once again the separability of the sets α and β, and the regressiveness of the function b_n, it follows that we can find the number b_u. This gives the value of $f(x)$. In a similar fashion one can determine the value of $f(x)$ in the event $x \in \beta$. From these remarks we can conclude that f will have a partial recursive extension. This completes the proof.

REMARK. We shall state without proof, two additional facts which can be established in the proof of Theorem 1.2. These are

(a) $$\delta \in \min(A, B) ,$$

(b) $$\delta \mid (\alpha + \beta) - \delta .$$

Since $\alpha + \beta \in A + B$, these facts imply that

(∗) $$\min(A, B) \leqq A + B .$$

In the proof of Theorem 1.2, A and B were assumed to be infinite regressive isols. However, it is easily seen that the relation denoted by (∗) is also true in the event either A or B is finite, for in this case $\min(A, B)$ assumes one of the values (A, B). From these remarks one has the following

THEOREM. *For all regressive isols A and B,*

$$A + B \in \Lambda_R \Longrightarrow \min(A, B) \leqq A + B .$$

The statement obtained by reversing the implication in the above theorem is false, for in the second note it is shown that there are two infinite regressive isols which are comparable relative to the $\leqq^*$ relation, hence their minimum assumes one of these two values, and yet whose sum is not regressive. According to Theorem 1.1, this also means that reversing the implication in Theorem 1.2 yields a false statement as well.

THEOREM 1.3. *There exist infinite regressive isols A and B which* *are not* $\overset{*}{\vee}$ *related.*

Proof. Let $\{p_i\}$ be an enumeration of partial recursive functions of one variable such that:

(a) every partial recursive function of one variable occurs at least once in $\{p_i\}$,

(b) $p_0(1) \neq 3$ and $p_0(3) \neq 1$.

We shall define two functions a_n and b_n such that the recursive equivalence types, $A = \text{Req}\,\rho a_n$ and $B = \text{Req}\,\rho b_n$ satisfy the conditions of the Theorem.

Put $a_0 = 1$ and $b_0 = 3$. We note that (b) implies

$$(1) \qquad p_0(a_0) \neq b_0 \quad \text{and} \quad p_0(b_0) \neq a_0 .$$

Let $t \geqq 1$ and suppose that $a_0, \cdots, a_{t-1}$ and $b_0, \cdots, b_{t-1}$ have already been defined. We define a_t and b_t by setting

$$a_t = j(a_{t-1}, u_t) ,$$
$$b_t = j(b_{t-1}, v_t) ,$$

where the numbers u_t and v_t will be defined in such a manner that

$$(2) \qquad p_t(a_t) \neq b_t \quad \text{and} \quad p_t(b_t) \neq a_t .$$

The definition of u_t *and* v_t. Set

$$\eta = \{u \mid j(a_{t-1}, u) \in \delta p_t\} ,$$
$$\zeta = \{v \mid j(b_{t-1}, v) \in \delta p_t\} .$$

We consider three cases:

Case I. $\eta' \neq \phi$. Let u be the smallest number belonging to η'. Then $p_t j(a_{t-1}, u)$ is undefined.

Subcase I.1. There exists a number v such that

$$p_t j(b_{t-1}, v) \neq j(a_{t-1}, u) .$$

Set

$$u_t = u ,$$
$$v_t = (\mu v)[p_t j(b_{t-1}, v) \neq j(a_{t-1}, u)] .$$

Subcase I.2. For all numbers v,

$$p_t j(b_{t-1}, v) = j(a_{t-1}, u) .$$

Consider the number $j(a_{t-1}, u + 1)$. Since j maps ε^2 one-to-one onto ε, it follows that $j(a_{t-1}, u+1) \neq j(a_{t-1}, u)$. Hence for all numbers v,

$$p_t j(b_{t-1}, v) \neq j(a_{t-1}, u+1) .$$

Clearly there exist numbers v' such that $j(b_{t-1}, v') \neq p_t(a_{t-1}, u+1)$. Set

$$u_t = u + 1 ,$$
$$v_t = (\mu v')[j(b_{t-1}, v') \neq p_t j(a_{t-1}, u+1)] .$$

Case II. $\zeta' \neq \phi$. Here we proceed in a fashion similar to Case I. The details are omitted.

Case III. $\eta' = \zeta' = \phi$, i.e., $\eta = \zeta = \varepsilon$, i.e.,

$$(\forall u)[j(a_{t-1}, u) \in \delta] \quad \text{and} \quad (\forall v)[j(b_{t-1}, v) \in \delta] ,$$

where $\delta = \delta p_t$. The numbers in the following four lists:

L1. $j(a_{t-1}, 0),\ j(a_{t-1}, 1), \cdots$

L2. $p_t j(b_{t-1}, 0),\ p_t j(b_{t-1}, 1), \cdots$

L3. $j(b_{t-1}, 0),\ j(b_{t-1}, 1), \cdots$

L4. $p_t j(a_{t-1}, 0),\ p_t j(a_{t-1}, 1), \cdots$

are therefore all defined. Since the function $j(x, y)$ is one-to-one, all numbers in L1 are distinct and all numbers in L3 are distinct.

*Subcase III.*1. L1 contains a number which does not occur in L2. Set

$$u_t = (\mu u)(\forall v)[j(a_{t-1}, u) \neq p_t j(b_{-1}, v)] .$$

Since all of the numbers in L3 are distinct, it follows that

$$(\exists v)[j(b_{t-1}, v) \neq p_t j(a_{t-1}, u_t)] .$$

Set

$$v_t = (\mu v)[j(b_{t-1}, v) \neq p_t j(a_{t-1}, u_t)] .$$

*Subcase III.*2. Every number of L1 occurs at least once in L2. Since L1 contains infinitely many numbers this implies that L2 contains infinitely many numbers. Hence, not only

$$(\forall u)(\exists v)[j(a_{t-1}, u) \neq p_t j(b_{t-1}, v)] ,$$

but also

$$(\forall u)(\exists \text{ infinitely many } v)[j(a_{t-1}, u) \neq p_t j(b_{t-1}, v)] .$$

This must be true in particular for $u = 0$. Thus there exists an infinite sequence $v_0, v_1, v_2, \cdots$ of distinct numbers such that

$$(\forall n)[j(a_{t-1}, 0) \neq p_t j(b_{t-1}, v_n)] .$$

Let

$$n^* = (\mu n)[j(b_{t-1}, v_n) \neq p_t j(a_{t-1}, 0)] .$$

Define

$$u_t = 0 ,$$
$$v_t = v_{n^*} .$$

This completes the definition of the numbers u_t and v_t, and hence also of the functions a_n and b_n. It is readily verified that the numbers a_t and b_t have been so defined as to satisfy (2), that is

$$p_t(a_t) \neq b_t \quad \text{and} \quad p_t(b_t) \neq a_t .$$

Combining this fact with (1) gives

(3) $$(\forall n)[p_n(a_n) \neq b_n \quad \text{and} \quad p_n(b_n) \neq a_n] .$$

Let

$$\alpha = \rho a_n \quad \text{and} \quad \beta = \rho b_n .$$

We claim:

(a) a_n and b_n are strictly increasing regressive functions and α and β are retraceable sets,

(b) $\alpha \mid \beta$,

(c) a_n and b_n are not $\overset{*}{\vee}$ related,

(d) α and β are immune sets.

Re (a): It follows from the definition of the function $j(x, y)$ that $x < j(x, y)$ for $x > 0$. Moreover, we have

$$a_0 > 0 \quad \text{and} \quad (\forall n)(\exists u)[a_{n+1} = j(a_n, u)] ,$$
$$b_0 > 0 \quad \text{and} \quad (\forall n)(\exists v)[b_{n+1} = j(b_n, v)] .$$

Hence

$$a_0 < a_1 < a_2 < \cdots \quad \text{and} \quad b_0 < b_1 < b_2 < \cdots ,$$

and therefore a_n and b_n are strictly increasing functions. Set

$$q(x) = \begin{cases} a_0, & \text{if} \quad x = a_0 , \\ k(x), & \text{if} \quad x \neq a_0 . \end{cases}$$

Clearly $q(x)$ is a recursive function and it can be readily shown that $q(x)$ is a regressing function of a_n. By replacing a_0 by b_0 in the definition of $q(x)$ yields a regressing function of b_n. Hence, a_n and b_n are each strictly increasing regressive functions and therefore α and β are retraceable sets.

Re (*b*): As a consequence of the definition of the functions a_n and b_n, we have

$$\alpha \subset \{z \mid z = 1 \vee (\exists n)[k^n(z) = 1]\} ,$$
$$\beta \subset \{z \mid z = 3 \vee (\exists n)[k^n(z) = 3]\} .$$

The sets appearing on the right sides are clearly r.e. Also, since $k(3) = 0$, $k(1) = 0$ and $k(0) = 0$, they are disjoint. Hence $\alpha \mid \beta$.

Re (*c*): Suppose that statement (c) were false; this would then mean $a_n \overset{*}{\vee} b_n$. Hence there would be a partial recursive function $p(x)$ such that

$$(4) \qquad (\forall n)[p(a_n) = b_n) \vee (p(b_n) = a_n)] .$$

Assume that the index of p in our enumeration is i, i.e., $p(x) = p_i(x)$. In view of (4), we would have

$$p_i(a_i) = b_i \cdot \text{ or } \quad p_i(b_i) = a_i .$$

However, according to (3) this statement must be false. This contradiction establishes the desired conclusion that a_n and b_n are not $\overset{*}{\vee}$ related.

Re (*d*): By part (a), each of the sets α and β is retraceable and hence is either recursive or immune. If one of these sets is recursive then the strictly increasing function ranging over the set would be a recursive function. Thus, if α were a recursive set then a_n would be a recursive function. In this event, we would have that

$$b_n \leqq^* n, \text{ since } b_n \text{ is a regressive function,}$$
$$n \leqq^* a_n, \text{ since } a_n \text{ is a regressive function,}$$

and, by the transitivity of the $\leqq^*$ relation, also that $b_n \leqq^* a_n$. By Proposition 1.1 (b), this means that $a_n \overset{*}{\vee} b_n$, which is not possible according to part (c). Therefore α must be an immune set. In a similar way it can be shown that β is also an immune set. This verifies (d).

To complete the proof, let

$$A = \text{Req}\, \alpha \quad \text{and} \quad B = \text{Req}\, \beta .$$

By statements (a) and (d) it follows that A and B are infinite regressive isols. In addition, combining statements (a) and (c) with the Remark following Definition 2 implies that A and B are not $\overset{*}{\vee}$ related. Hence A and B satisfy the requirements of the Theorem.

REMARK A. In [2, Theorem T2] it is shown that both the collection Λ_R of all regressive isols and the collection Λ_{CR} of all cosimple regressive isols are not closed under addition. We note that the first of these results can be obtained by combining Theorems 1.2 and 1.3.

REMARK B. It is readily seen from Definitions 1 and 2, that the $\overset{*}{\vee}$ relation for infinite regressive isols is both reflexive and symmetric. The following Corollary to Theorem 1.3 shows that $\overset{*}{\vee}$ is a not a transitive relation.

COROLLARY. *There exist infinite regressive isols A, B and W with $A \overset{*}{\vee} W$, $B \overset{*}{\vee} W$, while A and B are not $\overset{*}{\vee}$ related.*

Proof. Let A and B be any two infinite regressive isols which are not $\overset{*}{\vee}$ related. Set $W = \min(A, B)$. Then W is an infinite regressive isol with

$$W \leqq^* A \quad \text{and} \quad W \leqq^* B .$$

Hence, by Theorem 1.1

$$W \overset{*}{\vee} A \quad \text{and} \quad W \overset{*}{\vee} B .$$

According to our choice of A and B, the proof is complete.

Note 2. The main results of this note will establish the fact that $A \leqq^* B$ (where $A, B \in \Lambda_R$) represents neither a necessary condition nor a sufficient condition for the sum $A + B$ to belong to Λ_R. In the following discussion we will use the notion of the *degree of unsolvability of a regressive isol.* This concept is studied in [2]. If A is a regressive isol, then Δ_A will denote its degree of unsolvability.

THEOREM 2.1. *There exist regressive isols A and B with $A \leqq^* B$, yet whose sum $A + B$ is not regressive.*

Proof. Let P and Q denote two (infinite) regressive isols with different degrees of unsolvability, i.e., $\Delta_P \neq \Delta_Q$. Set

$$A = \min(P, Q) .$$

Then A is an infinite regressive isol such that

$$A \leqq^* P \quad \text{and} \quad A \leqq^* Q .$$

To complete the proof we need only show that at least one of the two isols $A+P$ and $A+Q$ is not regressive. To prove this fact, let us suppose otherwise, namely that both $A+P$ and $A+Q$ are regressive isols. Then according to [2, Proposition 17(d)], it follows that

$$\Delta_A = \Delta_P \quad \text{and} \quad \Delta_A = \Delta_Q \ ,$$

and therefore $\Delta_P = \Delta_Q$. This last equality contradicts our choice of P and Q. Hence, either $A+P$ or $A+Q$ is not regressive. If we define B to be P if $A+P \notin \Lambda_R$ and to be Q otherwise, then A and B will satisfy the requirements of the Theorem.

REMARK. It is proven in [2] that there are cosimple regressive isols with different degrees of unsolvability. Moreover, the minimum of two cosimple regressive isols is again a cosimple regressive isol. Thus, as a consequence of the previous proof, we see that the following result is also true.

THEOREM. *There exist cosimple regressive isols A and B with $A \leqq^* B$ yet whose sum $A+B$ is not regressive.*

THEOREM 2.2. *There exist regressive isols S and T which are incomparable relative to the $\leqq^*$ relation and whose sum is regressive.*

Proof. This shall be a constructive type of proof and we shall use a technique introduced in the proof of [4, Theorem 95]. The proof will progress in four steps.

Step I. In this step we shall define a particular function a_n from ε into ε, and show that it is strictly increasing and regressive.

Let $p_i(x)$ denote a function of the two variables i and x such that every one-to-one partial recursive function and no other function appears in the sequence $\{p_i\}$. For any numbers $t_0, \cdots, t_m, i$; $\max^* \{p_i(t_0), \cdots, p_i(t_m)\}$ is defined to be 0 if none of the $m+1$ numbers $p_i(t_0), \cdots, p_i(t_m)$ is defined; and is defined to be the maximum of those numbers $p_i(t_0), \cdots, p_i(t_m)$ which are defined; if at least one of them is defined.

The function a_n is defined by,

$$a_0 = 1 \ ,$$
$$a_{k+1} = j(a_k, u_{k+1}) \ , \quad \text{where}$$

$u_{k+1} = 0$, if either $k = 4n+1$ or $k = 4n+3$,

$u_{k+1} = (\mu y)[j(a_k, y) > \max^* \{p_n(a_0), \cdots, p_n(a_n)\}]$, if either $k = 4n$ or $k = 4n+2$.

It is readily seen that a_n is an everywhere defined function from ε into ε. Moreover, just as the function a_n in the proof of Theorem 1.3 was shown to be strictly increasing and regressive, it can be shown that

a_n is also strictly increasing and regressive.

Step II. Let the four sets δ_0, δ_1, δ_2 and δ_3 denote the ranges of the functions a_{4n}, a_{4n+1}, a_{4n+2} and a_{4n+3} respectively. Since each of the functions $4n$, $4n+1$, $4n+2$ and $4n+3$ is strictly increasing and recursive, it follows that each of the functions a_{4n}, a_{4n+1}, a_{n+2} and a_{4n+3} is regressive. Hence the four sets δ_0, δ_1, δ_2 and δ_3 are each regressive. We shall now prove:

(a) not $[\delta_0 \simeq \delta_1]$,
(b) not $[\delta_2 \simeq \delta_3]$,
(c) a_n ranges over an immune set.

Re (a): To prove statement (a), let us suppose that it is false. Then, by the enumeration in Step I, there would be a number i such that

$$\delta_0 \subset \delta p_i \quad \text{and} \quad p_i(\delta_0) = \delta_1 .$$

One consequence of this fact is

$$(1) \qquad (p_i(a_0), p_i(a_4), \cdots, p_i(a_{4i})) \subset \delta_1 .$$

By the definition of the function a_n, it follows that a_{4i+1} would exceed each of the numbers $p_i(a_0), p_i(a_4), \cdots, p_i(a_{4i})$. Since a_n is strictly increasing, the same would be true for a_{4j+1} with $j \geqq 1$. Hence from (1) we can conclude that

$$(p_i(a_0), p_i(a_4), \cdots, p_i(a_{4i})) \subset (a_1, a_5, \cdots, a_{4(i-1)+1}) .$$

However, the set on the left side has exactly $i+1$ members while the set on the right side has only i members. This contradicts the fact that p_i is a one-to-one function. This means that statement (a) must be true.

Re (b): We can prove statement (b) in a way similar to the one used to prove (a). Assuming that statement (b) is false implies that there is a number i such that

$$\delta_2 \subset \delta p_i, \quad \text{and} \quad p_i(\delta_2) = \delta_3 ,$$

and

$$(2) \qquad (p_i(a_2), p_i(a_6), \cdots, p_i(a_{4i+2})) \subset \delta_3 .$$

The definition of the function a_n implies that a_{4i+3} will exceed each of the numbers $p_i(a_2), p_i(a_6), \cdots, p_i(a_{4i+2})$, and since a_n is strictly increasing, the same will be true for a_{4j+3} with $j \geqq i$. Hence from (2) we can conclude that

$$(p_i(a_2), p_i(a_6), \cdots, p_i(a_{4i+2})) \subset (a_3, a_7, \cdots, a_{4(i-1)+3}) .$$

Yet the set on the left side has exactly $i + 1$ members while the set on the right side has exactly i members. This contradicts the fact that p_i is a one-to-one function. Therefore (b) must be a true statement.

Re (c): Since a_n is a strictly increasing regressive function it ranges over an infinite retraceable set. We know that this set will be either recursive or immune. But it is easily seen that if a_n ranges over an infinite recursive set then each of the sets δ_0 and δ_1 will also be infinite and recursive. According to statement (a), this is not possible. Hence a_n ranges over an immune set. This verifies (c) and also completes Step II.

Step III. Let

$$\sigma = \delta_0 + \delta_3 \quad \text{and} \quad \tau = \delta_1 + \delta_2 .$$

We shall now prove:

(d) σ and τ are infinite regressive sets,

(e) $\sigma \mid \tau$,

(f) not $[\sigma \leqq^* \tau]$,

(g) not $[\tau \leqq^* \sigma]$.

For this purpose, let

$$g(x) = \begin{cases} 4n , & \text{if} \quad x = 2n , \\ 4n + 3 , & \text{if} \quad x = 2n + 1 , \end{cases}$$

$$h(x) = \begin{cases} 4n + 1 , & \text{if} \quad x = 2n , \\ 4n + 2 , & \text{if} \quad x = 2n + 1 . \end{cases}$$

Then

$$(3) \qquad \rho a_{g(n)} = \sigma \quad \text{and} \quad \rho a_{h(n)} = \tau .$$

We also note that the functions g and h are each recursive and strictly increasing. In addition, their ranges are disjoint and the union of their ranges is ε.

Re (d): Since both g and h are strictly increasing, recursive functions and a_n is a regressive function it readily follows that both $a_{g(n)}$ and $a_{h(n)}$ are regressive function. By (3), this means that σ and τ are infinite regressive sets.

Re (e): From the two facts, a_n is a regressive function, and the ranges of the recursive functions g and h are disjoint, one can easily show that the two functions $a_{g(n)}$ and $a_{h(n)}$ are separated. This means that σ and τ are separated sets.

Re (*f*): Suppose that statement (f) were false, namely assume that $\sigma \leqq^* \tau$. According to Proposition (d), this implies that $a_{g(n)} \leqq^* a_{h(n)}$. Comparing the definitions of $g(x)$ and $h(x)$, we can conclude from this fact that

$$a_{4n} \leqq^* a_{4n+1} .$$

Clearly,

$$a_{4n+1} \leqq^* a_{4n} ,$$

and hence by Proposition (e),

$$a_{4n} \simeq a_{4n+1} .$$

According to Proposition (d), this implies that $\delta_0 \simeq \delta_1$ which is not possible in view of part (a). Therefore statement (f) is true.

Re (*g*): To verify (g) we can proceed as in the previous case. Suppose that statement (g) is false. This will imply that $a_{h(n)} \leqq^* a_{g(n)}$, and this fact gives

$$a_{4n+2} \leqq^* a_{4n+3} .$$

Clearly,

$$a_{4n+3} \leqq^* a_{4n+2} ,$$

and hence

$$a_{4n+2} \simeq a_{4n+3} .$$

This means that $\delta_2 \simeq \delta_3$ which is not possible in view of part (b). This contradiction establishes (g) and also completes Step III.

Step IV. Let

$$S = \text{Req}\,\sigma \quad \text{and} \quad T = \text{Req}\,\tau .$$

Both σ and τ are infinite subsets of the immune set ρa_n, and therefore are themselves immune sets. Also, by part (d), σ and τ are regressive. Hence

(i) S and T are infinite regressive isols.

Combining [2, Proposition P 10] and statement (f) and (g), implies that

(ii) S and T are incomparable relative to the $\leqq^*$ relation.

In view of (i) and (ii), in order to complete the proof it remains only to show that

(iii) $S + T \in \Lambda_R$.

Since σ and τ are separated sets, it follows that $\sigma + \tau \in S + T$. Moreover, $\sigma + \tau$ is a regressive set since $\sigma + \tau = \rho a_n$. Hence $S + T$ is a regressive isol. This verifies (iii) and completes the proof.

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SOLUTION OF AN INVARIANT SUBSPACE PROBLEM OF K. T. SMITH AND P. R. HALMOS

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The following theorem is proved.

Let T be a bounded linear operator on an infinite-dimensional Hilbert space H over the complex numbers and let $p(z) \neq 0$ be a polynomial with complex coefficients such that $p(T)$ is completely continuous (compact). Then T leaves invariant at least one closed linear subspace of H other than H or $\{0\}$.

For $p(z) = z^2$ this settles a problem raised by P. R. Halmos and K. T. Smith.

The proof is within the framework of Nonstandard Analysis. That is to say, we associate with the Hilbert space H (which, ruling out trivial cases, may be supposed separable) a larger space, $^*H$, which has the same formal properties within a language L. L is a higher order language but $^*H$ still exists if we interpret the sentences of L in the sense of Henkin. The system of *natural numbers* which is associated with $^*H$ is a nonstandard model of arithmetic, i.e., it contains elements other than the standard natural numbers. The problem is solved by reducing it to the consideration of invariant subspaces in a subspace of $^*H$ the number of whose dimensions is a nonstandard positive integer.

1. Introduction.

We shall prove:

MAIN THEOREM 1.1. *Let T be a bounded linear operator on an infinite-dimensional Hilbert space H over the complex numbers and let $p(z) \neq 0$ be a polynomial with complex coefficients such that $p(T)$ is completely continuous (compact). Then T leaves invariant at least one closed subspace of H other than H or $\{0\}$.*

For $p(z) = z^2$ this settles Problem No. 9 raised by Halmos in [2] and there credited to K. T. Smith. For this case, a first proof was given by one of us (A.R.) while the other (A.R.B.) provided an alternative proof which extends to the case considered in 1.1. The argument given below combines the two proofs, both of which are based on Nonstandard Analysis. The Nonstandard Analysis of Hilbert space was developed previously by A.R. as far as the spectral analysis of completely continuous self-adjoint operators (compare [7]) while A.R.B. has disposed of the spectral theorem for bounded self-adjoint operators

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by the same method. The general theory will be sketched here only as far as it is required for the proof of our main theorem.

Some of our arguments are adapted from the proofs of the theorem for $p(z) = z$, i.e., when T is itself completely continuous, which are due to von Neumann and Aronszajn for Hilbert space, as above, and to Aronszajn and K. T. Smith for general Banach spaces [1].

The particular version of Nonstandard Analysis which is convenient here relies on a higher order predicate language, L, which includes symbols for all complex numbers, all sets and relations of such numbers, all sets of such sets and relations, all relations of relations, etc. Quantification with respect to variables of all these types is permitted. Within this framework, a sequence of complex numbers, $y = s_n$, $n = 1, 2, 3, \cdots$, is given by a many-one relation $S(n, y)$ when n varies over the set of positive integers, P. The separable Hilbert space, H, may then be represented as a set of such sequences (i.e., as l_2) while a particular operator on H is identified with a relation of relations.

Let K be the set of sentences formulated in L which hold in the field of complex numbers, C. K includes sentences about, or involving, the sets of real numbers and of natural numbers, since these may be regarded as subsets of the complex numbers which are named in L. It also includes sentences about Hilbert space as represented above.

Nonstandard Analysis is based on the fact that, in addition to C, K possesses other models, which are proper extensions of C. We single out any one of them, $^*C$, calling it *the nonstandard model*, as opposed to the *standard model*, C. However, $^*C$ is a model of K only if the notions of set, relations, etc. are interpreted in $^*C$ in the sense of the higher order model theory of Henkin [3]. That is to say, the sets of sets, relations, etc., which are taken into account in the interpretation of a sentence in $^*C$ may (and will) be proper subsets of the corresponding sets over $^*C$ in the absolute sense. The sets, relations, etc. which are taken into account in the interpretation in $^*C$ will be called *admissible*.

The basic properties and notions of Nonstandard Analysis which are expounded in [4] and [5] are applicable here. Thus, an individual of $^*C$ (which will still be called a complex number) may or may not be an element of C, i.e., a complex number in the ordinary sense or *standard* number, briefly an *S-number*. Every *finite* complex number a is infinitely close to a unique standard complex number, ^0a. That is to say, if $|a|$ is smaller than some real S-number, then there exists a complex S-number, ^0a, the *standard part* of a, such that $|a - {}^0a|$ is smaller than all positive S-numbers. A number which is infinitely close to 0 is *infinitely small* or *infinitesimal*. In particular, 0 is the only S-number which is infinitesimal. A complex number a which is not finite, i.e., which is such that $|a|$ is greater than any S-number, is *infinite*. There exist elements of $^*C$ which are infinite.

Every set, relation, etc. in C possesses a natural extension to $^*C$. This is simply the set, relation, $\cdots$, in $^*C$ which is denoted by the same symbol in L. At our convenience, we may, or may not, denote it by the same symbol also in our notation (which is not necessarily part of L). Thus, we shall denote the extension of the set of positive integers, P, to $^*C$ by $^*P$ but if $\sigma = \{a_n\}$ is a sequence of complex numbers in C then we shall denote its extension to $^*C$ still by $\sigma = \{a_n\}$. According to the definition of an infinite number which was given above, the infinite positive integers in $^*C$ are just the elements of $^*P - P$.

The following results are basic (for the proofs see [5] and [6]).

THEOREM 1.2. *The sequence $\{a_n\}$ in C converges to a limit a (a an S-number) if and only if the extension of $\{a_n\}$ in $^*C$ satisfies the condition that $|a - a_n|$ is infinitesimal for all infinite n.*

THEOREM 1.3. *Let $\{a_n\}$ be an admissible sequence in $^*C$ such that a_n is infinitesimal for all finite n. Then there exists an infinite positive integer ω (i.e., $\omega \in {}^*P - P$) such that a_n is infinitesimal for all n smaller than ω.*

$\{a_n\}$ is called *admissible* in $^*C$ if the relation representing $\{a_n\}$ belongs to the set of relations which are admissible in the sense explained above. Admissible operators, etc., are defined in a similar way. 1.3. shows that the sequence $\{a_n\}$ which is defined by $a_n = 0$ for finite n and by $a_n = 1$ for infinite n is not admissible in $^*C$.

2. Nonstandard Hilbert space. The selected representation of the Hilbert space H consists of all sequences $\{s_n\}$ of complex numbers such that $\|\sigma\|^2 = \sum_{n=1}^{\infty} |s_n|^2$ converges. The corresponding space $^*H$ over $^*C$ consists of all admissible sequences $\{s_n\}$ in $^*C$ such that $\|\sigma\|^2 = \sum_{n=1}^{\infty} |s_n|^2$ converges, i.e., such that it satisfies the formal (classical) definition of convergence in L.

Among the points of $^*H$ are the extensions of points of H (as sequences). We identify the points of H with their extension in $^*H$ and may then regard H as a subset (though not an admissible subset) of $^*H$.

A point σ of $^*H$ is called *norm-finite* if $\|\sigma\|$ is a finite real number in the sense explained in section 1. σ is *near-standard* if $\|\sigma - \sigma^0\|$ is infinitesimal for some ${}^0\sigma \in H$. If such a ${}^0\sigma$ exists then it is determined uniquely by σ. It is called the *standard part* of σ.

Applying 1.2. to the partial sums of any point $\sigma = \{s_n\}$ in H, we obtain:

THEOREM 2.1. *For any* $\sigma = \{s_n\}$ *in* H *and any infinite positive integer* ω, *the sum* $\sum_{n=\omega}^{\infty} | s_n |^2$ *is infinitesimal.*

Next, we sketch the proof of:

THEOREM 2.2. *A point* $\sigma = \{s_n\}$ *in* $^*H$ *is near-standard if and only if it is norm-finite and if at the same time* $\sum_{n=\omega}^{\infty} | s_n |^2$ *is infinitesimal for all infinite* ω.

Suppose that $\| \sigma - {}^0\sigma \|$ is infinitesimal for some ${}^0\sigma$ in H. Then $\sigma \| = \| \sigma - {}^0\sigma + {}^0\sigma \| \leqq \| \sigma - {}^0\sigma \| + \| {}^0\sigma \| < 1 + \| {}^0\sigma \|$ so that σ is norm-finite. Also, let ${}^0\sigma = \{s'_n\}$, then $\sum_{n=\omega}^{\infty} | s'_n |^2$ is infinitesimal for infinite ω, by 2.1. Also, $\sum_{n=\omega}^{\infty} | s_n - s'_n |^2$ is infinitesimal since this sum cannot exceed $\| \sigma - {}^0\sigma \|^2$. But

$$\sum_{n=\omega}^{\infty} | s_n |^2 \leqq \left(\left(\sum_{n=\omega}^{\infty} | s_n - s'_n |^2 \right)^{1/2} + \left(\sum_{n=\omega}^{\infty} | s'_n |^2 \right)^{1/2} \right)^2 ,$$

showing that the conditions of 2.2 are necessary.

Supposing that they are satisfied, $\| \sigma \|$ is finite, hence $| s_n |$ is finite for any n and s_n possesses a standard part, ${}^0s_n$. Consider the sequence $\{{}^0s_n\}$ in C. It can be shown that $\sum_{n=1}^{\infty} |{}^0s_n |^2$ converges in C and hence, represents a point σ' in H and $^*H$. Thus, if $\sigma' = \{s'_n\}$ then $s'_n = {}^0s_n$ for finite n but not necessarily for infinite n. Since, for all finite k, $\sum_{n=1}^{k} | s_n - s'_n |^2 = \sum_{n=1}^{k} | s_n - {}^0s_n |^2$ is infinitesimal, it follows from 1.3 that $\sum_{n=1}^{k} | s_n - s'_n |^2$ is still infinitesimal for some infinite k, $k = \omega - 1$, say. On the other hand, $\sum_{n=\omega}^{\infty} | s_n |^2$ is infinitesimal by assumption, and $\sum_{n=\omega}^{\infty} | s'_n |^2$ is infinitesimal, by 2.1. The inequality

$$\| \sigma - \sigma' \|^2 = \sum_{n=1}^{\infty} | s_n - s'_n |^2 \leqq \sum_{n=1}^{\omega-1} | s_n - s'_n |^2 + \left(\left(\sum_{n=\omega}^{\infty} | s_n |^2 \right)^{1/2} + \left(\sum_{n=\omega}^{\infty} | s'_n |^2 \right)^{1/2} \right)^2 ,$$

then shows that $\| \sigma - \sigma' \|$ is infinitesimal, σ is near-standard with standard part ${}^0\sigma = \sigma'$.

The following theorem is proved in [7] for general topological spaces but under somewhat different conditions.

THEOREM 2.3. *Let* A *be a compact set of points in* H. *Then all points of* $^*A$ (*i.e., of the set which corresponds to* A *in* $^*H$) *are near-standard.*

Indeed, suppose that A is compact but that $\sigma \in {}^*A$ is not near-standard. Then there exists a standard positive r such that $\| \sigma - \tau \| > r$ for all $\tau \in H$. This is trivial if σ is not norm-finite. If σ is norm-finite, then by 2.2, there exists an infinite positive integer ω such that $\sum_{n=\omega}^{\infty} | s_n |^2 > 2r^2$ for some standard positive number r. For any $\tau = \{t_n\}$ in H, $\sum_{n=\omega}^{\infty} | t_n |^2$ is infinitesimal. Hence

$$|| \sigma - \tau || = \left(\sum_{n=1}^{\infty} | s_n - t_n |^2 \right)^{1/2} \geqq \left(\sum_{n=\omega}^{\infty} | s_n - t_n |^2 \right)^{1/2}$$

$$\geqq \left(\sum_{n=\omega}^{\infty} | s_n |^2 \right)^{1/2} - \left(\sum_{n=\omega}^{\infty} | t_n |^2 \right)^{1/2} > r .$$

On the other hand, since A is compact it possesses an r-net, i.e., for some finite number of points in A, $\tau_1, \cdots, \tau_m$, and for all ξ in A, $|| \xi - \tau_i || < r$ for some i, $1 \leqq i \leqq m$. But, for the specified $\tau_1, \cdots, \tau_m$, this is a property of H which can be formulated as a sentence of K. It follows that for all points ξ of $^*A$ also $|| \xi - \tau_i || < r$ for some i, $1 \leqq i \leqq m$. This contradiction proves the theorem.

3. Operators in nonstandard Hilbert space. An operator from H into H may be regarded as a relation between elements of H, i.e., between sequences of elements of C (which are themselves relations). The corresponding operator in $^*H$, which is denoted by the same symbol in L, will be denoted here also by T. This cannot give rise to any confusion. For if $\tau = T\sigma$ in H then $\tau = T\sigma$ also in $^*H$ since $\tau = T\sigma$ can be expressed by a sentence of K.

In particular, let T be a bounded linear operator defined on all of H. For the assumed representation of H by sequences, T has a matrix representation, $T = (a_{jk})$, $j, k = 1, 2, 3, \cdots$. The coefficients of this matrix satisfy the conditions:

3.1.
$$\sum_{k=1}^{\infty} | a_{jk} |^2 < \infty \qquad j = 1, 2, 3, \cdots$$
$$\sum_{j=1}^{\infty} | a_{jk} |^2 < \infty \qquad k = 1, 2, 3, \cdots$$

In $^*H$ these subscripts of (a_{jk}) vary also over the infinite positive integers. By 3.1 and 2.1., $\sum_{k=\omega}^{\infty} | a_{jk} |^2$ is infinitesimal for infinite ω, provided j is finite. This is not necessarily true for infinite j as shown by the matrix for the identity operator.

THEOREM 3.2. *Let T be a completely continuous (compact) linear operator on H. Then T maps every norm-finite point in $^*H$ on a near-standard point.*

Proof. If σ is norm-finite then $|| \sigma || < r$ for some positive S-number r. The sphere $B = \{\xi \mid || \xi || < r\}$ is bounded in H and is mapped by T on a set whose closure, A, is compact. If the corresponding sets in

$^*H$ are $^*B$ and $^*A$ respectively then $^*B$ contains σ (since σ satisfies the defining condition of B) and so $^*A$ contains $T\sigma$. But $^*A$ contains only near-standard points, by 2.3, so $T\sigma$ is near-standard, proving 3.2.

In a somewhat different setting [7] the converse of 3.2 is also true.

THEOREM 3.3. *If $T = (a_{jk})$ is a completely continuous linear operator on H, then a_{jk} is infinitesimal for all infinite k (j finite or infinite).*

Proof. For finite j, this follows from the fact that $\sum_{n=k}^{\infty} | a_{jk} |^2$ is then infinitesimal. For infinite j, define $\sigma = \{s_n\}$ by $s_n = 0$ for $n \neq k$ and by $s_k = 1$. Then $|| \sigma || = 1$, so $\tau = \{t_j\} = T\sigma$ must be near-standard, by 3.2, where $t_j = \sum_{n=1}^{\infty} a_{jn} s_n = a_{jk}$. But then $t_j = a_{jk}$ must be infinitesimal for infinite j, by 2.2.

An operator $T = (a_{jk})$ will be called *almost superdiagonal* if $a_{jk} = 0$ for $j > k + 1$, $k = 1, 2, 3, \cdots$. This definition depends on the specified basis of H.

THEOREM 3.4. *Let T be a bounded linear operator on H which is almost superdiagonal. Let*

3.5. $$p(z) = c_0 + c_1 z + \cdots + c_m z^m, \; c_m \neq 0, \; m \geqq 1$$

be a polynomial with standard complex coefficients such that $p(T)$ is completely continuous. Then there exists an infinite positive integer ω such that $a_{\omega+1,\omega}$ is infinitesimal.

Proof. Put $Q = (b_{jk}) = p(T)$. We show by direct computation that, for any $h \geqq 1$,

3.6. $$b_{h+m,h} = c_m a_{h+1,h} a_{h+2,h+1} a_{h+3,h+2} \cdots a_{h+m,h+m-1} \, .$$

By 3.3, $b_{h+m,h}$ is infinitesimal for all infinite h. Since c_m is not infinitesimal, one of the remaining factors on the right hand side of 3.6 must be infinitesimal, e.g, $a_{h+j+1,h+j}$, $0 \leqq j < m$. Setting $\omega = h + j$, we obtain the theorem.

4. **Projection operators.** Let E be any admissible closed linear subspace of $^*H$ within the nonstandard model under consideration. The corresponding projection operator, which reduces to the identity on E, will be denoted by P_E. Given E, we define a subset $^\circ E$ of H as follows. For any $\sigma \in H, \sigma \in {}^\circ E$ if and only if $|| \sigma - \sigma' ||$ is infinitesimal for some $\sigma' \in E$. Since, by a familiar property of projection operators, $|| \sigma - \sigma' || \geqq || \sigma - P_E \sigma ||$, it follows that $\sigma \in {}^\circ E$ if and only if $|| \sigma - P_E \sigma ||$ is infinitesimal. In that case, $\sigma = {}^\circ (P_E \sigma)$. More generally, if τ is a

near-standard element of E then ${}^\circ\tau \in {}^\circ E$.

The tools developed so far suffice to establish the following theorem, 4.1, as well as the subsequent theorems, 4.2 ond 4.3.

THEOREM 4.1. *Given E as above, the set ${}^\circ E$ is a closed linear subspace of H.*

Proof. Let σ_1, σ_2 be elements of ${}^\circ E$. There exist elements τ_1, τ_2 of E such that $\|\sigma_1 - \tau_1\|$ and $\|\sigma_2 - \tau_2\|$ are infinitesimal. Then $\tau_1 + \tau_2$ belongs to E and

$$\|(\sigma_1 + \sigma_2) - (\tau_1 + \tau_2)\| \leqq \|\sigma_1 - \tau_1\| + \|\sigma_2 - \tau_2\|$$

so that the left hand side of this inequality also is infinitesimal. Hence, $\sigma_1 + \sigma_2$ belongs to ${}^\circ E$. Again for $\sigma \in {}^\circ E$ and λ standard complex, there exists $\tau \in E$ such that $\|\sigma - \tau\|$ is infinitesimal. Then $\lambda\tau \in E$ and $\|\lambda\sigma - \lambda\tau\| = |\lambda| \|\sigma - \tau\|$ is infinitesimal and so $\lambda\sigma \in {}^\circ E$. This shows that ${}^\circ E$ is linear in the algebraic sense.

Now let $\sigma_n \to \sigma$, where the σ_n are defined for standard natural n and belong to ${}^\circ E$, and σ belongs to H. In order to prove that ${}^\circ E$ is closed we have to show that σ belongs to ${}^\circ E$. By assumption, the distances $\|\sigma_n - P_E\sigma_n\|$ are infinitesimal for all $n \in N$. Hence, by Theorem 1.3 there exists an infinite natural number ω such that $\|\sigma_n - P_E\sigma_n\|$ is infinitesimal for all $n < \omega$. The sequence of points $\{\sigma_n\}$ in ${}^\circ E \subseteqq H$ extends, in ${}^*H$, to a sequence of points defined for all $n \in {}^*N$. Moreover, by 1.2 above, the fact that $\sigma_n \to \sigma$ in H implies that $\|\sigma_n - \sigma\|$ is infinitesimal for all infinite n. Hence, for all infinite n less than ω, $\|\sigma - P_E\sigma_n\|$, which does not exceed

$$\|\sigma - \sigma_n\| + \|\sigma_n - P_E\sigma_n\| ,$$

also must be infinitesimal. But $P_E\sigma_n \in E$ and so $\sigma \in {}^\circ E$, as required. This completes the proof of 4.1.

Let ω be an infinite natural number. The closed linear subspace of ${}^*H$ which consists of all points $\sigma = \{s_n\}$ such that $s_n = 0$ for $n > \omega$ will be denoted by H_ω. The corresponding projection operator, which will be denoted by P maps any $\sigma = \{s_n\}$ in ${}^*H$ into the point $\sigma' = \{s'_n\}$, where $s'_n = s_n$ for $n \leqq \omega$ and $s'_n = 0$ for $n > \omega$. For any point $\sigma \in H$, $\|\sigma - P\sigma\| = (\sum_{n=\omega+1}^{\infty} |s_n|^2)^{1/2}$ is infinitesimal, by 2.1.

For any bounded linear operator T on H let $T' = PTP$, and let T_ω be the restriction of T' to H_ω. Then $\|T'\| \leqq \|P\|^2 \|T\| \leqq \|T\|$ and so $\|T_\omega\| \leqq \|T\|$.

THEOREM 4.2. *Let E be an admissible closed linear subspace of H_ω which is invariant for T_ω, i.e., $T_\omega E \subseteqq E$. Then ${}^\circ E$ is* *invariant for* T, $T\,{}^\circ E \subseteq {}^\circ E$.

Proof. Let $\sigma \in {}^\circ E$, then we have to show that $T\sigma \in {}^\circ E$. By assumption, there exists a $\tau \in E$ such that $\|\sigma - \tau\|$ is infinitesimal. Then $T_\omega \tau \in E$, i.e., $PT\tau \in E$. Thus, in order to show that $T\sigma$ is infinitely close to E, we only have to establish that the quantity $a = \|T\sigma - PT\tau\|$ is infinitesimal. Now

$$\begin{aligned} a = \|T\sigma - PT\tau\| &= \|T\sigma - PT\sigma + PT(\sigma - \tau)\| \\ &\leq \|T\sigma - PT\sigma\| + \|P\|\,\|T\|\,\|\sigma - \tau\| \end{aligned}$$

and $\|T\|$ is a standard real number, while $\|P\| \leq 1$ and $\|\sigma - \tau\|$ is infinitesimal. At the same time $T\sigma$ is a point of H and so the difference $T\sigma - PT\sigma$ is infinitesimal, as shown above. It follows that a is infinitesimal, and this is sufficient for the proof of 4.2.

The number of dimensions of H_ω as defined within the language L is ω, $d(H_\omega) = \omega$. In this sense, H_ω is "finite-dimensional". Similarly, with every admissible closed linear subspace E of H_ω, there is associated a natural number $d(E)$ in ${}^*C$, which may be finite or infinite, and which has the properties of a dimension to the extent to which these can be expressed as sentences of K.

THEOREM 4.3. *Let E and E_1 be two admissible closed linear subspaces of H_ω such that $E \subseteq E_1$ and $d(E_1) = d(E) + 1$. Then ${}^\circ E \subseteq {}^\circ E_1$ and any two points of ${}^\circ E_1$ are linearly dependent modulo ${}^\circ E$.*

Proof. Since $E \subseteq E_1$, it is trivial that ${}^\circ E \subseteq {}^\circ E_1$. Now suppose that ${}^\circ E_1$ contains two points σ_1 and σ_2 which are linearly independent modulo ${}^\circ E$. Then σ_1 and σ_2 are infinitely close to points τ_1, τ_2 of E_1, respectively. Since the dimension of E_1 exceeds that of E only by one, there must be a representation

$$4.4. \qquad \tau_2 = \lambda\tau_1 + \tau$$

or vice versa, where $\tau \in E$ and λ is an element of ${}^*C$. Now if λ were infinitesimal (including $\lambda = 0$) τ_2 would be infinitely close to E, and so σ_2 would be infinitely close to E and would belong to ${}^\circ E$. This is contrary to the assumption that σ_1 and σ_2 are linearly independent modulo ${}^\circ E$. If λ were infinite, then the relation

$$\tau_1 = \lambda^{-1}\tau_2 - \lambda^{-1}\tau$$

(in which λ^{-1} is infinitesimal and $\lambda^{-1}\tau$ belongs to E) would show that σ_1 belongs to ${}^\circ E$. Note that both τ_1 and τ_2 are norm-finite since they are infinitely close to the standard points σ_1 and σ_2, respectively.

We conclude that λ possesses a standard part, ${}^\circ\lambda$, and that ${}^\circ\lambda \neq 0$.

Also, $\tau = \tau_2 - \lambda\tau_1$ is infinitely close to $\sigma = \sigma_2 - {}^\circ\lambda\sigma_1$, since

$$\begin{aligned} \|\tau - \sigma\| &= \|\tau_2 - \lambda\tau_1 - (\sigma_2 - {}^\circ\lambda\sigma_1)\| \\ &\leqq \|\tau_2 - \sigma_2\| + |\lambda|\,\|\tau_1 - \sigma_1\| + |\lambda - {}^\circ\lambda|\,\|\sigma_1\| \end{aligned}$$

so that $\|\tau - \sigma\|$ is infinitesimal. It follows that σ belongs to ${}^\circ E$ and that σ_1 and σ_2 are linearly dependent modulo ${}^\circ E$. This contradiction proves the theorem.

5. Proof of the main theorem. We are now ready to prove 1.1. To begin with, we work in the standard model, i.e., in an ordinary Hilbert space H over the complex numbers, C. Our method, like that of [1] is based on the fact that in a *finite-dimensional* space, of dimension μ say, any linear operator possesses a chain of invariant subspaces

5.1. $$E_0 \subseteqq E_1 \subseteqq E_2 \subseteqq \cdots \subseteqq E_\mu$$

where $d(E_j) = j$, $0 \leqq j \leqq \mu$, so that $E_0 = \{0\}$.

The proof of 1.1. is trivial [1] unless for every $\sigma \neq 0$ in H, the set $A = \{\sigma, T\sigma, T^2\sigma, \cdots, T^n\sigma, \cdots\}$ is linearly independent algebraically and generates the entire space. Assuming from now on that this is the case, we choose σ such that $\|\sigma\| = 1$, and we replace A by an equivalent orthonormal set $B = \{\sigma = \eta_1, \eta_2, \eta_3, \cdots \eta_n, \cdots\}$ by the Gram-Schmidt method. Then $\{\sigma, T\sigma, \cdots, T^{n-1}\sigma\}$ and $\{\eta_1, \eta_2, \cdots \eta_n\}$ are linearly dependent upon each other. We deduce without difficulty that T is almost superdiagonal with respect to the basis B. Representing any $\tau \in H$ by the sequence $\{t_n\}$, where $t_n = (\tau, \eta_n)$, we may then identify H with the sequence space considered in the preceding sections. Thus, if $T = (a_{jk})$ in this representation, then $a_{jk} = 0$ for $j > k + 1$, $k = 1, 2, 3, \cdots$ and, passing to ${}^*C$ and ${}^*H$, there exists an infinite positive integer ω such that $a_{\omega+1,\omega}$ is infinitesimal, by 3.4. ω will be kept fixed from now on, and for it we consider the space H_ω and the operators P and $T' = PTP$ introduced in Section 4 above.

Let $\xi = \{x_i\}$ be any norm-finite element of ${}^*H$. Consider the difference

$$(TP - T')\xi = (I - P)TP\,\xi = \zeta = \{z_n\}\,.$$

We obtain by direct computation that $z_{\omega+1} = a_{\omega+1,\omega}x_\omega$, and $z_n = 0$ for $n \neq \omega + 1$. Hence $\|\zeta\| \leqq |a_{\omega+1,\omega}|\,\|\xi\|$, so that ζ is infinitesimal. Using the equivalence relation $\tau_1 \sim \tau_2$ for points of ${}^*H$ such that $\|\tau_1 - \tau_2\|$ is infinitesimal, we have shown that $TP\xi \sim T'\xi$, where the points on both sides of this equivalence are norm-finite. We then prove by induction that:

5.2. $T^r P\xi \sim (T')^r \xi$ for norm-finite ξ, $r = 1, 2, 3, \cdots$.

The case $r = 1$ has just been disposed of. Suppose 5.2 proved for $r - 1$, $r \geqq 2$. Then

$$T^r P\xi \sim T(T')^{r-1}\xi = TP(T')^{r-1}\xi \sim T'(T')^{r-1}\xi = (T')^r\xi$$

where we have made use of the first equivalence for $(T')^{r-1}\xi$ in place of ξ. Applying 5.2 to the monomials of $p(T)$, and taking into account that $P\xi \sim \xi$ for $\xi \in H_\omega$, we obtain

5.3. $p(T)\xi \sim p(T')\xi$ for norm-finite ξ in H_ω.

Let T_ω be the restriction of T' to H_ω, as in Section 4. Since H_ω is "finite" more precisely ω-dimensional in the sense of Nonstandard Analysis, there exists a chain of subspaces as in 5.1 with $\mu = \omega$, such that $T_\omega E_j \subseteqq E_j$, $j = 0, 1, 2, \cdots, \omega$. The E_j are also linear subspaces of $^*H$. They are finite-dimensional, hence closed, in the sense of Nonstandard Analysis, i.e., they satisfy the formal condition of closedness as expressed within the language L. Let P_j be the projection operator from $^*H$ onto E_j, $j = 0, 1, 2, \cdots, \omega$, so that $P_\omega = P$.

Suppose $p(z)$ is given by 3.5. For any $\xi \neq 0$ in H, $p(T)\xi$ must be different from 0 otherwise $\xi, T\xi, \cdots, T^n\xi$ would be linearly dependent, contrary to assumption. Choose ξ in H with $\|\xi\| = 1$. Since $\xi \sim P\xi$, $p(T)\xi \sim p(T)P\xi$, so $p(T)P\xi$ is not infinitesimal and by 5.3, $p(T)P\xi$ and hence $p(T')\xi$ is not infinitesimal. Thus, $\|p(T')\xi\| > r$ for some standard positive r. Consider the expressions

5.4. $r_j = \|p(T')\xi - p(T')P_j\xi\|$, $j = 0, 1, 2, \cdots, \omega$,

and note that $r_j \leqq \|p(T')\|\,\|\xi - P_j\xi\|$. We have $r_0 = \|p(T')\xi\|$ so $r_0 > r$. Also $\|\xi - P_\omega\xi\| = \|\xi - P\xi\|$ is infinitesimal, hence $r_\omega < r/2$. It follows that there exists a smallest positive integer λ with may be finite or infinite, such that $r_\lambda < r/2$ but $r_{\lambda-1} \geqq r/2$.

With every E_j, we associate the closed linear subspace $^\circ E_j$ of H which was defined in Section 4. Now $^\circ E_{\lambda-1}$ cannot coincide with H, more particularly, it cannot include ξ. For if it did, then $\|\xi - P_{\lambda-1}\xi\|$ would be infinitesimal, so $r_{\lambda-1}$, which is bounded by $\|p(T')\|\,\|\xi - P_{\lambda-1}\xi\|$ would be infinitesimal, contrary to the choice of λ.

On the other hand $^\circ E_\lambda$ cannot reduce to $\{0\}$. Consider the point $\eta = p(T')P_\lambda\xi$. $\eta \in E_\lambda$ since $P_\lambda\xi \in E_\lambda$ and E_λ is invariant under $p(T_\omega)$ and, equivalently, under $p(T')$. Also, since $P_\lambda\xi \in H_\omega$,

$$\eta = p(T')P_\lambda\xi \sim p(T)P_\lambda\xi ,$$

where the right-hand side is near-standard, by 3.2, since $P_\lambda\xi$ is norm-finite and $p(T)$ is completely continuous. It follows that η possesses a standard part, $^{\circ}\eta$, and that $^{\circ}\eta$ belongs to $^{\circ}E_{\lambda}$. Again, $^{\circ}\eta = 0$ would imply that η is infinitesimal. Hence, by 5.4

$$r_{\lambda} \geqq \| p(T')\xi \| - \| p(T')P_{\lambda}\xi \| > r - \zeta$$

where ζ is infinitesimal. Hence $r_{\lambda} > r/2$, contrary to the choice of λ. We conclude that $^{\circ}E_{\lambda}$ contains a point different from 0, i.e., $^{\circ}\eta$.

Both $^{\circ}E_{\lambda-1}$ and $^{\circ}E_{\lambda}$ are invariant for T, by 4.2. If neither were a proper invariant subspace of H for T we should have $^{\circ}E_{\lambda-1} = \{0\}$, $^{\circ}E_{\lambda} = H$. But this contradicts 4.3, proving 1.1.

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INVARIANT SUBSPACES OF POLYNOMIALLY COMPACT OPERATORS

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This paper is a comment on the solution of an invariant subspace problem by A. R. Bernstein and A. Robinson [2]. The theorem they prove can be stated as follows: if A is an operator on a Hilbert space H of dimension greater than 1, and if p is a nonzero polynomial such that $p(A)$ is compact, then there exists a nontrivial subspace of H invariant under A. ("Operator" means bounded linear transformation; "Hilbert space" means complete complex inner product space; "compact" means completely continuous; "subspace" means closed linear manifold; "nontrivial", for subspaces, means distinct from $\{0\}$ and from H.) The Bernstein-Robinson proof has two aspects: it is an ingenious adaptation of the proof by N. Aronszajn and K. T. Smith of the corresponding theorem for compact operators [1], and it makes strong use of metamathematical concepts such as nonstandard models of higher order predicate languages. The purpose of this paper is to show that by appropriate small modifications the Bernstein-Robinson proof can be converted (and shortened) into one that is expressible in the standard framework of classical analysis.

A quick glance at the problem is sufficient to show that there is no loss of generality in assuming the existence of a unit vector e such that the vectors $e, Ae, A^2e, \cdots$ are linearly independent and have H for their (closed linear) span. (This comment appears in both [1] and [2].) The Gram-Schmidt orthogonalization process applied to the sequence $\{e, Ae, A^2e, \cdots\}$ yields an orthonormal basis $\{e_1, e_2, e_3, \cdots\}$ with the property that the span of $\{e, \cdots, A^{n-1}e\}$ is the same as the span of $\{e_1, \cdots, e_n\}$ for each positive integer n. It follows that if $a_{mn} = (Ae_n, e_m)$, then $a_{mn} = 0$ unless $m \leqq n + 1$; in other words, in the matrix of A all entries more than one step below the main diagonal must vanish. The matrix entries of the kth power of A are given by $a_{mn}^{(k)} = (A^k e_n, e_m)$. A straightforward induction argument, based on matrix multiplication, yields the result that $a_{mn}^{(k)} = 0$ unless $m \leqq n + k$, and

$$a_{n+k,n}^{(k)} = \Pi_{1 \leqq j \leqq k} a_{n+j,\, n+j-1} .$$

(With the usual understanding about an empty product having the value 1, the result is true for $k = 0$ also.) This result for powers has an implication for polynomials. If the degree of p (the only polynomial

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needed) is k ($\geqq 1$), and if the matrix entries of $p(A)$ are given by $a_{mn}^{(p)} = (p(A)e_n, e_m)$, then $a_{n+k,n}^{(p)}$ is a constant multiple (by the leading coefficient of p) of $a_{n+k,n}^{(k)}$. Since $\| p(A)e_n \| \to 0$ as $n \to \infty$ (because of the compactness of $p(A)$), there exists an increasing sequence $\{k(n)\}$ of positive integers (in fact a sequence with no gaps of length greater than the degree of p) such that the corresponding subdiagonal terms $a_{k(n)+1,k(n)}$ tend to 0 as n tends to ∞. (This very useful conclusion is one of the analytic tools used in [2], where it is described in terms of "infinite positive integers".)

If H_n is the span of $\{e_1, \cdots, e_{k(n)}\}$, then $\{H_n\}$ is an increasing sequence of finite-dimensional subspaces of H whose span is H. If P_n is the projection with range H_n, then $P_n \to 1$ (the identity operator) strongly. Since, for each n, the operator P_nAP_n leaves H_n invariant, it follows that, for each n, there exists a chain of subspaces invariant under P_nAP_n,

$$\{0\} = H_n^{(0)} \subset H_n^{(1)} \subset \cdots \subset H_n^{(k(n))} = H_n ,$$

with $\dim H_n^{(i)} = i$, $i = 0, 1, \cdots, k(n)$. (The consideration of such chains is essential in both [1] and [2].)

If $\{f_n\}$ and $\{g_n\}$ are sequences of vectors in H, it is convenient to write $f_n \sim g_n$ to mean that $\|f_n - g_n\| \to 0$ as $n \to \infty$. Assertion: if $\{f_n\}$ is a bounded sequence of vectors in H, then

$$AP_nf_n \sim P_nAP_nf_n . \tag{1}$$

(Intuitively: H_n is approximately invariant under A.) The proof is a straightforward computation, based on the fact that $P_nf = \sum_{j=1}^{k(n)} (f, e_j)e_j$ whenever $f \in H$. Since $AP_nf_n - P_nAP_nf_n = \sum_{j=1}^{k(n)} (f_n, e_j) \sum_{i=k(n)+1}^{\infty} a_{ij}e_i$, since the largest j here is $k(n)$ and the smallest i is $k(n)+1$, and since $a_{ij} = 0$ unless $i \leqq j+1$, it follows that $\| AP_nf_n - P_nAP_nf_n \| \leqq \| f_n \| \cdot | a_{k(n)+1,k(n)} |$.

The conclusion (1) can be generalized to higher exponents:

$$A^kP_nf_n \sim (P_nAP_n)^kf_n , \qquad k = 1, 2, 3, \cdots ; \tag{2}$$

the proof is by induction on k and is omitted. For $k = 0$, (2) says that $\| P_nf_n - f_n \| \to 0$, which is a stringent condition on the bounded sequence $\{f_n\}$; if that condition is satisfied, then (2) implies that

$$p(A)P_nf_n \sim p(P_nAP_n)f_n . \tag{3}$$

Return now to the unit vector e. Since $P_ne = e$ for each n, it follows that $p(P_nAP_n)e \sim p(A)e$. Since $p(A)e \neq 0$ (because the vectors $e, Ae, A^2e, \cdots$ are linearly independent), it follows that

$$\varepsilon = \lim_n \| p(P_nAP_n)e \| = \| p(A)e \| > 0 .$$

Consider, for each n, the numbers

$$\| p(P_nAP_n)e - p(P_nAP_n)P_n^{(0)}e \| ,$$
$$\| p(P_nAP_n)e - p(P_nAP_n)P_n^{(1)}e \| ,$$
$$\cdots$$
$$\| p(P_nAP_n)e - p(P_nAP_n)P_n^{(k(n))}e \| ,$$

where $P_n^{(i)}$ is the projection with range $H_n^{(i)}$. Since $P_n^{(0)}$ is the zero projection, the first of these numbers tends to ε. Since, on the other hand, $P_n^{(k(n))} = P_n$, the last of these numbers is always 0. In view of these facts it is possible to choose for each n (with possibly a finite number of exceptions) a positive integer $i(n)$, $1 \leqq i(n) \leqq k(n)$, such that

$$\| p(P_nAP_n)e - p(P_nAP_n)P_n^{(i(n)-1)}e \| \geqq \frac{\varepsilon}{2} , \tag{4}$$

and

$$\| p(P_nAP_n)e - p(P_nAP_n)P_n^{(i(n))}e \| < \frac{\varepsilon}{2} ; \tag{5}$$

the simplest way to do it is to let $i(n)$ be the smallest positive integer for which these inequalities are true. (The construction of this particular "infinite positive integer" i is the second major analytic insight in [2].)

Since both $\{P_n^{(i(n)-1)}\}$ and $\{P_n^{(i(n))}\}$ are bounded sequences of operators, there exists an increasing sequence $\{n_j\}$ of positive integers such that both $\{P_{n_j}^{(i(n_j)-1)}\}$ and $\{P_{n_j}^{(i(n_j))}\}$ are weakly convergent. Write, for typographical convenience, $Q_j^- = P_{n_j}^{(i(n_j)-1)}$ and $Q_j^+ = P_{n_j}^{(i(n_j))}$. Let M^- be the set of all those vectors f in H for which $Q_j^- f \to f$ (strongly), and, similarly, let M^+ be the set of those vectors f for which $Q_j^+ f \to f$ (strongly). The purpose of what follows is to prove that both M^- and M^+ are subspaces of H, that both are invariant under A, and that at least one of them is nontrivial.

Since linear combinations are continuous, it follows that M^- is a linear manifold. To prove that M^- is closed, suppose that g is in the closure of M^-; it is to be proved that $g \in M^-$, i.e., that $Q_j^- g \to g$. Given a positive number δ, find f in M^- so that $\| f - g \| < \delta/3$, and then find j_0 so that $\| Q_j f - f \| < \delta/3$ whenever $j \geqq j_0$. It follows that if $j \geqq j_0$, then $\| Q_j^- g - g \| \leqq \| Q_j^- g - Q_j^- f \| + \| Q_j^- f - f \| + \| f - g \| < \delta$. This proves that M^- is closed; the proof for M^+ is the same.

To prove that M^- is invariant under A, suppose that $f \in M^-$, so that $Q_j^- f \to f$, and infer, first, that $AQ_j^- f \to Af$, just because A is bounded, and, second, that $Q_j^- AQ_j^- f \sim Q_j^- Af$, because Q_j^- is uniformly bounded. Then reason as follows: $Q_j^- Af \sim Q_j^- AQ_j^- f = Q_j^- P_{n_j} AP_{n_j} Q_j^- f$ (because $Q_j^- \leqq P_{n_j}$) $= P_{n_j} AP_{n_j} Q_j^- f$ (because the range of Q_j^- is invariant under $P_{n_j}AP_{n_j}) \sim AP_{n_j}Q_j^-f$ (by (1)) $= AQ_j^-f \to Af$. This proves that M^- is invariant; the proof for M^+ is the same.

The next step is to prove that $M^- \neq H$; this is done by proving that e does not belong to M^-. For this purpose observe first that the operators $p(P_nAP_n)$ are uniformly bounded. (Observe that

$$\|(P_nAP_n)^k\| \leqq \|P_nAP_n\|^k \leqq \|A\|^k$$

and use the polynomial whose coefficients are the absolute values of the coefficients of p.) Now use (4):

$$\frac{\varepsilon}{2} \leqq \|p(P_{n_j}AP_{n_j})\| \cdot \|e - Q_j^-e\| .$$

Since $\|p(P_{n_j}AP_{n_j})\|$ is bounded from above, its reciprocal is bounded away from zero, and, consequently, $\|e - Q_j^-e\|$ is bounded away from zero, which makes the convergence $Q_j^-e \to e$ impossible.

The corresponding step for M^+ says that $M^+ \neq \{0\}$; the proof is quite different. The choice of the sequence $\{n_j\}$ implies that the sequence $\{Q_j^+e\}$ is weakly convergent; the compactness of $p(A)$ implies, therefore, that the sequence $\{p(A)Q_j^+e\}$ is strongly convergent to, say, f. The proof that follows consists of two parts: (i) $f \neq 0$, (ii) $f \in M^+$. Part (i): $p(A)Q_j^+e \sim p(P_{n_j}AP_{n_j})Q_j^+e$ (by (3)), which is within $\varepsilon/2$ of $p(P_{n_j}AP_{n_j})e$ (by (5)), whose norm tends to ε; it follows that $\|p(A)Q_j^+e\|$ cannot tend to 0, and hence that $f \neq 0$. Part (ii): $Q_j^+f \sim Q_j^+p(A)Q_j^+e$ (since Q_j^+ is uniformly bounded) $\sim Q_j^+p(P_{n_j}AP_{n_j})Q_j^+e$ (by (3), and, again, uniform boundedness) $= p(P_{n_j}AP_{n_j})Q_j^+e$ (because the range of Q_j^+ is invariant under $p(P_{n_j}AP_{n_j})) \sim p(A)Q_j^+e$ (by (3)) $\to f$ (by definition).

If $M^+ \neq H$, all is well; it remains to be proved that if $M^+ = H$, then $M^- \neq \{0\}$. If $M^+ = H$, then $Q_j^+f \to f$ for all f, and, a fortiori, $Q_j^+f \to f$ weakly. At the same time the sequence $\{Q_j^-\}$ is known to be weakly convergent to, say, Q^-. The operators Q_j^- and Q_j^+ are projections such that $Q_j^- \leqq Q_j^+$ and such that $Q_j^+ - Q_j^-$ has rank 1. It follows that, for each j, there exists a unit vector f_j such that $(Q_j^+ - Q_j^-)f = (f, f_j)f_j$ for all f. Observe now that Q_j^-e cannot tend weakly to e, for, if it did, then it would tend strongly to e (an elementary property of projections), and that was proved to be not so. This implies that $Q^-e \neq e$, or, equivalently, that $(1 - Q^-)e \neq 0$. Can the numbers $|(e, f_j)|$ be arbitrarily small? Since $|((Q_j^+ - Q_j^-)e, g)| \leqq |(e, f_j)| \cdot \|g\|$ for all g, an affirmative answer would imply that $((1 - Q^-)e, g) = 0$ for all g, so that $(1 - Q^-)e = 0$—a contradiction. The fact so obtained (that the numbers $|(e, f_j)|$ are bounded away from zero) makes it possible to prove that $M^- \neq \{0\}$; it turns out that if $g \perp (1 - Q^-)e$, then $g \in M^-$. Indeed, since $(e, f_j)(f_j, g) \to ((1 - Q^-)e, g) = 0$, it follows that $(f_j, g) \to 0$, and hence that $(f, f_j)(f_j, g) \to 0$ for all

f. This implies that $((1 - Q^{-})f, g) = 0$ for all f, and hence that $(1 - Q^{-})g = 0$. In other words, $Q_j^{-}g \to g$ weakly, and therefore strongly (the same property of projections that was alluded to above); from this it follows, finally, that $g \in M^{-}$.

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NEW INFINITE CLASSES OF PERIODIC JACOBI-PERRON ALGORITHMS

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The question whether a system of $n-1$ real algebraic numbers ($n = 2, 3, \cdots$) chosen from an algebraic field of degree not higher than n, yields periodicity by Jacobi's Algorithm is still as open and challenging as hundred years ago. The present paper gives an affirmative answer to this problem in the following case: let $K(w)$ be an algebraic number field generated by $w = (D^n - d : m)^{1/n}$, where m, n, d, D are natural numbers satisfying the conditions $m \geqq 1$, $n \geqq 3$, $d \mid D$, $1 \leqq d \leqq D/2(n-1)$. Then $n-1$ numbers can be chosen from $K(w)$, so that their Jacobi Algorithm becomes purely periodic. The length of the period equals n^2 (or n, if $d = m = 1$). This is the longest period of a periodic Jacobi Algorithm ever known. In three corollaries the following special cases are investigated

$$w = (D^n - d^r)^{1/n}, \qquad (r = 0, 1, \cdots, n)$$
$$w = (D^n - d^r D)^{1/n}, \qquad (r = 0, 1, \cdots, n-2)$$
$$w = (D^n - pd/m)^{1/n}. \qquad (n = p^u,\ p \text{ a prime},\ u = 1, 2, \cdots, m \text{ as before})$$

In all these three cases the Algorithm of Jacobi remains purely periodic with length equal to n^2.

The main tools in proving these results are the polynomials

$$f_s(w, D-1) = \sum_0^s \binom{n-s-1+i}{i} w^{s-i}(D-1)^i,$$
$$F_s(w, D) = \sum_0^s \binom{n-s-1+i}{i} w^{s-i} D^i, \qquad (s = 1, \cdots, n-1)$$

of which each is an inverse function of the other.

This paper reveals new infinite classes of Periodic Jacobi Algorithms, adding more and wider specific cases to already existing results explored by the author in his previous works. For any given real number $a^{(0)}$ Euclid's Algorithm, namely

$$a^{(0)} = b^{(0)} + \frac{1}{a^{(1)}}, \cdots, a^{(v)} = b^{(v)} + \frac{1}{a^{(v+1)}}, \qquad (v = 0, 1, \cdots)$$

where $b^{(v)} = [a^{(v)}]$ is the greatest integer not exceeding $a^{(v)}$, leads up to Ordinary Continued Fractions. This Algorithm was generalized by Jacobi [1], and its theory masterfully developed by Perron [2] for any

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number of $n-1$ real numbers ($n \geqq 3$) in the following way.

Let $a_k^{(0)}$ ($k = 1, 2, \cdots, n-1$) be any set of $n-1$ real numbers; from this set (infinitely many) new sets $a_k^{(v)}$ ($v = 0, 1, \cdots;$) of $n-1$ real numbers each are being formed by the recursion formula

$$(1) \qquad a_{n-1}^{(v+1)} = \frac{1}{a_1^{(v)} - b_1^{(v)}}\,; \qquad a_{k-1}^{(v+1)} = (a_k^{(v)} - b_k^{(v)})a_{n-1}^{(v+1)}\,, \qquad (v = 0, 1, \cdots;\ k = 2, 3, \cdots, n-1)$$

where again $b_k^{(v)} = [a_k^{(v)}]$ is the greatest integer not exceeding $a_k^{(v)}$. For $n = 2$ Jacobi-Perron's Algorithm (henceforth denoted by JAPAL) is Euclid's Algorithm, namely $a_1^{(v+1)} = 1 : (a_1^{(v)} - b_1^{(v)})$. The JAPAL is called periodic, if there exist two nonnegative integers t, m such that

$$(2) \qquad a_k^{(v+m)} = a_k^{(v)}\,, \qquad (k = 1, 2, \cdots, n-1;\ v = t, t+1, \cdots)$$

whereby the t lines

$$a_1^{(v)}, a_2^{(v)}, \cdots, a_{n-1}^{(v)} \qquad (v = 0, 1, \cdots, t-1)$$

are called the preperiod of the JAPAL, t its length, and the m lines

$$a_1^{(v)}, a_2^{(v)}, \cdots, a_{n-1}^{(v)} \qquad (v = t, t+1, \cdots, m+t-1)$$

are called the period of the JAPAL, m its length: the sum $m+t$ is called the length of the JAPAL. For $t = 0$ the JAPAL is called purely periodic. Whether or not there exist, for any $n > 2$, remarkable classes of sets of $n-1$ real numbers whose JAPAL becomes periodic, could not be decided by Perron.

In eight previous papers [3] I succeeded to prove that the JAPAL becomes periodic for certain sets of $n-1$ Algebraic Irrationals of degree n. Some specific results announced in my papers are the following:

Let D, d, m, n be natural numbers such that

$$n \geqq 3\,; \quad m \geqq 1\,; \quad d \mid D\,; \quad D \geqq dC \quad (C \text{ a positive constant})$$

and let w denote one of the following irrationals—

$$w = (D^n + d)^{1:n}\,; \quad (D^n + d : m)^{1:n}\,; \quad (D^n + d^k D)^{1:n}\,; \quad (D^n - d)^{1:n}\,,$$

then the JAPAL of the $n-1$ numbers

$$w, w^2, \cdots, w^{n-1}$$

becomes periodic with the lengths $2n-1$; $2n-1$; $2n-1$; $n^2 + (n-1)^2$ respectively. Trying to enlarge the family of infinite algebraic fields $K(w)$ containing sets of $n-1$ numbers whose JAPAL becomes periodic, I naturally asked for the periodicity of $(D^n - d : m)^{1:n}$, $(D^n - d^k)^{1:n}$

$(k = 0, 1, \cdots, n)$, $(D^n - d^k D)^{1:n}$ $(k = 0, 1, \cdots, n-2)$ and succeeded to establish it. The results are announced in this paper. My previous results thus become a special case of $(D^n - d : m)^{1:n}$ $(m = 1)$; but here I use much more refined methods to prove periodicity.

II. Statement of the main theorem.

In order to state the main result of this paper it is advisable to introduce the following new notations:

DEFINITION 1. A matrix of n rows and $n-1$ columns of the form

$$\left\| \begin{matrix} A_1, & A_2, & \cdots, & A_{n-2}, & A_{n-1} \\ 0, & 0, & \cdots, & 0, & 1 \\ 0, & 0, & \cdots, & 0, & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0, & 0, & \cdots, & 0, & 1 \end{matrix} \right\| \tag{3}$$

will be called a fugue. The first row of the fugue will be called its accumulator, and the numbers

$$A_1, A_2, \cdots, A_{n-1}$$

the first, second, $\cdots$, $n-1$st element of the fugue's accumulator.

DEFINITION 2. The meaning of a combined sigma-sign is given by the formula

$$\sum_{i=0/c}^{t-1/n} a_i = c \sum_{i=0}^{t-1} a_1 + \sum_{i=t}^{n} a_i \, . \tag{4}$$

We are now able to state

THE MAIN THEOREM. *Let m, n, d, D be natural numbers satisfying the following conditions*

$$m \geqq 1 \, ; \quad n \geqq 3 \, ; \quad d \mid D \, ; \quad 1 \leqq d \leqq D : 2(n-1) \, .$$

Let us further denote

$$\begin{aligned} w &= (D^n - d : m)^{1:n} \, , \\ f_s(w, D-1) &= \sum_{i=0}^{s} \binom{n-1-s+i}{i} w^{s-i} (D-1)^i \, , \quad (s = 1, \cdots, n-1) \end{aligned} \tag{5}$$

then the JAPAL of the $n-1$ numbers

[1] For $n = 2$ we get Euclid's Algorithm leading up to the periodic Continued fractions of a quadratic irrational. We shall demonstrate the validity of the Main Theorem in this case, too.

$$f_1(w, D-1), f_2(w, D-1), \cdots, f_{n-1}(w, D-1)$$

is purely periodic and its primitive length is n^2. The period consists of n fugues. The $n-1$ elements of the accumulator of the first fugue have the form

$$(6) \quad A_k = -1 + \sum_{i=0}^{k} \binom{n-1-k+i}{i} D^{k-i}(D-1)^i \; ; \; (k = 1, \cdots, n-1) \; .$$

The accumulator of the sth fugue $(s = 2, \cdots, n-1)$ has the form: the first $n-s$ elements have the form

$$(6b) \quad A_k = -1 + \sum_{i=0}^{k} \binom{n-1-k+i}{i} D^{k-i}(D-1)^i \; ; \; (k = 1, \cdots, n-s)$$

the following $s-1$ elements have the form

$$(6c) \quad A_{n-s+t} = -1 + \sum_{i=0/m:d}^{t-1/n-s+t} (-1)^i \binom{s-1-t+i}{i} \binom{n}{s-t+i} D^{n-s+t-i} \; ; \quad (t = 1, 2, \cdots, s-1) \; .$$

The $n-1$ elements of the accumulator of the nth fugue have the form

$$(6d) \quad A_{n-s+t} = -1 + (-1)^t \binom{n-1}{t} + (m:d) \sum_{i=0}^{t-1} (-1)^i \binom{n-1-t+i}{i} \binom{n}{t-i} D^{t-i} \; . \quad (t = 1, 2, \cdots, n-1)$$

In the case of $m = d = 1$ the primitive length of the period is n. The period consists here of one fugue, and the elements of its accumulator have the form (6).

In the quadratic case $(n = 2)$ we have, according to the Main Theorem, as can be easily calculated by the reader,

$$w = (D^2 - d:m)^{1:2} \; ; \quad 2d \leqq D \; ; \quad d \mid D \; ,$$
$$f_1(w, D-1) = w + D - 1 \; ;$$

the accumulator of the first fugue has the form

$$A_1 = 2(D-1) \; ;$$

the accumulator of the second fugue has the form

$$A_1 = 2(mD:d-1) \; ;$$

therefore we have the development in a periodic continued fraction:

$$(6e) \quad (D^2 - (d:m))^{1:2} + D - 1 = [\overline{2(D-1), 1, 2(mD:d-1), 1}] \; .$$

Illustration for (6e): $D = 12$; $d = 3$; $m = 10$,

$$\sqrt{143{,}7} + 11 = [\overline{22, 1, 78, 1}] .$$

Two conclusions which follow directly from the Main Theorem are the following corollaries:

COROLLARY 1. *Let n, d, D be natural numbers satisfying the following conditions:*

$$n \geqq 3 ; \quad d \mid D ; \quad 1 \leqq d \leqq D : 2(n-1) ,$$

and let denote

$$w = (D^n - d^r)^{1:n} , \qquad (r = 0, 1, \cdots, n)$$

$$\text{(5a)} \qquad f_s(w, D-d) = d^{-s} \sum_{i=0}^{s} \binom{n-1-s+i}{i} w^{s-i}(D-d)^i ;$$

$$(s = 1, \cdots, n-1)$$

then the JAPAL of the $n-1$ numbers

$$f_1(w, D-d), f_2(w, D-d), \cdots, f_{n-1}(w, D-d)$$

is purely periodic and its primitive length is n^2. (the case $d = 1$ is excluded). The period consists of n fugues. The $n-1$ elements of the accumulator of the first fugue have the form:

$$\text{(7)} \quad A_k = -1 + d^{-k} \sum_{i=0}^{k} \binom{n-1-k+i}{i} D^{k-i}(D-d)^i ; \quad (k = 1, \cdots, n-1)$$

the elements of the accumulator of the sth *fugue* ($s = 2, 3, \cdots, n-1$) *have the form: the first $n-s$ elements have the form—*

$$\text{(7a)} \quad A_k = -1 + d^{-k} \sum_{i=0}^{k} \binom{n-1-k+i}{i} D^{k-i}(D-d)^i ; \quad (k = 1, \cdots, n-s)$$

the following $s-1$ elements have the form:

$$\text{(7b)} \quad A_{n-s+t} = -1 + \sum_{i=0/d^{n-r}}^{t-1/n-s+t} (-1)^i \binom{s-1-t+i}{i} \binom{n}{s-t+i} \left(\frac{D}{d}\right)^{n-s+t-i} .$$

$$(t = 1, 2, \cdots, s-1)$$

the $n-1$ elements of the accumulator of the nth *fugue have the form:*

$$A_{n-s+t} = -1 + (-1)^t \binom{n-1}{t}$$

$$\text{(7c)} \qquad + d^{n-r} \sum_{i=0}^{t-1} (-1)^i \binom{n-1-t+i}{i} \binom{n}{t-i} \left(\frac{D}{d}\right)^{n-s+t-i} .$$

$$(t = 1, 2, \cdots, n-1)$$

COROLLARY 2. *Let n, d, D be natural numbers satisfying the following conditions*:

$$n \geqq 3\,; \quad d \mid D\,; \quad 2d(n-1) \leqq D \leqq d^{n-r}\,, \qquad (r = 0, \cdots, n-2)$$

and let denote

$$w = (D^n - d^r D)^{1:n}\,, \qquad (r = 0, 1, \cdots, n-2)$$

$$\text{(5b)} \quad f_s(w, D-d) = d^{-s} \sum_{i=0}^{s} \binom{n-1-s+i}{i} w^{s-i}(D-d)^i\,; \; (s=1, 2, \cdots, n-1)\,.$$

Then the JAPAL of the $n-1$ numbers

$$f_1(w, D-d), f_2(w, D-d), \cdots, f_{n-1}(w, D-1)$$

is purely periodic and its primitive length is n^2. The period consists of n fugues. The $n-1$ elements of the accumulator of the first fugue have the form:

$$\text{(8)} \quad A_k = -1 + d^{-k} \sum_{i=0}^{k} \binom{n-1-k+i}{i} D^{k-i}(D-d)^i\,; \; (k=1, 2, \cdots, n-1)\,.$$

The $n-1$ elements of the accumulator of the sth *fugue have the form*: $(s = 2, 3, \cdots, n-1)$ *the first $n-s$ elements have the form*—

$$\text{(8a)} \quad A_k = -1 + d^{-k} \sum_{i=0}^{k} \binom{n-1-k+i}{i} D^{k-i}(D-d)^i\,, \; (k=1, 2, \cdots, n-s)\,.$$

The following $s-1$ elements have the form

$$\text{(8b)} \quad A_t = -1 + \sum_{i=0/d^{n-r}:D}^{t-1/n-s+t} (-1)^i \binom{s-t-1+i}{i} \binom{n}{s-t+i} \left(\frac{D}{d}\right)^{n-s+t-i}; \qquad (t = 1, 2, \cdots, s-1)$$

The $n-1$ elements of the accumulator of the nth *fugue have the form*:

$$\text{(8c)} \quad \begin{aligned} A_t = -1 + \sum_{i=0/d^{n-r}:D}^{t-1/n-s+t} (-1)^i \binom{s-1-t+i}{i} \binom{n}{s-t+i} \left(\frac{D}{d}\right)^{n-s+t-i} \\ + (-1)^t \binom{n-1}{t}\,. \qquad (t = 1, 2, \cdots, n-1) \end{aligned}$$

It is obvious that all the elements of the accumulators (6) to (8a) are integers. We shall prove that the elements of the accumulators (8b), (8c) are integers, too. To this end we have to prove that

$$(d^{n-r} : D)(D : d)^{n-s-t-i}$$

are integers. Denoting $n - s + t - i = u$, we have $1 \leqq u \leqq n - 1$; further

$$(d^{n-r} : D)(D : d)^u = D^{u-1} : d^{u-n+r} .$$

Since $d \mid D$, we have to prove $u - n + r \leqq u - 1$. But

$$u - n + r \leqq u - n + n - 2 = u - 2 .$$

III. Auxiliary functions-notations and identities. The essential tools used here to prove the Main Theorem and its Corollaries are the following functions:

$$f_s(w, D - 1) = \sum_{i=0}^{s} \binom{n - s - 1 + i}{i} w^{s-i}(D - 1)^i ; \tag{9}$$

$$(s = 1, \cdots, n - 1),\ f_0(w, D - 1) = 1 .$$

$$F_s(w, D) = \sum_{i=0}^{s} \binom{n - 1 - s + i}{i} w^{s-i} D^i ; \tag{10}$$

$$(s = 1, \cdots, n - 1),\ F_0(w, D) = 1 .$$

$$g_{n-s,t}(w, D) = \sum_{i=0/m:d}^{t-1/n-s+t} (-1)^i \binom{s - t - 1 + i}{i} F_{n-s+t-i}(w, D) ; \tag{11}$$

$$(s = 2, 3, \cdots, n ;\ t = 1, 2, \cdots, s - 1)$$

For any polynomial $P_s(w, D)$ in w, D with integers c_i as coefficients, namely

$$P_s(w, D) = \sum_{i=0}^{s} c_i w^{s-i} D^i ; \qquad (s = 1, \cdots, n - 1),\ P_0(w, D) = 1 ; \tag{12}$$

the following abbreviations will be used

$$P_s(w, D) = P_s ; \qquad (s = 1, \cdots, n - 1;\ P_0 = 1) . \tag{12a}$$

$$P_s(D, D) = \bar{P}_s ; \qquad (s = 1, \cdots, n - 1;\ \bar{P}_0 = 1) . \tag{12b}$$

$$\frac{P_s(D, D) - P_s(w, D)}{P_1(D, D) - P_1(w, D)} = \frac{\bar{P}_s - P_s}{\bar{P}_1 - P_1} = {}^{(1)}P_s ; \tag{12c}$$

$$(s = 1, \cdots, n - 1;\ {}^{(1)}P_0 = 0) .$$

The following identities are essential for the proof of the Main Theorem and its corollaries:

$$f_s(D - 1, D - 1) = \binom{n}{s}(D - 1)^s ; \qquad (s - 0, 1, \cdots, n - 1) . \tag{13}$$

Proof of (13). We have from (9):

$$f_s(D-1, D-1) = \sum_{i=0}^{s} \binom{n-1-s+i}{i} (D-1)^{s-i}(D-1)^i$$
$$= (D-1)^s \sum_{i=0}^{s} \binom{n-1-s+i}{i} = (D-1)^s \binom{n}{s} .$$

(13a) $$\bar{F}_s = \binom{n}{s} D^s ; \qquad (s = 0, 1, \cdots, n-1)$$

Proof of (13a). This is completely analogous to proof of (13).

(14) $${}^{(1)}F_s = F_{s-1} ; \qquad (s = 1, 2, \cdots, n-1) .$$

Proof of (14). We have from (10)—

$$F_1 = \sum_{i=0}^{1} \binom{n-2+i}{i} w^{1-i} D^i = w + (n-1)D ;$$
$$\bar{F}_1 = D + (n-1)D = nD ;$$
$$F_1 - \bar{F}_1 = w - D .$$

We thus have to prove

$$F_s - \bar{F}_s = (w - D)F_{s-1} .$$

We have

$$F_s - \bar{F}_s = \sum_{i=0}^{s} \binom{n-1-s+i}{i} w^{s-i} D^i - \binom{n}{s} D^s ;$$

$$(w-D)F_{s-1} = (w-D) \sum_{i=0}^{s-1} \binom{n-s+i}{i} w^{s-1-i} D^i$$
$$= \sum_{i=0}^{s-1} \binom{n-s+i}{i} w^{s-i} D^i - \sum_{i=0}^{s-1} \binom{n-s+i}{i} w^{s-i-1} D^{i+1}$$
$$= \sum_{i=0}^{s-1} \binom{n-s+i}{i} w^{s-i} D^i - \sum_{i=1}^{s} \binom{n-s+i-1}{i-1} w^{s-i} D^i$$
$$= w^s + \sum_{i=1}^{s-1} \left(\binom{n-s+i}{i} - \binom{n-s+i-1}{i-1} \right) w^{s-i} D^i - \binom{n-1}{s-1} D^s$$
$$= w^s + \sum_{i=1}^{s-1} \binom{n-s-1+i}{i} w^{s-i} D^i + \binom{n-1}{s} D^s - \binom{n}{s} D^s$$
$$= \sum_{i=0}^{s} \binom{n-s-1+i}{i} w^{s-i} D^i - \binom{n}{s} D^s .$$

(15) $$f_s = \sum_{i=0}^{s} (-1)^i \binom{n-1-s+i}{i} F_{s-i} ; \qquad (s = 0, 1, \cdots, n-1) .$$

Proof of (15). If we arrange the expression on the right hand of the equation

$$f_s = \sum_{i=0}^{s} (-1)^i \binom{n-1-s+i}{i} F_{s-i}$$
$$= \sum_{i=0}^{s} (-1)^i \binom{n-1-s+i}{i} \sum_{j=0}^{s-i} \binom{n-1-s+i+j}{j} w^{s-i-j} D^j$$

in descending powers of w, we get

$$f_s = \sum_{i=0}^{s} w^{s-i} \sum_{j=0}^{i} (-1)^j \binom{n-s-1+j}{j} \binom{n-s-1+i}{i-j} D^{i-j} .$$

Now the identity holds:

$$\binom{n-s-1+j}{j}\binom{n-s-1+i}{i-j} = \frac{(n-s-1+j)!(n-s-1+i)!}{(n-s-1)!j!(i-j)!(n-s-1+j)!}$$
$$= \frac{(n-s-1+i)!}{j!(n-s-1)!(i-j)!} = \frac{(n-s-1+i)!i!}{i!(n-s-1)!j!(i-j)!}$$
$$= \binom{n-s-1+i}{i}\binom{i}{j} .$$

In view of this identity we get

$$f_s = \sum_{i=0}^{s} w^{s-1} \binom{n-s-1+i}{i} \sum_{j=0}^{i} (-1)^j \binom{i}{j} D^{i-j}$$
$$= \sum_{i=0}^{s} \binom{n-s-1+i}{i} w^{s-i} (D-1)^i .$$

(16) $$^{(1)}f_s = \sum_{i=0}^{s-1} (-1)^i \binom{n-1-s+i}{i} F_{s-1-i} . \qquad (s = 1, 2, \cdots, n-1) .$$

Proof of (16). We have from (15):

$$f_s = \sum_{i=0}^{s-1} (-1)^i \binom{n-1-s+i}{i} F_{s-i} + (-1)^s \binom{n-1}{s} ,$$
$$\bar{f}_s = \sum_{i=0}^{s-1} (-1)^i \binom{n-1-s+i}{i} \bar{F}_{s-i} + (-1)^s \binom{n-1}{s} .$$

In view of these two formulas and according to (14), we get

$$^{(1)}f_s = \frac{\bar{f}_s - f_s}{D-w} = \sum_{i=0}^{s-1} (-1)^i \binom{n-1-s+i}{i} (\bar{F}_{s-i} - F_{s-i}) : (D-w)$$
$$= \sum_{i=0}^{s-1} (-1)^i \binom{n-1-s+i}{i} F_{s-1-i} .$$

$$(17) \qquad {}^{(1)}f_s - {}^{(1)}f_{s-1} = f_{s-1} . \qquad (s = 1, 2, \cdots, n-1) .$$

Proof of (17). We have, on the basis of (16)

$$\begin{aligned}
&{}^{(1)}f_s - {}^{(1)}f_{s-1} \\
&= \sum_{i=0}^{s-1} (-1)^i \binom{n-1-s+i}{i} F_{s-1-i} \\
&\quad - \sum_{i=0}^{s-2} (-1)^i \binom{n-s+i}{i} F_{s-2-i} \\
&= F_{s-1} + \sum_{i=1}^{s-1} (-1)^i \binom{n-1-s+i}{i} F_{s-1-i} \\
&\quad - \sum_{i=0}^{s-2} (-1)^i \binom{n-s+i}{i} F_{s-2-i} \\
&= F_{s-1} - \sum_{i=0}^{s-2} (-1)^i \binom{n-s+i}{i+1} F_{s-2-i} \\
&\quad - \sum_{i=0}^{s-2} (-1)^i \binom{n-s+i}{i} F_{s-2-i} \\
&= F_{s-1} - \sum_{i=0}^{s-2} (-1)^i \left(\binom{n-s+i}{i+1} + \binom{n-s+i}{i} \right) F_{s-2-i} \\
&= F_{s-1} - \sum_{i=0}^{s-2} (-1)^i \binom{n-s+i+1}{i+1} F_{s-2-i} \\
&= F_{s-1} + \sum_{i=1}^{s-1} (-1)^i \binom{n-s+i}{i} F_{s-1-i} \\
&= \sum_{i=0}^{s-1} (-1)^i \binom{n-s+i}{i} F_{s-1-i} = f_{s-1} .
\end{aligned}$$

$$(18) \qquad {}^{(1)}g_{n-s,t} = \sum_{i=0/m:d}^{t-1/n-s+t-1} (-1)^i \binom{s-t-1+i}{i} F_{n-s+t-1-i}$$

$$(s = 2, 3, \cdots, n;\ t = 1, 2, \cdots, s-1) .$$

Proof of (18). This follows directly from (16), if we interprete ${}^{(1)}g_{n-s,t}$ as $(\bar{g}_{n-s,t} - g_{n-t,s}) : (D - w)$. (It will be shown later that this interpretation is in accordance with the general notation of ${}^{(1)}P_s$.

$$(19) \qquad {}^{(1)}g_{n-s,1} - {}^{(1)}f_{n-s} = g_{n-(s+1),1} . \qquad (s = 1, 2, \cdots, n-1) .$$

Proof of (19). We have from (16), (18):

$$\begin{aligned}
&{}^{(1)}g_{n-s,1} - {}^{(1)}f_{n-s} \\
&= \frac{m}{d} F_{n-s} + \sum_{i=1}^{n-s} (-1)^i \binom{s-2+i}{i} F_{n-s-i}
\end{aligned}$$

$$- \sum_{i=0}^{n-s-1} (-1)^i \binom{s-1+i}{i} F_{n-s-1-i}$$

$$= \frac{m}{d} F_{n-s} + \sum_{i=0}^{n-s-1} (-1)^{i+1} \binom{s-1+i}{i+1} F_{n-s-1-i}$$

$$- \sum_{i=0}^{n-s-1} (-1)^i \binom{s-1+i}{i} F_{n-s-1-i}$$

$$= \frac{m}{d} F_{n-s} + \sum_{i=0}^{n-s-1} (-1)^{i+1} \left(\binom{s-1+i}{i+1} + \binom{s-1+i}{i} \right) F_{n-s-1-i}$$

$$= \frac{m}{d} F_{n-s} + \sum_{i=0}^{n-s-1} (-1)^{i+1} \binom{s+i}{i+1} F_{n-s-1-i}$$

$$= \frac{m}{d} F_{n-s} + \sum_{i=1}^{n-s} (-1)^i \binom{s-1+i}{i} F_{n-s-i}$$

$$= \sum_{i=0/m:d}^{0/n-s} (-1)^i \binom{s-1+i}{i} F_{n-s-i} = g_{n-(s+1),1} .$$

(20) $$^{(1)}g_{n-s,t+1} - {}^{(1)}g_{n-s,t} = g_{n-(s+1),t+1} .$$

$$(s = 2, \cdots, n-1;\ t = 1, \cdots, s-1) .$$

Proof of (20). We have from (18):

$$^{(1)}g_{n-s,t+1} - {}^{(1)}g_{n-s,t}$$

$$= \sum_{i=0/m:d}^{t/n-s+t} (-1)^i \binom{s-t-2+i}{i} F_{n-s+t-i}$$

$$- \sum_{i=0/m:d}^{t-1/n-s+t-1} (-1)^i \binom{s-t-1+i}{i} F_{n-s+t-1-i}$$

$$= \frac{m}{d} \sum_{i=0}^{t} (-1)^i \binom{s-t-2+i}{i} F_{n-s+t-i}$$

$$- \frac{m}{d} \sum_{i=0}^{t-1} (-1)^i \binom{s-t-1+i}{i} F_{n-s+t-1-i}$$

$$+ \sum_{i=t+1}^{n-s+t} (-1)^i \binom{s-t-2+i}{i} F_{n-s+t-i}$$

$$- \sum_{i=t}^{n-s+t-1} (-1)^i \binom{s-t-1+i}{i} F_{n-s+t-1-i}$$

$$= \frac{m}{d} \sum_{i=0}^{t} (-1)^i \binom{s-t-2+i}{i} F_{n-s+t-i}$$

$$- \frac{m}{d} \sum_{i=1}^{t} (-1)^{i-1} \binom{s-t-2+i}{i-1} F_{n-s+t-i}$$

$$+ \sum_{i=t+1}^{n-s+t} (-1)^i \binom{s-t-2+i}{i} F_{n-s+t-i}$$

$$- \sum_{i=t+1}^{n-s+t} (-1)^{i-1} \binom{s-t-2+i}{i-1} F_{n-s+t-i}$$

$$= \frac{m}{d} F_{n-s+t} + \frac{m}{d} \sum_{i=1}^{t} (-1)^i \binom{s-t-1+i}{i} F_{n-s+t-i}$$

$$+ \sum_{i=t+1}^{n-s+t} (-1)^i \binom{s-t-1+i}{i} F_{n-s+t-i}$$

$$= \sum_{i=0/m:d}^{t/n-s+t} (-1)^i \binom{s-t-1+i}{i} F_{n-s+t-i} = g_{n-(s+1),t+1} .$$

(20a) $$^{(1)}g_{n-s,t} - {}^{(1)}g_{n-s,q} = \sum_{i=0}^{t-1-q} g_{n-(s+1),t-i} . \qquad (q < t)$$

Proof of (20a). We have from (20)

$$^{(1)}g_{n-s,t} - {}^{(1)}g_{n-s,q} = \sum_{i=0}^{t-1-q} ({}^{(1)}g_{n-s,t-i} - {}^{(1)}g_{n-s,t-i-1})$$

$$= \sum_{i=0}^{t-1-q} {}^{(1)}g_{n-(s+1),t-i}$$

(20b) $$^{(1)}g_{0,t} - {}^{(1)}g_{0,t-1} = \frac{m}{d} f_{t-1} . \qquad (t = 1, 2, \cdots, n-1) .$$

Proof of (20b). From (18) we derive:

$$^{(1)}g_{0,t} = \frac{m}{d} \sum_{i=0}^{t-1} (-1)^i \binom{n-t-1+i}{i} F_{t-1-i} ,$$

$$^{(1)}g_{0,t-1} = \frac{m}{d} \sum_{i=0}^{t-2} (-1)^i \binom{n-t+i}{i} F_{t-2-i} ;$$

$$^{(1)}g_{0,t} - {}^{(1)}g_{0,t-1}$$

$$= \frac{m}{d} \sum_{i=0}^{t-1} (-1)^i \binom{n-t-1+i}{i} F_{t-1-i}$$

$$- \frac{m}{d} \sum_{i=0}^{t-2} (-1)^i \binom{n-t+i}{i} F_{t-2-i}$$

$$= \frac{m}{d} F_{t-1} + \frac{m}{d} \sum_{i=1}^{t-1} (-1)^i \binom{n-t-1+i}{i} F_{t-1-i}$$

$$+ \frac{m}{d} \sum_{i=1}^{t-1} (-1)^i \binom{n-t-1+i}{i-1} F_{t-1-i}$$

$$= \frac{m}{d} F_{t-1} + \frac{m}{d} \sum_{i=1}^{t-1} (-1)^i \binom{n-t+i}{i} F_{t-1-i}$$

$$= \frac{m}{d} \sum_{i=0}^{t-1} (-1)^i \binom{n-t+i}{i} F_{t-1-i} = \frac{m}{d} f_{t-1} .$$

(20c) $$\frac{g_{n-s,s-1} - \bar{g}_{n-s,s-1} + 1}{D - w} = g_{n-(s+1),s} . \qquad (s = 2, 3, \cdots, n-1) .$$

Proof of (20c). We have from (18) and the definition of w

$$\frac{g_{n-s,s-1} - \bar{g}_{n-s,s-1} + 1}{D - w} = \frac{1}{D - w} - {}^{(1)}g_{n-s,s-1}$$
$$= \frac{w^{n-1} + w^{n-2}D + \cdots + D^{n-1}}{D^n - w^n} - {}^{(1)}g_{n-s,s-1}$$
$$= \frac{F_{n-1}}{D^n - (D^n - d : m)} - {}^{(1)}g_{n-s,s-1}$$
$$= \frac{m}{d}F_{n-1} - {}^{(1)}g_{n-s,s-1}$$
$$= \frac{m}{d}F_{n-1} - \sum_{i=0/m:d}^{s-2/n-2} (-1)^i F_{n-2-i}$$
$$= \frac{m}{d}F_{n-1} + \sum_{i=1/m:d}^{s-1/n-1} (-1)^i F_{n-1-i}$$
$$= \sum_{i=0/m:d}^{s-1/n-1} (-1)^i F_{n-1-i} = g_{n-(s+1),s} .$$

(20d) $$\frac{g_{0,n-1} - \bar{g}_{0,n-1} + 1}{(m : d)(D - w)} = f_{n-1} .$$

Proof of (20d). We have from previous proofs and formulas

$$\frac{1}{(m : d)(D - w)} = F_{n-1} ; \quad \frac{\bar{g}_{0,n-1} - g_{0,n-1}}{(m : d)(D - w)} = \frac{{}^{(1)}g_{n-n,n-1}}{m : d}$$
$$= \frac{d}{m} \sum_{i=0/m:d}^{n-2} (-1)^i F_{n-2-i} = \sum_{i=0}^{n-2} (-1)^i F_{n-2-i} ;$$

therefore

$$\frac{g_{0,n-1} - \bar{g}_{0,n-1} + 1}{(m : d)(D - w)} = F_{n-1} - \sum_{i=0}^{n-2} (-1)^i F_{n-2-i}$$
$$= \sum_{i=0}^{n-1} (-1)^i F_{n-1-i} = f_{n-1} .$$

(17a) $${}^{(1)}f_s - {}^{(1)}f_q = \sum_{i=0}^{s-q-1} f_{s-1-i} . \qquad (1 \leqq q \leqq s-1) .$$

Proof of (17a). We have from (17)

$${}^{(1)}f_s - {}^{(1)}f_q = \sum_{i=0}^{s-q-1} ({}^{(1)}f_{s-i} - {}^{(1)}f_{s-i-1}) = \sum_{i=0}^{s-q-1} f_{s-1-i} .$$

(20e) $${}^{(1)}g_{n-s,t} - {}^{(1)}f_q = \sum_{i=0}^{t-1} g_{n-(s+1)t-i} + \sum_{i=0}^{n-s-q-1} f_{n-s-1-i} . \quad (1 \leqq q \leqq n-s) .$$

Proof of (20e). This follows immediately combining (17a), (19), (20a).

IV. Inequalities. In this chapter we shall establish magnitude relations between the auxiliary functions f_s, F_s, $g_{n-s,t}$. We first note that

$$(21)\qquad D-1<w<D;\quad (D-1)^k<w^k<D^k. \qquad (k=1,2,\cdots,n-1)$$

(21) follows directly from the definition of w. From (21) and the definition of f_s and F_s follows further

$$(22)\qquad f_s(D-1,D-1)<f_s<F_s<\bar{F}_s;\quad f_s<\bar{f}_s<\bar{F}_s. \qquad (s=1,2,\cdots,n-1)$$

$$(23)\qquad \left(1+\frac{1}{D-1}\right)^{n-2}<1{,}65 \quad \text{for } 2(n-1)\leqq D.$$

Proof of (23). Since $D\geqq 2d(n-1)$, $d\geqq 1$, we have $D\geqq 2(n-1)$, $D-1\geqq 2(n-1)-1>2(n-2)$. Therefore

$$\left(1+\frac{1}{D-1}\right)^{n-2}<\left(1+\frac{1}{2(n-2)}\right)^{n-2}=\left(\left(1+\frac{1}{2(n-2)}\right)^{2(n-2)}\right)^{1:2}$$
$$<e^{1:2}=1{,}64872\cdots<1{,}65.$$

$$(24)\qquad \bar{F}_i\leqq F_{i+1}(D-1,D-1). \qquad (i=0,1,\cdots,n-2).$$

Proof of (24). We have to prove, following (13a):

$$\binom{n}{i}D^i\leqq\binom{n}{i+1}(D-1)^{i+1},$$
$$\left(1+\frac{1}{D-1}\right)^i\leqq\frac{n-i}{i+1}(D-1),$$
$$D-1\geqq\frac{i+1}{n-i}\left(1+\frac{1}{D-1}\right)^i.$$

We prove *a fortiori*, since $(i+1):(n-i)$ is an increasing function,

$$D-1\geqq\frac{n-1}{2}\cdot 1{,}65>\frac{i+1}{n-i}\left(1+\frac{1}{D-1}\right)^i;$$

but

$$D-1\geqq 2d(n-1)-1\geqq 2(n-1)-1>1{,}65\frac{n-1}{2}.$$

$$(24\text{a})\qquad F_s<F_{s+t}. \qquad (s=0,\cdots,n-2;\ t=1,\cdots,n-1;\ 1\leqq s+t\leqq n-1)$$

Proof of (24a). It follows from (22), (24)

$$F_s<\bar{F}_s<F_{s+1}(D-1,D-1)<F_{s+1},$$

so that

$$F_s < F_{s+1} < F_{s+2} < \cdots < F_{s+t} .$$

(24b) $$f_s < f_{s+t} .$$
$$(s = 0, \cdots, n-2;\ t = 1, \cdots, n-1;\ 1 \leqq s+t \leqq n-1) .$$

Proof of (24b). It follows from (22), (24)

$$f_s < \bar{f}_s < \bar{F}_s < F_{s+1}(D-1, D-1) = f_{s+1}(D-1, D-1) < f_{s+1} .$$

(25) $$2F_{n-2} < \frac{1}{d} F_{n-1} .$$

Proof of (25). We have to prove

$$2 \sum_{i=0}^{n-2} (i+1) w^{n-2-i} D^i < \frac{1}{d} \sum_{i=0}^{n-1} w^{n-1-i} D^i$$

and prove *a fortiori*

$$2 \sum_{i=0}^{n-2} (i+1) w^{n-2-i} D^i < \frac{2(n-1)}{D} \sum_{i=0}^{n-1} w^{n-1-i} D^i$$
$$\leqq \frac{1}{d} \sum_{i=0}^{n-1} w^{n-1-i} D^i .$$

We thus have to prove

$$\sum_{i=0}^{n-2} (i+1) w^{n-2-i} D^i < \frac{(n-1) w^{n-1}}{D} + \sum_{i=0}^{n-2} (n-1) w^{n-2-i} D^i ,$$

which is always true, since $i + 1 \leqq n - 1$, $0 < (n-1)w^{n-1}/D$,

(26) $${}^{(1)}f_s \leqq F_{s-1} . \qquad (s = 1, 2, \cdots, n-1) .$$

Proof of (26). We have from (15)

$${}^{(1)}f_s = F_{s-1}$$
$$- \sum_{i=1}^{(s-1)/2} \left(\binom{n-s+2i-2}{2i-1} F_{s-2i} - \binom{n-s+2i-1}{2i} F_{s-2i-1} \right)$$
$$- e \binom{n-2}{s-1} , \qquad (e = 0, \text{ when } s \text{ is odd}, = 1 \text{ otherwise})$$

so that ${}^{(1)}f_s \leqq F_{s-1}$, if we can prove that the expression under the sigma sign is not negative. We shall therefore prove

$$\binom{n-s+2i-2}{2i-1} F_{s-2i} > \binom{n-s+2i-1}{2i} F_{s-2i-1} ,$$

or

$$F_{s-2i} > \frac{n-s+2i-1}{2i} F_{s-2i-1} \,,$$

and prove *a fortiori*

$$F_{s-2i} > \binom{n}{s-2i}(D-1)^{s-2i} \geqq \frac{n-s+2i-1}{2i}\binom{n}{s-2i-1}D^{s-2i-1}$$
$$> \frac{n-s+2i-1}{2i} F_{s-2i-1} \,.$$

We thus have to prove

$$\binom{n}{s-2i}(D-1)^{s-2i} \geqq \frac{n-s+2i-1}{2i} D^{s-2i-1} \,,$$

or

$$(D-1)^{s-2i} \geqq \frac{(n-s+2i-1)(s-2i)}{2i(n-s+2i+1)} D^{s-2i-1} \,,$$

or

$$(D-1) \geqq \frac{n-s+2i-1}{n-s+2i+1} \cdot \frac{s-2i}{2i}\left(1+\frac{1}{D-1}\right)^{s-2i-1} .$$

But from $D \geqq 2d(n-1)$ we have

$$D-1 > \frac{n-3}{2} \cdot 1{,}65 > \frac{n-s+2i-1}{n-s+2i+1} \cdot \frac{s-2i}{2i}\left(1+\frac{1}{D-1}\right)^{s-2i-1} .$$

(27) $\quad {}^{(1)}g_{n-s,t} < (m:d)F_{n-s+t-1} \,. \quad (s = 2, 3, \cdots, n;\ t = 1, 2, \cdots, s-1) \,.$

Proof of (27). We have from (18) for $t = 2r+1$

$$\begin{aligned} {}^{(1)}g_{n-s,t} &= (m:d)F_{n-s+t-1} \\ &-(m:d)\sum_{i=1}^{r}\left(\binom{s-t+2i-2}{2i-1}F_{n-s+t-2i} - \binom{s-t+2i-1}{2i}F_{n-s+t-2i-1}\right) \\ &-\sum_{i=2r+1}^{n-s+2r}\left(\binom{s-t+2i-2}{2i-1}F_{n-s+t-2i} - \binom{s-t+2i-1}{2i}F_{n-s+t-2i-1}\right) . \end{aligned}$$

We shall now prove that the expressions under both the sigma signs are nonnegative, so that ${}^{(1)}g_{n-s,t} < (m:d)F_{n-s+t-1}$. We have to prove

$$\binom{s-t+2i-2}{2i-1}F_{n-s+t-2i} > \binom{s-t+2i-1}{2i}F_{n-s+t-2i-1} \,,$$

or

$$F_{n-s+t-2i} > \frac{s-t+2i-1}{2i} F_{n-s+t-2i-1} \,.$$

We shall prove *a fortiori*

$$F_{n-s+t-2i} > \binom{n}{s-t+2i}(D-1)^{n-s+t-2i}$$
$$\geqq \frac{s-t+2i-1}{2i}\binom{n}{s-t+2i+1}D^{n-s+t-2i-1}$$
$$> \frac{s-t+2i-1}{2i}F_{n-s+t-2i-1} .$$

We have to prove

$$\binom{n}{s-t+2i}(D-1)^{n-s+t-2i} \geqq \frac{s-t+2i-1}{2i}\binom{n}{s-t+2i+1}D^{n-s+t-2i-1} ,$$

or

$$D-1 \geqq \frac{s-t+2i-1}{s-t+2i+1}\cdot\frac{n-s+t-2i}{2i}\cdot\left(1+\frac{1}{D-1}\right)^{n-s+t-2i-1} ,$$

which follows immediately from $2(n-1)-1 \leqq D-1$ and the upper and lower bounds of s, t, as at the end of the previous proof.

For $t = 2r+2$ we have

$$\begin{aligned}
{}^{(1)}g_{n-s,t} &= (m:d)F_{n-s+t-1} \\
&-(m:d)\sum_{i=1}^{r}\left(\binom{s-t+2i-2}{2i-1}F_{n-s+t-2i} - \binom{s-t+2i-1}{2i}F_{n-s+t-2i-1}\right) \\
&-\left((m:d)\binom{s-2}{t-1}F_{n-s} - \binom{s-1}{t}F_{n-s-1}\right) \\
&-\sum_{i=2r+3}^{n-s+2r+1}\left(\binom{s-t+2i-2}{2i-1}F_{n-s+t-2i} - \binom{s-t+2i-1}{2i}F_{n-s+t-2i-1}\right) ,
\end{aligned}$$

so that in order to prove (27) in this case of $t = 2r+2$ we have only to add the proof of

$$(m:d)\binom{s-2}{t-1}F_{n-s} - \binom{s-1}{t}F_{n-s-1} \geqq 0 .$$

Since $m \geqq 1$, we prove *a fortiori*

$$\frac{1}{d}\binom{s-2}{t-1}F_{n-s} \geqq \binom{s-1}{t}F_{n-s-1} ,$$

or

$$F_{n-s} \geqq \frac{(s-1)d}{t}F_{n-s-1} .$$

We prove *a fortiori*

$$F_{n-s} > \binom{n}{s}(D-1)^{n-s} \geqq \frac{(s-1)d}{t}\binom{n}{s+1}D^{n-s-1} ,$$

or

$$D-1 \geqq \frac{s-1}{s+1}\cdot\frac{n-s}{t}\cdot d\left(\frac{1}{D-1}\right)^{n-s-1} ,$$

which follows immediately from $D-1 \geqq 2d(n-1)-1 > d(n-2)\cdot 1{,}65$.

(28) $\quad g_{n-s,t} < (m:d)F_{n-s+t} . \qquad (s=2,3,\cdots,n;\ t=1,2,\cdots,s-1)$.

Proof of (28). This is completely analogous to proof of (27).

(29) $\quad [f_s] = -1+\bar{f}_s . \qquad (s=1,2,\cdots,n-1)$.

Proof of (29). We have to prove

$$\text{(A)}\quad -1+\bar{f}_s < f_s ; \qquad \text{(B)}\quad f_s < \bar{f}_s .$$

To prove (A) we have to show that

$$\bar{f}_s - f_s < 1 , \qquad \text{or, dividing by } D-w>0 ,$$
$${}^{(1)}f_s < (m:d)F_{n-1} .$$

From (25), (26) we have

$${}^{(1)}f_s \leqq F_{s-1} \leqq F_{n-2} < (m:d)F'_{n-1} .$$

(B) follows from (22). Thus (29) is proved.

(30) $\quad [g_{n-s,t}] = -1+\bar{g}_{n-s,t} . \qquad (s=2,\cdots,n;\ t=1,\cdots,s-1)$

Proof of (30). We have to prove

$$\text{(A)}\quad -1+\bar{g}_{n-s,t} < g_{n-s,t} ; \qquad \text{(B)}\quad g_{n-s,t} < \bar{g}_{n-s,t} .$$

To prove (A) we have to show

$$\bar{g}_{n-s,t} - g_{n-s,t} < 1 , \qquad \text{or, dividing by } D-w ,$$
$${}^{(1)}g_{n-s,t} < (m:d)F_{n-1} .$$

But from (27) we have

$${}^{(1)}g_{n-s,t} < (m:d)F_{n-s+t-1} \leqq (m:d)F_{n-2} < (m:d)F_{n-1} .$$

To prove (B) we have to show, after dividing by $D-w$

$${}^{(1)}g_{n-s,t} > 0 .$$

But for $s < n$

$$^{(1)}g_{n-s,t} \geqq {}^{(1)}g_{n-s,t} - {}^{(1)}g_{n-s,t-1} = g_{n-(s+1),t} > 0 ;$$

and for $s = n$

$$^{(1)}g_{0,t} \geqq {}^{(1)}g_{0,t} - {}^{(1)}g_{0,t-1} = (m : d)f_{t-1} > 0 ,$$

(that the expressions $g_{n-s,t}$ are positive entities will become clear later, while carrying out the JAPAL for the f_i).

(31) $$\frac{f_i - \bar{f}_i + 1}{f_j - \bar{f}_j + 1} < 1 . \qquad (j < i < n - 1;\ j = 1, \cdots, n - 2) .$$

Proof of (31). It was shown that the denominator is positive. We therefore have to prove, after dividing by $D - w$

$$^{(1)}f_i - {}^{(1)}f_j > 0 ,$$

which follows directly from (17a).

(32) $$\frac{(D - w)f_j}{f_s - \bar{f}_s + 1} < 1 . \quad (j = 0, 1, \cdots, s - 2;\ s = 2, 3, \cdots, n - 1) .$$

Proof of (32). We have to show

$$(D - w)f_j < f_s - \bar{f}_s + 1 , \qquad \text{or, dividing by } D - w ,$$
$$f_j + {}^{(1)}f_s < (m : d)F_{n-1} .$$

But

$$f_j + {}^{(1)}f_s < f_{s-2} + F_{s-1} < F_{n-3} + F_{n-2} < 2F_{n-2} < (m : d)F_{n-1} .$$

(33) $$1 < \frac{f_{i-1} - \bar{f}_{i-1} + 1}{f_i - \bar{f}_i + 1} < 2 . \qquad (i = 1, \cdots, n - 1) .$$

Proof of (33). We have to prove, since the left hand inequality was proved in (31)

$$f_{i-1} - \bar{f}_{i-1} + 1 < 2(f_i - \bar{f}_i) + 2 ,$$

or carrying over and dividing m by $D - w$

$$2\cdot{}^{(1)}f_i - {}^{(1)}f_{i-1} < (m : d)F_{n-1} .$$

But

$$2\cdot{}^{(1)}f_i - {}^{(1)}f_{i-1} \leqq 2\cdot{}^{(1)}f_i \leqq 2F_{n-2} < \frac{m}{d}F_{n-1} .$$

(34) $$\frac{g_{n-s,t} - \bar{g}_{n-s,t} + 1}{f_q - \bar{f}_q + 1} < 1 .$$
$$(s = 2, 3, \cdots, n;\ t = 1, 2, \cdots, s - 1;\ q = 1, 2, \cdots, n - s) .$$

Proof of (34). We have to prove

$$g_{n-s,t} - \bar{g}_{n-s,t} + 1 < f_q - \bar{f}_q + 1 ,$$

or, dividing by $D - w$,

$$^{(1)}g_{n-s,t} - {}^{(1)}f_q > 0 ,$$

which follows directly from (20e).

(35) $$\frac{g_{n-s,t} - \bar{g}_{n-s,t} + 1^{2)}}{g_{n-s,j} - \bar{g}_{n-s,j} + 1} < 1 . \qquad (1 \leqq j < t \leqq s - 1 \leqq n - 1)$$

Proof of (35). We have to prove

$$g_{n-s,t} - \bar{g}_{n-s,t} + 1 < g_{n-s,j} - \bar{g}_{n-s,j} + 1 ,$$

or, after carrying over and dividing by $D - w$

$$^{(1)}g_{n-s,t} - {}^{(1)}g_{n-s,j} > 0 ,$$

which follows directly from (20a).

(36) $$1 < \frac{f_{n-s} - \bar{f}_{n-s} + 1}{g_{n-s,1} - \bar{g}_{n-s,1} + 1} < 2 . \qquad (s = 2, 3, \cdots, n) .$$

Proof of (36). We have to prove

(A) $g_{n-s,1} - \bar{g}_{n-s,1} + 1 < f_{n-s} - \bar{f}_{n-s} + 1 ,$

(B) $f_{n-s} - \bar{f}_{n-s} + 1 < 2(g_{n-s,1} - \bar{g}_{n-s,1}) + 2 .$

To prove (A) we have to show, after carrying over and dividing by $D - w$

$$^{(1)}g_{n-s,1} - {}^{(1)}f_{n-s} > 0 ;$$

But from (19) we have

$$^{(1)}g_{n-s,1} - {}^{(1)}f_{n-s} = g_{n-(s+1),1} , \qquad \text{for } s < n .$$

For $n = s$ we have

$$^{(1)}g_{0,1} - {}^{(1)}g_{0,0} = (m : d)f_0 > 1 .$$

To prove (B) we have to show, after carrying over and dividing by $D - w$

$$2 \cdot {}^{(1)}g_{n-s,1} - {}^{(1)}f_{n-s} < (m : d)F_{n-1} .$$

But

$$2 \cdot {}^{(1)}g_{n-s,1} - {}^{(1)}f_{n-s} \leqq 2 \cdot {}^{(1)}g_{n-s,1} < 2F_{n-s} \leqq 2F_{n-2} < (m : d)F_{n-1} .$$

(37) $$\frac{(D - w)f_{r-1}}{g_{n-s,t} - \bar{g}_{n-s,t} + 1} < 1 .$$

$$(r = 1, 2, \cdots, n - s;\ s = 2, 3, \cdots, n - 1;\ t = 1, 2, \cdots, s - 1) .$$

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[2] While carrying out the JAPAL in the following chapter, it will become clear that the numerator and denominator are positive entities.

Proof of (37). We have to prove

$$(D-w)f_{r-1} < g_{n-s,t} - \bar{g}_{n-s,t} + 1\,,$$

or, after carrying over and dividing by $D-w$

$${}^{(1)}g_{n-s,t} + f_{r-1} < (m:d)F_{n-1}\,,$$

or *a fortiori*

$${}^{(1)}g_{n-s,t} + f_{r-1} < F_{n-s+t-1} + F_{r-1} < 2F_{n-2} < (m:d)F_{n-1}\,.$$

(38) $$1 < \frac{g_{n-s,t} - \bar{g}_{n-s,t} + 1}{g_{n-s,t+1} - \bar{g}_{n-s,t+1} + 1} < 2\,.$$
$$(s = 2, 3, \cdots, n;\; t = 1, 2, \cdots, s-2)\,.$$

Proof of (38). We have to prove

(A) $g_{n-s,t} - \bar{g}_{n-s,t} + 1 > g_{n-s,t+1} - \bar{g}_{n-s,t+1} + 1\,,$

(B) $g_{n-s,t} - \bar{g}_{n-s,t} + 1 < 2(g_{n-s,t+1} - \bar{g}_{n-s,t+1}) + 2\,.$

To prove (A) we have to show, after carrying over and dividing by $D-w$

$${}^{(1)}g_{n-s,t+1} - {}^{(1)}g_{n-s,t} > 0\,,$$

which follows from

$${}^{(1)}g_{n-s,t+1} - {}^{(1)}g_{n-s,t} \mp g_{n-(s+1),t+1} > 0$$

for $s \leqq n-1$. For $s = n$ the proof is exactly as before.

To prove (B) we have to show, after carrying over and dividing by $D-w$

$$2\cdot{}^{(1)}g_{n-s,t+1} - {}^{(1)}g_{n-s,t} < (m:d)F_{n-1}\,,$$

which follows from

$$2\cdot{}^{(1)}g_{n-s,t+1} - {}^{(1)}g_{n-s,t} \leqq 2\cdot{}^{(1)}g_{n-s,t+1}$$
$$< 2F_{n-s+t} \leqq 2F_{n-2} \leqq (m:d)F_{n-1}\,.$$

V. The JAPAL of the $f_1, f_2, \cdots, f_{n-1}$. We shall now carry out the JAPAL of the numbers $f_1, f_2, \cdots, f_{n-1}$ and thus complete the proof of the Main Theorem. To this end I shall introduce still a few more new conceptions.

DEFINITION 4. The set of $n-1$ numbers $a_i^{(v)}$ $(i = 1, \cdots, n-1;$ $v = 0, 1, \cdots)$ shall be called the vth *generator* of the JAPAL, the number $a_i^{(v)}$ its ith element; the set of $n-1$ numbers

$$b_i^{(v)} = [a_i^{(v)}]$$

(i, v as before) shall be called the vth *genus* of the JAPAL, $b_i^{(v)}$ its ith element. The key to the final proof of the Main Theorem now rests with the

LEMMA. *Let the $n-1$ elements of the vth generator fulfill the following conditions:*

(A) $a_i^{(v)} = P_i(w, D)$; $P_1(w, D) = w + c_1 D$; $P_0(w, D) = \bar{P}_0(w, D) = 1$;

(B) $$[a_i^{(v)}] = -1 + \bar{P}_i(w, D).$$

($i = 1, 2, \cdots, n-1$; c_1 *a nonnegative integer*)

(C) $$0 < \frac{P_{t+k} - \bar{P}_{t+k} + 1}{P_t - \bar{P}_t + 1} < 1$$

$$(t = 1, \cdots, n-2;\ k = 1, \cdots, n-2;\ t + k \leqq n-1)$$

(39) $$0 < \frac{(D-w)({}^{(1)}P_{q-1} - {}^{(1)}P_{q-2})}{P_t - \bar{P}_t + 1} < 1\ ;$$

$$(q = 2, 3, \cdots, t;\ t = 2, 3, \cdots, n-1)$$

$$1 < \frac{P_{t-1} - \bar{P}_{t-1} + 1}{P_t - \bar{P}_t + 1} < 2\ . \qquad (t = 1, \cdots, n-1)\ .$$

Then the n genera

$$b_1^{(v+k)}, b_2^{(v+k)}, \cdots, b_{n-1}^{(v+k)} \qquad (k = 0, 1, \cdots, n-1)$$

form a fugue, and the elements of the $v+n$th generator, namely the $a_i^{(v+n)}$ ($i = 1, 2, \cdots, n-1$) have the form

(40) $$a_i^{(v+n)} = {}^{(1)}P_{i+1} - {}^{(1)}P_i\ ; \qquad (i = 1, \cdots, n-2)$$
$$a_{n-1}^{(v+n)} = \frac{P_{n-1} - \bar{P}_{n-1} + 1}{D - w}\ .$$

Proof of the lemma. In view of (39) (A), (B) and following formula (1) the elements of the $v+1$st generator have the form

(40a) $$a_i^{(v+1)} = \frac{P_{i+1} - \bar{P}_{i+1} + 1}{P_1 - \bar{P}_1 + 1}\ , \qquad (i = 1, \cdots, n-2)\ ;$$
$$a_{n-1}^{(v+n)} = \frac{1}{P_1 - \bar{P}_1 + 1}\ .$$

Since the elements of the $v+1$st generator fulfill the conditions (39) (C), the elements of the $v+1$st genus have the form

(40b) $$b_i^{(v+1)} = 0\ ; \quad (i = 1, 2, \cdots, n-2) \quad b_{n-1}^{(v+1)} = 1\ .$$

On the basis of (40a), (40b) and reminding from (39) that $\bar{P}_1 - P_1 = (D-w) = (D-w)({}^{(1)}P_1 - {}^{(1)}P_0)$ we obtain, following (1), for the $n-1$

elements of the $v+2$nd generator

$$a_i^{(v+2)} = \frac{P_{i+2} - \bar{P}_{i+2} + 1}{P_2 - \bar{P}_2 + 1}\,; \qquad (i = 1, 2, \cdots, n-3)$$

(40c)
$$a_{n-2}^{(v+2)} = \frac{(D-w)({}^{(1)}P_1 - {}^{(1)}P_0)}{P_2 - \bar{P}_2 + 1}\,,$$

$$a_{n-1}^{(v+2)} = \frac{P_1 - \bar{P}_1 + 1}{P_2 - \bar{P}_2 + 1}\,.$$

Now the elements $a_i^{(v+2)}$ $(i = 1, 2, \cdots, n-1)$ again satisfy conditions (39) (C), and therefore the elements of the $v+2$nd genus have the form

(40d) $$b_i^{(v+2)} = 0\,; \quad (i = 1, \cdots, n-2) \quad b_{n-1}^{(v+2)} = 1\,.$$

In view of (40c), (40d) and (1) the elements of the $v+3$rd generator have the form

$$a_i^{(v+3)} = \frac{P_{i+3} - \bar{P}_{i+3} + 1}{P_3 - \bar{P}_3 + 1}\,, \qquad (i = 1, 2, \cdots, n-4)$$

$$a_{n-3}^{(v+3)} = \frac{(D-w)({}^{(1)}P_1 - {}^{(1)}P_0)}{P_3 - \bar{P}_3 + 1}\,,$$

(40e)
$$a_{n-2}^{(v+3)} = \frac{(D-w)({}^{(1)}P_2 - {}^{(1)}P_1)}{P_3 - \bar{P}_3 + 1}\,,$$

$$a_{n-1}^{(v+3)} = \frac{P_2 - \bar{P}_2 + 1}{P_3 - \bar{P}_3 + 1}\,.$$

Continuing these considerations one arrives quite easily and by induction at the conclusion that the $v+t$th generator takes the form

$$a_i^{(v+t)} = \frac{P_{i+t} - \bar{P}_{i+t} + 1}{P_t - \bar{P}_t + 1}\,, \qquad (i = 1, \cdots, n-t-1)$$

(40f)
$$a_{n-t-1-j}^{(v+t)} = \frac{(D-w)({}^{(1)}P_j - {}^{(1)}P_{j-1})}{P_t - \bar{P}_t + 1}\,, \qquad (j = 1, \cdots, t-1)$$

$$a_{n-1}^{(v+t)} = \frac{P_{t-1} - \bar{P}_{t-1} + 1}{P_t - \bar{P}_t + 1}\,,$$

and that the $n-1$ elements of the $v+t$th genus have the form

(40g) $$b_i^{(v+t)} = 0\,; \quad (i = 1, \cdots, n-2) \quad b_{n-1}^{(v+t)} = 1\,.$$

Following the formulas (40f), (40g) and (1) we obtain that the $n-1$ elements of the $v+n-1$st generator have the form

$$a_i^{(v+n-1)} = \frac{(D-w)({}^{(1)}P_i - {}^{(1)}P_{i-1})}{P_{n-1} - \bar{P}_{n-1} + 1}\,, \qquad (i = 1, \cdots, n-2)$$

(40h)
$$a_{n-1}^{(v+n-1)} = \frac{P_{n-2} - \bar{P}_{n-2} + 1}{P_{n-1} - \bar{P}_{n-1} + 1}\,,$$

and that, on ground of (39) (C) the elements of the $v+n-1$st genus have the form

$$b_i^{(v+n-1)} = 0\ ; \quad (i = 1, 2, \cdots, n-2) \quad b_{n-1}^{(v+n-1)} = 1\ . \tag{40i}$$

Thus the n genera

$$b_1^{(v+j)}, b_2^{(v+j)}, \cdots, b_{n-1}^{(v+j)}\ , \qquad (j = 0, 1, \cdots, n-1)$$

indeed form a fugue as was stated in the lemma. Now we have from (40h)

$$a_1^{(v+n-1)} = \frac{(D-w)({}^{(1)}P_1 - {}^{(1)}P_0)}{P_{n-1} - \bar{P}_{n-1} + 1} = \frac{D-w}{P_{n-1} - \bar{P}_{n-1} + 1}\ , \tag{40j}$$

so that on the basis of (40h), (40i), (40j) we receive for the $n-1$ elements of the $v+n$th generator

$$\begin{aligned} a_i^{(v+n)} &= {}^{(1)}P_{i+1} - {}^{(1)}P_i\ , \qquad (i = 1, 2, \cdots, n-2) \\ a_{n-1}^{(v+n)} &= \frac{P_{n-1} + \bar{P}_{n-1} + 1}{D-w}\ . \end{aligned} \tag{40k}$$

By this the lemma is completely proved.

We are now able to prove the main Theorem quite easily in the following steps:
(1) Let be

$$P_i(w, D) = f_i(w, D) = a_i^{(0)}\ . \qquad (i = 1, 2, \cdots, n-1)\ . \tag{41}$$

Following (29), (31), (32), (33) the functions f_i $(i = 1, \cdots, n-1)$ indeed fulfill the conditions (39) (A), (B), (C). Therefore, following the lemma, we get for the $n+1$st generator, which is the first generator of the second fugue of the JAPAL

$$\begin{aligned} a_i^{(n)} &= {}^{(1)}f_{i+1} - {}^{(1)}f_i\ , \qquad (i = 1, 2, \cdots, n-2) \\ a_{n-1}^{(n)} &= \frac{f_{n-1} - \bar{f}_{n-1} + 1}{D-w}\ , \end{aligned}$$

so that on the ground of

$$\begin{aligned} \frac{f_{n-1} - \bar{f}_{n-1} + 1}{D-w} &= \frac{1}{D-w} - {}^{(1)}f_{n-1} \\ &= (m:d)F_{n-1} - \sum_{i=0}^{n-2}(-1)^i F_{n-2-i} \\ &= \sum_{i=0/m:d}^{0/n-1}(-1)^i F_{n-1-i} = g_{n-2,1} \end{aligned}$$

and on the basis of (17) we have for the $n-1$ elements of the first generator of the second fugue of the JAPAL

(41a) $$a_i^{(n)} = f_i \, ; \qquad (i = 1, 2, \cdots, n-2) \; a_{n-1}^{(n)} = g_{n-2,1} \, .$$

(2) Let

(41b) $$P_i(w, D) = f_i \, ; \qquad (i = 1, 2, \cdots, n-2) \; P_{n-1}(w, D) = g_{n-2,1} \, .$$

Following the formulas (29) to (37) the functions of (41b) fulfill the conditions (39), and therefore the elements of the first generator of the third fugue have the form

(42) $$\begin{aligned} &a_i^{(2n)} = {}^{(1)}f_{i+1} - {}^{(1)}f_i \, ; && (i = 1, 2, \cdots, n-3) \\ &a_{n-2}^{(2n)} = {}^{(1)}g_{n-2,1} - {}^{(1)}f_{n-2} \, ; \qquad a_{n-1}^{(2n)} = \frac{g_{n-2,1} - \bar{g}_{n-2,1} + 1}{D - w} \, . \end{aligned}$$

Following the formulas (17), (19), (20c) we get for the functions (42)

(42a) $$\begin{aligned} &a_i^{(2n)} = f_i \, ; && (i = 1, 2, \cdots, n-3) \\ &a_{n-2}^{(2n)} = g_{n-3,1} \, ; \qquad a_{n-1}^{(2n)} = g_{n-3,2} \, . \end{aligned}$$

In the same way we get from (42a) that the $n-1$ elements of the first generator of the fourth fugue have the form

(43) $$\begin{aligned} &a_i^{(3n)} = f_i \, ; \qquad (i = 1, 2, \cdots, n-4) \quad a_{n-3}^{(3n)} = g_{n-4,1} \, ; \\ &a_{n-2}^{(3n)} = g_{n-4,2} \, ; \qquad a_{n-1}^{(3n)} = g_{n-4,3} \, . \end{aligned}$$

Continuing this process of the JAPAL we get from (43) that the elements of the first generator of the sth fugue have the form ($s = 2, 3, \cdots, n$)

(44) $$\begin{aligned} &a_i^{((s-1)n)} = f_i \, ; && (i = 1, 2, \cdots, n-s) \\ &a_{n-s+t}^{((s-1)n)} = g_{n-s,t} \, . && (t = 1, 2, \cdots, s-1) \, . \end{aligned}$$

From (44) we finally deduce, for $n = s$, that the elements of the first generator of the nth fugue have the form

(45) $$a_i^{((n-1)n)} = g_{0,t} \, . \qquad (t = 1, 2, \cdots, n-1) \, .$$

But we have from (11)

$$\begin{aligned} g_{n-n,1} = g_{0,1} &= \sum_{i=0/m:d}^{0/1} (-1)^i \binom{n-2+i}{i} F_{1-i} \\ &= (m:d)F_1 - (n-1)F_0 = (m:d)F_1 - (n-1) \, , \end{aligned}$$

so that

$$\bar{g}_{0,1} - g_{0,1} = (m:d)(\bar{F}_1 - F_1) = (m:d)(D - w) \, .$$

With this and on the basis of the lemma, we get from (45) that the elements of the first generator of the $n+1$st fugue have the form

$$(46)\qquad \begin{aligned} a_i^{(n^2)} &= ({}^{(1)}g_{0,i+1} - {}^{(1)}g_{0,i})(m:d)\ ; \qquad (i=1,\cdots,n-2) \\ a_{n-1}^{(n^2)} &= \frac{g_{0,n-1} - \bar{g}_{0,n-1} + 1}{(m:d)(D-w)}\ . \end{aligned}$$

Now according to (20b), (20d) we have

$$(46a)\qquad a_i^{(n^2)} = f_i\ ; \qquad (i=1,2,\cdots,n-2) \quad a_{n-1}^{(n^2)} = f_{n-1}\ .$$

From (41) and (46a) we have

$$(47)\qquad a_i^{(n^2)} = a_i^{(0)}\ , \qquad (i=1,2,\cdots,n-1)$$

so that the $n-1$ elements of the first generator of the first fugue are identical with the $n-1$ elements of the first generator of the $n+1$st fugue. Thus (47) shows that the JAPAL of the f_i ($i=1,2,\cdots,n-1$) is purely periodic with the length n^2 (n fugues), as stated by the Main Theorem.

Now since

$$(48)\qquad \bar{f}_i = \sum_{i=0}^{s} \binom{n-1-s+i}{i} D^{s-i}(D-1)^i\ . \qquad (i=1,\cdots,n-1)$$

$$(48a)\qquad \begin{aligned} \bar{g}_{n-s,t} &= \sum_{i=0/m:d}^{t-1/n-s+t} (-1)^i \binom{s-t+i}{i} \bar{F}_{n-s+t-i} \\ &= \sum_{i=0/m:d}^{t-1/n-s+t} (-1)^i \binom{s-t+i}{i} \binom{n}{s-t+i} D^{n-s+t-i}\ , \\ &\qquad (s=2,3,\cdots,n\ ;\quad t=1,2,\cdots,s-1)\ , \end{aligned}$$

and since we have for the elements of the various genera of the JAPAL either

$$-1+\bar{f}_i \qquad \text{or} \qquad -1+\bar{g}_{n-s,t}$$

the pattern of the accumulators of the n fugues of the JAPAL as indicated in the formulas (6) to (6d) becomes immediately obvious. If $m=d=1$ we have

$$g_{n-2,1} = \sum_{i=0}^{0/n-1} (-1)^i F_{n-1-i} = \sum_{i=0}^{n-1} (-1)^i F_{n-1-i} = f_{n-1}\ .$$

We therefore get from (41a) that in this case the elements of the first generator of the second fugue have the form

$$a_i^{(n)} = f_i\ , \qquad (i=1,2,\cdots,n-1)$$

so that here

$$a_i^{(u)} = a_i^{(0)}\,, \qquad (i = 1, 2, \cdots, n-1) \tag{48b}$$

as stated in the Main Theorem, which, through this final remark, is completely proved.

Proof of Corollary 1. We make the following substitutions in $w = (D^n - d : m)^{1:n}$: Let T, t be natural numbers such that $t \mid T$, $t \geqq 1$, let denote

$$D = T : t\,; \quad d = 1\,; \quad m = t^{n-k}\,. \qquad (k = 0, 1, \cdots, n)\,. \tag{49}$$

Following the conditions of the Main Theorem, we have here

$$1 \leqq t \leqq T : 2(n-1)\,.$$

Further w takes the form

$$w = W : t\,; \qquad W = (T^n - t^k)^{1:n}\,. \tag{49a}$$

The functions $f_s, F_s, g_{n-s,t}$ take the form

$$f_s = t^{-s} \sum_{i=0}^{s} \binom{n-1-s+i}{i} W^{s-i}(T-t)^i\,; \qquad (s = 1, \cdots, n-1)$$

$$F_s = t^{-s} \sum_{i=0}^{s} \binom{n-1-s+i}{i} W^{s-i} T^i\,; \qquad (s = 1, \cdots, n-1) \tag{49b}$$

$$g_{n-s,u} = \sum_{i=0/t^{n-k}}^{u-1/n-s+u} (-1)^i \binom{s-u-1+i}{i} F_{n-s+u-i}$$

$$(s = 2, 3, \cdots, n\,;\; u = 1, 2, \cdots, s-1)\,.$$

If we substitute again in (49a), (49b)

$$D \text{ for } T\,; \qquad d \text{ for } t\,; \qquad w \text{ for } W\,, \tag{49c}$$

we get from the Main Theorem, that the JAPAL of the $n-1$ numbers

$$f_s = d^{-s} \sum_{i=0}^{s} \binom{n-1-s+i}{i} w^{s-i}(D-d)^i\,, \qquad w = (D^n - d^k)^{1:n}$$

$$(s = 1, 2, \cdots, n-1\,;\; k = 0, 1, \cdots, n)$$

takes the form as indicated in Corollary 1.

Proof of Corollary 2. Here we make the following substitutions in w. Let T, t be natural numbers, $t \mid T$; let

$$D = T : t\,; \quad d = 1\,; \quad m = t^{n-r} : T\,. \qquad (r = 0, 1, \cdots, n-2)\,. \tag{50}$$

The reader should note that the condition that m is a natural number is necessary only for the purpose that the elements of the accumulators in (6c), (6d) be integers. For the proof of the Main Theorem we made use only of the fact that $m \geqq 1$. The elements of the accumulators in (6c), (6d) may be integers even if m is not an integer, as was proved at the end of Chapter II. From $1 \leqq m = t^{n-r} : T$, we derive

$$T \leqq t^{n-r} , \tag{50a}$$

and from $1 \leqq D : 2d(n-1)$ and (50a)

$$2t(n-1) \leqq T \leqq t^{n-r} ; \quad t \geqq (2(n-1))^{1:(n-r-1)} . \tag{50b}$$

From $t \mid T$ and (50a) we derive the condition of (50), namely $r = 0, 1, \cdots, n-2$. For $r = n-2$ we have $T = t^2$. w takes the form

$$w = W : t ; \quad W = (T^n - t^r T)^{1:n} . \qquad (r = 0, \cdots, n-2) . \tag{50c}$$

If we again substitute

$$D \text{ for } T ; \quad d \text{ for } t ; \quad w \text{ for } W \tag{50d}$$

and follow the proof of Corollary 1, the proof of Corollary 2 will be completed.

COROLLARY 3. *Let d, D, u, m be natural numbers and p a prime number such that*

$$d \mid D ; \quad u, m \geqq 1 ; \quad dp \leqq D : 2(p^u - 1) , \tag{51}$$

and let denote

$$w = (D^{pu} - pd : m)^{p-u} ,$$

$$f_s(w, D-1) = \sum_{i=0}^{s} \binom{n-1-s+i}{i} w^{s-i}(D-1)^i . \tag{52}$$

$$(s = 1, \cdots, n-1) .$$

Then the JAPAL of the $n-1$ numbers

$$f_1(w, D-1), f_2(w, D-1), \cdots, f_{(p^u-1)}(w, D-1)$$

is purely periodic and its primitive length is p^{2u}. The period consists of p^u fugues, each fugue being a matrix of p^u rows and $p^u - 1$ columns. The accumulators of the fugues have the form as those in the Main Theorem, namely (6) *to* (6d), *where d is substituted by pd and n by p^u.*

Proof of Corollary 3. All we have to prove is to show that all those integers which appear in the accumulators and are multiples of d are also multiples of p. This concerns all the numbers

$$(-1)^i\binom{s-1-t+i}{i}\binom{p^u}{s-t+i}D^{n-s+t-i}\,, \tag{52a}$$

$$(i=0,1,\cdots,t-1;\; t=1,2,\cdots,s-1;\; s=2,3,\cdots,p^u)$$

where the decisive point is the relation

$$1 \leqq s-t+i \leqq p^u-1\,. \tag{52b}$$

But since, as is well known,

$$p\,\Big|\binom{p^u}{k} \qquad \text{for } u=1,2,\cdots;\; k=1,2,\cdots,p^u-1 \tag{52c}$$

it follows from (52c) in view of (52b) that the numbers in (52a) are all multiples of p.

We leave it to the reader to prove the interesting fact, that each element of all the accumulators (6) to (6d) appearing in the Main Theorem are multiples of p, if $n=p^u$ (p prime, $u=1,2,\cdots$)

VI. Illustrations. (1) To illustrate the Main Theorem let us take $n=5$. Then the Main Theorem would sound:

Let d, D, m be natural numbers such that

$$d\mid D\,;\qquad m\geqq 1\,;\qquad 1\leqq d\leqq D:8\,.$$

Let

$$w=(D^5-d:m)^{1:5}\,,$$

$$f_s=\sum_{i=0}^{s}\binom{4-s+i}{i}w^{s-i}(D-1)^i\,. \qquad (s=1,2,3,4)\,.$$

Then the JAPAL of the 4 numbers

$$w+4(D-1)\,;\qquad w^2+3w(D-1)+6(D-1)^2\,;$$
$$w^3+2w^2(D-1)+3w(D-1)^2+4(D-1)^3\,;$$
$$w^4+w^3(D-1)+w^2(D-1)^2+w(D-1)^3+(D-1)^4$$

is purely periodic and its primitive length is 25. The period consists of five fugues, and the accumulators of these fugues have the form:

First fugue

$$5(D-1)\,;\qquad 5(D-1)(2D-1)\,;\qquad 5(D-1)(2D^2-2D+1)\,;$$
$$5D(D-1)(D^2-D+1)\,;$$

Second fugue

$$5(D-1)\,;\qquad 5(D-1)(2D-1)\,;\qquad 5(D-1)(2D^2-2D+1)\,;$$
$$5D\Big(\frac{mD^3}{d}-2D^2+2D-1\Big)\,;$$

Third fugue

$$5(D-1)\,;\qquad 5(D-1)(2D-1)\,;\qquad 5D\Big(\frac{2mD^3}{d}-4D^2+3D-1\Big)\,;$$
$$5D\Big(\frac{mD^2}{d}(D-2)+2D-1\Big)\,;$$

Fourth fugue

$$5(D-1)\,;\quad 5\Big(\frac{2mD^2}{d}-3D+1\Big)\,;\quad 5\Big(\frac{2mD^2}{d}(D-2)\Big)+15D-5\,,$$
$$5D\Big(\frac{mD}{d}(D^2-2D+2)-1\Big)\,;$$

Fifth fugue

$$5\Big(\frac{mD}{d}-1\Big)\,;\quad 5\Big(\frac{mD}{d}(2D-3)+1\Big)\,;\quad 5\frac{mD}{d}\Big((2D^2-4D+3)-1\Big)\,;$$
$$\frac{5mD}{d}(D^3-2D^2+2D-1)\,.$$

In the case of $n=5$, $m=d=1$, the JAPAL of the 4 numbers

$$f_s=\sum_{i=0}^{s}\binom{4-s+i}{i}w^{s-i}(D-1)^i\,;\qquad (s=1,2,3,4)\ \ w=(D^5-1)^{1:5}$$

is purely periodic and its primitive length is 5. It consists of one fugue, the accumulator of which has the form

$$5(D-1)\,;\qquad 5(D-1)(2D-1)\,;\qquad 5(D-1)(2D^2-2D+1)\,;$$
$$5D(D-1)(D^2-D+1)\,.$$

To illustrate Corollary 3 we shall take $p=2$; $u=2$. Then Corollary 3 would sound:

Let d, D be natural numbers such that $d\,|\,D$, $d\leqq D:12$ and let $w=(D^4-2d)$; then the JAPAL of the three numbers

$$w+3(D-1)\,;\qquad w^2+2w(D-1)+3(D-1)^2\,;$$
$$w^3+w^2(D-1)+w(D-1)^2+(D-1)^3$$

is purely periodic and its primitive length is 16. The period consists of four fugues, the accumulators of which have the form

First fugue

$$4(D-1)\ ; \qquad 2(D-1)(3D-1)\ ; \qquad 2(D-1)(2D^2-D+1)\ ;$$

Second fugue

$$4(D-1)\ ; \qquad 2(D-1)(3D-1)\ ; \qquad 2\left(\frac{mD^3}{d}-3D^2+2D-1\right)\ ;$$

Third fugue

$$4(D-1)\ ; \qquad \frac{3mD^2}{d}-8D+2\ ; \qquad \frac{mD^2}{d}(2D-3)+2(2D-1)\ ;$$

Fourth fugue

$$2\left(\frac{mD}{d}-2\right)\ ; \qquad \frac{mD}{d}(3D-4)+2\ ; \qquad \frac{mD}{d}(2D^2-3D+2)-2\ .$$

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PERMANENT OF THE DIRECT PRODUCT OF MATRICES

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Let A and B be nonnegative matrices of orders m and n respectively. In this paper we derive some properties of the permanent of the direct product $A \times B$ of A with B. Specifically we prove that

$$\operatorname{per}(A \times B) \geqq (\operatorname{per}(A))^n (\operatorname{per}(B))^m$$

with equality if and only if A or B has at most one nonzero term in its permanent expansion. We also show that every term in the permanent expansion of $A \times B$ is expressible as the product of n terms in the permanent expansion of A and m terms in the permanent expansion of B, and conversely. This implies that a minimal positive number $K_{m,n}$ exists such that

$$\operatorname{per}(A \times B) \leqq K_{m,n} (\operatorname{per}(A))^n (\operatorname{per}(B))^m$$

for all nonnegative matrices A and B of orders m and n respectively. A conjecture is given for the value of $K_{m,n}$.

Definitions. Let $A = [a_{ij}]$ be a matrix of order m with entries from a field F. The *permanent* of A is defined by

$$\operatorname{per}(A) = \Sigma a_{1i_1} a_{2i_2} \cdots a_{mi_m} ,$$

where the summation extends over all permutations $(i_1, i_2, \cdots, i_m)$ of the integers $1, 2, \cdots, m$. The set of elements

$$a_{1i_1}, a_{2i_2}, \cdots, a_{mi_m}$$

where $(i_1, i_2, \cdots, i_m)$ is a permutation of $1, 2, \cdots, m$ is called a *permutation array* of A, while their product

$$a_{1i_1} a_{2i_2} \cdots a_{mi_m}$$

is a *permutation product* of A. The permanent of A is then the sum of all the permutation products of A. The *term rank* $\rho(A)$ of the matrix A is defined to be the maximal order of a minor of A with a nonzero term in its determinant expansion. By a theorem of König [3] it is also equal to the minimal number of lines (rows and columns) which collectively contain all the nonzero entries of A. Obviously $\rho(A) = m$ if and only if A has a nonzero permutation product. A good discussion of these concepts is given by H.J. Ryser in [3].

If B is another matrix of order n with entries from the field F,

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then the *direct product* (or *Kronecker product*) of A with B is defined by

$$A \times B = \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1m}B \\ a_{21}B & a_{22}B & \cdots & a_{2m}B \\ \vdots & \vdots & & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mm}B \end{bmatrix}.$$

It is a matrix of order mn. The submatrix of $A \times B$ given by

$$[a_{ij}B] \qquad (1 \leqq i, j \leqq m)$$

is called the *(i, j)-block* of $A \times B$ or sometimes simply a *block* of $A \times B$. Direct products are discussed by C.C. MacDuffee in [2]. We mention those properties which will be of use to us. First it is readily verified that an associative law is satisfied, so that $A_1 \times A_2 \times \cdots \times A_k$ can be defined unambiguously. If C and D are matrices of orders m and n respectively, then

$$(A \times B)(C \times D) = AC \times BD . \tag{1.1}$$

Thus if $PAP_1 = A_1$ where P and P_1 are permutation matrices of order m and if $QBQ_1 = B_1$ where Q and Q_1 are permutation matrices of order n, then

$$(P \times Q)(A \times B)(P_1 \times Q_1) = A_1 \times B_1 . \tag{1.2}$$

This says that *permutations of the rows and columns of A and B induce permutations of the rows and columns of $A \times B$.*

It follows by inspection that a permutation matrix P of order mn exists such

$$P^T(A \times B)P = B \times A , \tag{1.3}$$

where P^T denotes the transpose of P. That is, *the rows and columns of $A \times B$ can be simultaneously permuted to give $B \times A$.* From this we immediately obtain

$$\operatorname{per}(A \times B) = \operatorname{per}(B \times A) . \tag{1.4}$$

A formula for the determinant of $A \times B$ is given by

$$\det(A \times B) = (\det(A))^n(\det(B))^m . \tag{1.5}$$

The definition of the determinant is very similar to that of the permanent, the only difference being that in the determinant we assign a certain sign to the permutation products. It is therefore natural to ask whether (1.5) has a counterpart for the permanent. It is this question that we consider for nonnegative matrices A and B. A *nonnegative matrix* is one whose entries are nonnegative real numbers. Such matrices are discussed by Gantmacher in [1].

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2. Preliminary theorems. A well-known theorem due to Frobenius and König asserts (in our terminology) that all permutation products of A are zero if and only if A contains an s by t submatrix of 0's with $s + t = m + 1$, m being the order of A. We divide the opposite situation where A has at least one nonzero permutation product into two cases.

THEOREM 2.1. *Let A be a matrix of order m with entries from a field F. Then A has precisely one nonzero permutation product if and only if by permutations of its rows and columns it may be brought to the triangular form:*

$$\begin{bmatrix} a_{11} & & & & \\ & a_{22} & & 0 & \\ & & \cdot & & \\ & & & \cdot & \\ & * & & \cdot & \\ & & & & a_{mm} \end{bmatrix} \tag{2.1}$$

*where $a_{11}a_{12} \cdots a_{mm} \neq 0$. In (2.1.), 0 denotes all zeros while * denotes arbitrary elements.*

Proof. We need only prove the necessity. Thus suppose A has precisely one nonzero permutation product. If $m = 2$, the result is readily verified. We may regard A as the incidence matrix of a collection of subsets $S_1, S_2, \cdots, S_m$ of the elements $x_1, x_2, \cdots, x_m$. Here x_j is a member of S_i if and only if $a_{ij} \neq 0$. The fact that A has precisely one nonzero permutation product means the subsets $S_1, S_2, \cdots, S_m$ have exactly one system of distinct representatives. If all of the subsets $S_1, S_2, \cdots, S_m$ contained two or more elements, then by [3, Thm. 1.2, p. 48] there would be at least two systems of distinct representatives. Hence one of the subsets $S_1, S_2, \cdots, S_m$ must contain only one element. Therefore by permuting the rows and columns of A we may assume A has the form

$$\left[\begin{array}{c|c} a_{11} & 0\ 0 \cdots 0 \\ \hline * & \\ * & \\ \vdots & A^1 \\ * & \end{array}\right]$$

where $a_{11} \neq 0$ and $*$ denotes arbitrary elements. Now A^1 can have only one nonzero permutation product otherwise A would have more than one. Applying induction to A^1, we obtain desired result.

COROLLARY 2.2. *Let A be a $(0, 1)$-matrix of order m. Then* $\operatorname{per}(A) = 1$ *if and only if the row and columns of A can be permuted to yield a triangular matrix with* 1's *on the main diagonal and* 0's *above the main diagonal.*

THEOREM 2.3. *Let A be a matrix of order m with entries from an arbitrary field F. Then A has more than one nonzero permutation product if and only if the rows and columns of A can be permuted to give a configuration of the form*:

$$\left[\begin{array}{cccc|cccc} a_{11} & a_{12} & & & & & & \\ & \ddots & \ddots & & & & & \\ & & \ddots & a_{r-1,r} & & & & \\ a_{r1} & & & a_{r,r} & & & & \\ \hline & & & & a_{r+1,r+1} & & & \\ & & & & & \ddots & & \\ & & & & & & \ddots & \\ & & & & & & & a_{mm} \end{array}\right] \tag{2.2}$$

where $2 \leqq r \leqq m$ and each a_{ij} designated above is not zero. All entries not designated are arbitrary.

Proof.[1] Suppose A has more than one nonzero permutation product. By permuting rows and columns we may assume to begin with that the elements on the main diagonal of A are nonzero. The conclusion now follows by using the well-known fact that if Q is a permutation matrix then there exists another permutation matrix P such that P^TQP is the direct sum of full cycle permutation matrices.

THEOREM 2.4. *Let A and B be nonnegative matrices of order n. Then*

$$\operatorname{per}(AB) \geqq \operatorname{per}(A)\operatorname{per}(B)\ . \tag{2.3}$$

[1] The author is indebted to the referee for improving the exposition here.

Strict inequality occurs in (2.3) *if and only if there exists an integer* i *with* $1 \leqq i \leqq n$ *having the property that if* A_i *denotes the matrix* A *with column* i *deleted and* B_i *denotes the matrix* B *with row* i *deleted, then*

$$\text{per}\,(A_iB_i) > 0\,. \tag{2.4}$$

Proof. Every permutation product of AB is the sum of n^n terms, each of which consists of the product of n elements of A and n elements of B. Consider a term

$$a_{1i_1}b_{i_1j_1}\cdots a_{ni_n}b_{i_nj_n} \tag{2.5}$$

of per (A) per (B). Here $(i_1, \cdots, i_n)$ and $(j_1, \cdots, j_n)$ are permutations of the integers $1, \cdots, n$. The expression (2.5) is a term in the permutation product of AB arising from the elements of AB in positions $(1, j_1), \cdots, (1, j_n)$. From this and the fact that A and B are nonnegative matrices, (2.3) follows.

Strict inequality occurs in (2.3) if and only if some permutation product of AB contains a nonzero term of the form

$$a_{1i_1}b_{i_1j_1}\cdots a_{ni_n}b_{i_nj_n} \tag{2.6}$$

where $(j_1, \cdots, j_n)$ is a permutation of the integers $1, \cdots, n$ and where $1 \leqq i_s \leqq n$ for $s = 1, \cdots, n$, but $(i_1, \cdots, i_n)$ is not a permutation of of $1, \cdots, n$. Thus there exists at least one integer k between 1 and n such that i_j is different from k for $j = 1, \cdots, n$. Let A_k be the matrix obtained by crossing out column k of A and B_k the matrix obtained by crossing out row k of B. Then a nonzero term of the form (2.6) occurs if and only if per $(A_kB_k) > 0$. This establishes the theorem.

3. Main theorems.

We now prove the main result of this paper.

THEOREM 3.1. *Let* A *and* B *be nonnegative matrices of orders* m *and* n *respectively. Then*

$$\text{per}\,(A \times B) \geqq (\text{per}\,(A))^n(\text{per}\,(B))^m\,. \tag{3.1}$$

Equality occurs in (3.1) *if and only if* A *or* B *has at most one nonzero permutation product.*

Proof. We have

$$A \times B = (A \times I_n)(I_m \times B)\,,$$

where I_m and I_n are identity matrices of orders m and n respectively. Hence by Theorem 2.4,

$$\text{per}(A \times B) \geqq \text{per}(A \times I_n)\,\text{per}(I_m \times B)\,. \tag{3.2}$$

But

$$\text{per}(I_m \times B) = (\text{per}(B))^m\,,$$

since $I_m \times B$ is the direct sum of B taken m times. Also

$$\text{per}(A \times I_n) = \text{per}(I_n \times A) = (\text{per}(A))^n$$

by (1.4). This establishes (3.1).

We now investigate the circumstance of equality in (3.1). We remark that equality occurs in (3.1) is and only if equality occurs in (3.2). Necessary and sufficient conditions that equality occur in (3.2) are given in Theorem 2.4. In proving that equality occurs under the conditions stated in the theorem we may assume by (1.4) that A has at most one nonzero permutation product. If all permutation products of A are zero, then per $(A) = 0$ and the term rank $\rho(A)$ of A satisfies $\rho(A) < m$. It then follows by an easy application of König's Theorem that

$$\rho(A \times B) \leqq \rho(A)n < mn\,.$$

Therefore per $(A \times B) = 0$, and equality occurs in (3.1). It A has precisely one nonzero permutation product, then according to Theorem 2.1 the rows and columns of A can be permuted to give the triangular matrix

$$\begin{bmatrix} a_{11} & & \\ \cdot & 0 & \\ & \cdot & \\ * & \cdot & \\ & & a_{mm} \end{bmatrix} \tag{3.3}$$

where per $(A) = a_{11} \cdots a_{mm} \neq 0$. Since permutations of the rows and columns of A induce in a natural way permutations of the rows and columns of $A \times B$, we may assume A has the form (3.3). From this it follows equality occurs in (3.1).

Conversely, suppose both A and B have at least two nonzero permutations products. Since permutations of the rows and columns of A and B give rise to permutations of the rows and columns of $A \times B$, we may assume by Theorem 2.3 that

$$A = \left[\begin{array}{ccc|ccc} a_{11} & a_{12} & & & & \\ & \ddots & \ddots & & & \\ & & a_{r-1,r+1} & & & \\ a_{r1} & & a_{rr} & & & \\ \hline & & & a_{r+1,r+1} & & \\ & & & & \ddots & \\ & & & & & a_{mm} \end{array}\right] \tag{3.4}$$

and

$$B = \left[\begin{array}{ccc|ccc} b_{11} & b_{12} & & & & \\ & \ddots & \ddots & & & \\ & & b_{s-1,s} & & & \\ b_{s1} & & b_{ss} & & & \\ \hline & & & b_{s+1,s+1} & & \\ & & & & \ddots & \\ & & & & & b_{nn} \end{array}\right] . \tag{3.5}$$

Here $2 \leqq r \leqq m$ and $2 \leqq s \leqq n$. Each a_{ij} and b_{kl} designated is not zero while all other entries not designated are arbitrary. Consider the matrices $A \times I_n$ and $I_m \times B$. Cross out column one of $A \times I_n$ to obtain the matrix $(A \times I_n)_1$ and cross out row one of $I_m \times B$ to obtain the matrix $(I_m \times B)_1$. The matrix $(A \times I_n)_1(I_m \times B)_1$ is of order mn and we consider it to be partitioned into m^2 blocks (submatrices) of size n by n in the natural way. Just as in the direct product we shall speak of the (i, j)-block of $(A \times I_n)_1(I_m \times B)_1$, $1 \leqq i, j \leqq m$. In each of the (k, k)-blocks, $k = 1, \cdots, r-1$ select the last $n-1$ main diagonal elements; in the (r, r)-block select the elements in positions $(1, 2), \cdots, (s-1, s)$, $(s+1, s+1), \cdots, (n, n)$; in each of the $(j, j+1)$-blocks, $j = 1, \cdots, r-1$ select the first main diagonal element; in the $(r, 1)$-block select the element in position $(s, 1)$; and finally in each the (i, i)-blocks, $i = r+1, \cdots, n$ select all of the main diagonal elements. It can be verified that each of the elements selected is different from zero and that the collection form a permutation array of $(A \times I_n)_1(I_m \times B)_1$. Hence their product is a nonzero permutation product of $(A \times I_n)_1(I_m \times B)_1$ and

$$\operatorname{per}\left((A \times I_n)_1(I_m \times B)_1\right) > 0 .$$

By Theorem 2.4 strict inequality occurs in (3.2) and thus in (3.1). This concludes the proof of the theorem.

COROLLARY 3.2. *Suppose both A and B have nonzero permanents.*

Then equality occurs in (3.1) *if and only if by permuting rows and columns A or B can be brought to triangular form with nonzero elements on the main diagonal and zeros above the main diagonal.*

Proof. This is a direct consequence of Theorems 2.1 and 3.1.

COROLLARY 3.3. *If A and B are matrices of* 0's *and* 1's, *then equality occurs in* (3.1) *if and only if* $\text{per}(A) = 0$ *or* 1 *or* $\text{per}(B) = 0$ *or* 1.

Theorem 3.1 may be generalized to include the direct product of any finite number matrices in the following way.

THEOREM 3.4. *Let* $A_1, A_2, \cdots, A_n$ *be nonnegative matrices of orders* $m_1, m_2, \cdots, m_n$ *respectively, and let*

$$e_i = \prod_{\substack{j=1 \\ j\neq i}}^{n} m_j \qquad (i = 1, 2, \cdots, n).$$

Then

$$\text{per}(A_1 \times A_2 \times \cdots \times A_n) \geqq (\text{per}(A_1))^{e_1}(\text{per}(A_2))^{e_2} \cdots (\text{per}(A_n))^{e_n}. \tag{3.6}$$

Equality occurs in (3.6) *if and only if* $\text{per}(A_i) = 0$ *for some* $i = 1, 2, \cdots, n$ *or* $(n-1)$ *of the matrices* $A_1, A_2, \cdots, A_n$ *have exactly one nonzero permutation product.*

Proof. Inequality (3.6) follows from Theorem 3.1 and an obvious induction on n. Suppose $\text{per}(A_i) = 0$ for some $i = 1, 2, \cdots, n$. Using the fact that the direct product operation is associative and (1.4.), we may assume that $\text{per}(A_n) = 0$. Then by Theorem 3.1 we obtain $\text{per}(A_1 \times A_2 \times \cdots A_n) = 0$ and equality occurs in (3.6). Suppose $n-1$ of the matrices $A_1, A_2, \cdots, A_n$ have precisely one nonzero permutation product. By associativity and (1.4) we may assume $A_1, A_2, \cdots, A_{n-1}$ do. Then $A_1 \times A_2 \times \cdots \times A_{n-1}$ also has exactly one nonzero permutation product. Applying Theorem 3.1 we obtain equality in (3.6).

Conversely suppose $A_1, A_2, \cdots, A_n$ all have nonzero permanents and at least two have more than one nonzero permutation product. By associativity, (1.3), and (1.4) we may assume A_1 and A_2 do. Then by theorem 3.1 we have

$$\begin{aligned} &\text{per}(A_1 \times A_2 \times \cdots \times A_n) \\ &\qquad \geqq (\text{per}(A_1 \times A_2))^{m_3\cdots m_n}(\text{per}(A_3 \times \cdots \times A_n))^{m_1 m_2} \\ &\qquad > (\text{per}(A_1))^{e_1}\,\text{per}(A_2)^{e_2}\,(\text{per}(A_3 \times \cdots \times A_n))^{m_1 m_2} \\ &\qquad \geqq (\text{per}(A_1))^{e_1}\,(\text{per}(A_2))^{e_2} \cdots (\text{per}(A_n))^{e_n}. \end{aligned}$$

This establishes the theorem.

Inequality (3.1) and the more general (3.6) containing many inequalities obtained by specializing the matrices concerned. For instance if in (3.1) we let $A = J_m$, the matrix of 1's of order m, and $B = J_n$, the matrix of 1's of order n, we obtain

$$(mn)! \geqq (m!)^n(n!)^m \,. \tag{3.7}$$

Equality occurs in (3.7) if and only if $m = 1$ or $n = 1$.

The following theorem is basic.

THEOREM 3.5. *Let A and B be matrices of orders m and n respectively with entries from a field F. Then every permutation product of $A \times B$ is expressible (in general, not uniquely) as the product of n permutation products of A and m permutation products of B. Conversely, the product of n permutation products of A and m permutation products of B is a permutation product of $A \times B$.*

Proof. Consider an arbitrary permutation product of $A \times B$ and a permutation array which gives rise to this product. Suppose this permutation array contains c_{ij} entries from the (i, j)-block of $A \times B$, $1 \leqq i, j \leqq m$. Form the matrix

$$C = [c_{ij}] \qquad (i, j = 1, 2, \cdots, m) \,.$$

C is a matrix of order m whose entries are nonnegative integers. Since A and B are square matrices, we have

$$\sum_{j=1}^{m} c_{ij} = n \qquad (i = 1, 2, \cdots, m) \,,$$

and

$$\sum_{i=1}^{m} c_{ij} = n \qquad (j = 1, 2, \cdots, m) \,.$$

Then by [3, Th. 5.2, p. 56] we have

$$C = c_1P_1 + c_2P_2 + \cdots + c_tP_t$$

where $c_1, c_2, \cdots, c_t$ are positive integers with $c_1 + c_2 + \cdots c_t = n$ and where $P_1, P_2, \cdots, P_t$ are distinct permutation matrices of order m. Each permutation matrix P_k corresponds in a natural way to a permutation array of A. Let this array be denoted by

$$a_{1\sigma_k(1)},\, a_{2\sigma_k(2)},\, \cdots,\, a_{m\sigma_k(m)}$$

where $\sigma_k(1), \sigma_k(2), \cdots, \sigma_k(m)$ is a permutation of $1, 2, \cdots, m$. Then the mn a's that appear in the given permutation product of $A \times B$ can be arranged as:

$$(a_{1\sigma_1(1)}a_{2\sigma_1(2)} \cdots a_{m\sigma_1(m)})^{c_1} \cdots (a_{1\sigma_t(1)}a_{2\sigma_t(2)} \cdots a_{m\sigma_t(m)})^{c_t} .$$

Since there exists a permutation matrix P such that $P^T(A \times B)P = B \times A$, it follows that the specified permutation array of $A \times B$ is also a permutation array of $B \times A$. Therefore in a similar manner the b's in the given permutation product of $A \times B$ can be expressed as the product of m permutation products of B.

Conversely, it is easy to verify that the product of n permutation products of A is a permutation product of $A \times I_n$ and the product of m permutation products of B is a permutation product of $I_m \times B$. The matrix product

$$(A \times I_n)(I_m \times B) = A \times B$$

yields the desired permutation product of $A \times B$.

COROLLARY 3.6. *There exists a minimal positive real number $K_{m,n}$, depending only on m and n, such that*

$$\text{per}\,(A \times B) \leqq K_{m,n}\,(\text{per}\,(A))^n\,(\text{per}\,(B))^m$$

for any two nonnegative matrices A and B of orders m and n respectively.

Proof. By the theorem all of the distinct terms in per $(A \times B)$ appear at least once in the product $(\text{per}\,(A))^n\,(\text{per}\,(B))^m$. Since there are in total $(mn)!$ terms in per $(A \times B)$, we have

$$\text{per}\,(A \times B) \leqq (mn)!\,(\text{per}\,(A))^n\,(\text{per}\,(B))^m$$

for all nonnegative matrices A and B of orders m and n respectively. This shows the existence of the constant $K_{m,n}$.

The constant $K_{m,n}$ is given by the equation

$$K_{m,n} = \text{l.u.b.}\,\frac{\text{per}\,(A \times B)}{(\text{per}\,(A))^n(\text{per}\,(B))^m} \tag{3.8}$$

where the least upper bound is taken over all nonnegative matrices A and B of orders m and n respectively with nonzero permanents. The ratio

$$\frac{\text{per}\,(A \times B)}{(\text{per}\,(A))^n\,(\text{per}(B))^m}$$

is homogeneous in the sense that if a row or column of A or B is multiplied by a positive real number then the ratio is unchanged. This allows one to assume A and B are *row stochastic* (the sum of the elements of each row is one) in determining $K_{m,n}$. Also by continuity considerations only positive matrices A and B need be considered. Hence we may replace the l.u.b. in (3.8) by the l.u.b. over all positive row stochastic matrices A and B of orders m and n respectively. If in (3.8) we let $A = J_m$, the matrix of 1's of order m, and $B = J_n$, the matrix of 1's or order n, we obtain

$$K_{m,n} \geqq \frac{(mn)!}{(m!)^n(n!)^m} . \tag{3.9}$$

We *conjecture* here that (3.9) is actually an equality and therefore that

$$\text{per}\,(A \times B) \leqq \frac{(mn)!}{(m!)^n(n!)^m}(\text{per}\,(A))^n\,(\text{per}\,(B))^m$$

for all nonnegative matrices A and B of orders m and n respectively. There is limited evidence to suggest that this is true. For instance it can be verified for $m = n = 2$, i.e. $K_{2,2} = 3/2$.

To conclude this section we give the following interpretation of Theorem 3.5. Let $S_1, S_2, \cdots, S_m$ be m subsets of the elements $a_1, a_2, \cdots, a_m$ and $T_1, T_2, \cdots, T_n$ be n subsets of the elements $b_1, b_2, \cdots, b_n$. Form the incidence matrices $A = [a_{ij}]$ and $B = [b_{kl}]$ of orders m and n respectively. Here $a_{ij} = 1$ if a_j is a member of S_i and 0 otherwise. Similarly for B. For $i = 1, 2, \cdots, m$ and $j = 1, 2, \cdots, n$ define $S_i \times T_j$ to be a subset of all the ordered pairs

$$(a_r, b_s) \qquad (r = 1, 2, \cdots, m;\ s = 1, 2, \cdots, n).$$

We have (a_r, b_s) is a member of $S_i \times T_j$ if and only if a_r is a member of S_i and b_s is a member of T_j. The incidence matrix of this collection of subsets is $A \times B$. Theorem 3.5 applied to this situation says that if we have a system of distinct representatives (SDR) of the collection of subsets

$$S_i \times T_j \qquad (i = 1, 2, \cdots, m; j = 1, 2, \cdots n), \tag{3.10}$$

then the first components of the members of this SDR can be arranged into n SDR's of the collection $S_1, S_2, \cdots, S_m$ and the second components can be arranged into m SDR's of the collection $T_1, T_2, \cdots, T_n$. Conversely, n SDR's of $S_1, S_2, \cdots, S_m$ and m SDR's of $T_1, T_2, \cdots, T_n$ can be paired up in at least one way to form an SDR of the collection of subsets in (3.10).

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PSEUDOCOMPACTNESS AND UNIFORM CONTINUITY IN TOPOLOGICAL GROUPS

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This work contains a number of theorems about pseudocompact groups. Our first and most useful theorem allows us to decide whether or not a given (totally bounded) group is pseudocompact on the basis of how the group sits in its Weil completion. A corollary, which permits us to answer a question posed by Irving Glicksberg (Trans. Amer. Math. Soc. 90 (1959), 369–382) is: The product of any set of pseudocompact groups is pseudocompact. Following James Kister (Proc. Amer. Math. Soc. 13 (1962), 37–40) we say that a topological group G has property U provided that each continuous function mapping G into the real line is uniformly continuous. We prove that each pseudocompact group has property U.

Sections 2 and 3 are devoted to solving the following two problems: (a) In order that a group have property U, is it sufficient that each bounded continuous real-valued function on it be uniformly continuous? (b) Must a nondiscrete group with property U be pseudocompact? Theorem 2.8 answers (a) affirmatively. Question (b), the genesis of this paper, was posed by Kister (loc. cit.). For a large class of groups the question has an affirmative answer (see 3.1); but in 3.2 we offer an example (a Lindelöf space) showing that in general the answer is negative.

Much of the content of this paper is summarized by Theorem 4.1, in which we list a number of properties equivalent to pseudocompactness for topological groups. We conclude with an example of a metrizable, non totally bounded Abelian group on which each uniformly continuous real-valued function is bounded.

Conventions and definitions. All topological groups considered here are assumed to be Hausdorff. The algebraic structure of the groups we consider is virtually immaterial; in particular, our groups are permitted to be non-Abelian.

A topological group G is said to be totally bounded if, for each neighborhood U of the identity, a finite number of translates of U covers G. It has been shown in [10] by Weil that each totally bounded group is a dense topological subgroup of a compact group and that this compactification is unique to within a topological isomorphism leaving G fixed pointwise. We refer to this compactification of G as the Weil

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completion of G, and we reserve the symbol $\bar{G}$ to denote it.

Kister's property U was defined in the summary above. In the same vein, we say that a topological group has property BU if each bounded continuous real-valued function on G is uniformly continuous. The uniform structure on G referred to implicitly in the definitions of properties U and BU should be taken to be either the left uniform structure, defined as in 4.11 of [7], or the right uniform structure. It often happens that these structures do not coincide, and in this case there is a left uniformly continuous real-valued function on G which is not right uniformly continuous. Nevertheless it is easy to see that every [bounded] continuous real-valued function on G is left uniformly continuous if and only if every [bounded] continuous real-valued function on G is right uniformly continuous. Hence the definitions of properties U and BU are unambiguous.

Our topological vocabulary is that of the Gillman-Jerison text [5]. The following definition, which is useful in § 2, is in consonance with 4J of [5]: A topological space is a P-space provided that each of its G_δ subsets is open.

1. Pseudocompact groups. The Weil completion of a topological group plays a fundamental rôle in many of the arguments which follow. Our first result shows that each pseudocompact group admits such a completion.

THEOREM 1.1. *Each pseudocompact group is totally bounded.*

Proof. If the topological group G is not totally bounded, then there is a neighborhood U of the identity e in G and a sequence $\{x_k\}$ of points in G for which

$$x_k \notin \bigcup_{n<k} x_n U$$

for all k. We choose a symmetric neighborhood V of e for which $V^4 \subset U$, and we select for each positive integer k a nonnegative continuous function f_k on G such that

$$f_k(x_k) = k \text{ and } f_k \equiv 0 \text{ off } x_k V .$$

Using the local finiteness of the sequence $\{x_k V\}$, it is easy to check that the real-valued function f defined on G by the relation

$$f(x) = \sum_{k=1}^{\infty} f_k(x)$$

is continuous. Since f is unbounded, the group G is not pseudocompact.

Discussion. Although pseudocompact groups have not (so far as we can determine) been in their own right an object of detailed study, various authors have considered an example in one connection or another. If $\{X_\alpha\}_{\alpha \in A}$ is a set of separable metric spaces — which we here take to be topological groups — then the set

$$Y = \{x \in \Pi X_\alpha : x_\alpha \text{ is the identity in } X_\alpha \text{ for all but countably many } \alpha \text{ in } A\}$$

is an example of what Corson in [3] calls a Σ-space. Corson shows in his Theorem 2 that each continuous real-valued function on Y admits a continuous extension to ΠX_α. It follows that if each X_α is compact, then Y is pseudocompact and ΠX_α is the Stone-Čech compactification of Y. This and other interesting results were obtained (also in the product-space context) by Glicksberg in [6]. Kister examined in [8] the case in which each X_α is a compact topological group.

Like every pseudocompact space, the Σ-space Y defined above meets each nonempty G_δ subset of its Stone-Čech compactification. The appropriate group-theoretic analogue of this topological characterization of pseudocompactness is given in the following theorem. The reader will notice instantly that this theorem yields information about the Stone-Čech compactification of a pseudocompact group; we shall incorporate this observation into Theorem 4.1.

The Baire sets in a topological space X are those subsets of X belonging to the smallest σ-algebra containing all zero-sets in X.

THEOREM 1.2. *Let G be a totally bounded group and let*

$$\mathscr{N} = \{N : N \text{ is a closed, normal subgroup of } \bar{G} \text{ and } N \text{ is a } G_\delta \text{ set in } \bar{G}\} .$$

Then the following assertions are equivalent:

(a) *G is pseudocompact;*

(b) *each translate of each element of $\mathscr{N}$ meets G;*

(c) *each nonempty Baire subset of $\bar{G}$ meets G;*

(d) *each nonempty G_δ subset of $\bar{G}$ meets G;*

(e) *each continuous real-valued function on G admits a continuous extension to $\bar{G}$.*

Proof. (a) $\Rightarrow$ (b). If (b) fails, then $x_0 N \cap G = \varnothing$ for some x_0 in $\bar{G}$ and some N in $\mathscr{N}$. Since N is clearly not open, the quotient group $\bar{G}/N$ is infinite. Like any compact, first countable group, $\bar{G}/N$ is metrizable. Choosing an unbounded real-valued continuous function f on $\bar{G}/N\backslash\{x_0N\}$ and defining g on $\bar{G}\backslash x_0N$ by the relation

$$g(x) = f(xN) ,$$

we see that g is unbounded and continuous. The restriction of g to G is unbounded, and hence (a) fails.

(b) $\Rightarrow$ (c). This implication follows trivially from the following fact, a special case of Lemma 2.4 of [9]: If E is a Baire subset of $\bar{G}$, then $E = EN$ for some N in $\mathscr{N}$.

(c) $\Rightarrow$ (d). Since $\bar{G}$ is completely regular, each nonempty G_δ subset of $\bar{G}$ contains a nonempty zero-set of $\bar{G}$.

(d) $\Rightarrow$ (b). This is clear.

(b) $\Rightarrow$ (e). Let f be a real-valued continuous function on G, and let $\mathscr{B}$ be a countable base for the topology on the line. For each B in $\mathscr{B}$ there is clearly an open subset U_B of $\bar{G}$ for which

$$f^{-1}(B) = U_B \cap G\ .$$

By 1.6 and 2.4 of [9], there is an element N_B of $\mathscr{N}$ for which

$$cl_{\bar{G}} U_B = N_B \cdot cl_{\bar{G}} U_B\ .$$

Setting $N = \bigcap_{B \in \mathscr{B}} N_B$, we clearly have $N \in \mathscr{N}$ and $cl_{\bar{G}} U_B = N \cdot cl_{\bar{G}} U_B$ for each B in $\mathscr{B}$.

We next prove:

(*) If $x_1 \in G$, $x_2 \in G$, and $x_1^{-1}x_2 \in N$, then $f(x_1) = f(x_2)$.

If (*) fails, we can find neighborhoods B_1 and B_2 of $f(x_1)$ and $f(x_2)$ respectively such that $B_1 \in \mathscr{B}$, $B_2 \in \mathscr{B}$, and $clB_1 \cap clB_2 = \varnothing$. Since f is continuous on G, we have

$$cl_G f^{-1}(B_1) \cap cl_G f^{-1}(B_2) = \varnothing\ ,$$

i.e.,

$$cl_G(U_{B_1} \cap G) \cap cl_G(U_{B_2} \cap G) = \varnothing\ .$$

Now $x_1 \in N \cdot cl_G(U_{B_1} \cap G)$; hence

$$x_2 \in N \cdot cl_G(U_{B_1} \cap G) \subset N \cdot cl_{\bar{G}}(U_{B_1}) = cl_{\bar{G}}(U_{B_1})\ ,$$

so that $x_2 \in cl_G(U_{B_1} \cap G)$. Of course $x_2 \in cl_G(U_{B_2} \cap G)$, and this contradiction completes the proof of (*).

With (*) and hypothesis (b) at our disposal, it is easy to define an extension $\bar{f}$ of f: given x_0 in $\bar{G}$, we choose any x in $x_0N \cap G$ and set $\bar{f}(x_0) = f(x)$.

To check the continuity of $\bar{f}$ at an arbitrary point x_0 in $\bar{G}$, we choose $\varepsilon > 0$. We will produce a neighborhood U of the identity in $\bar{G}$ with the property that $|\bar{f}(x_0) - \bar{f}(y_0)| < \varepsilon$ whenever $y_0 \in x_0U$. Indeed, choose $x \in x_0N \cap G$ and let V be a neighborhood of the identity in $\bar{G}$ such that

$$|f(x) - f(y)| < \varepsilon \text{ whenever } y \in xV \cap G\ .$$

Now let U be any neighborhood of the identity in $\bar{G}$ for which $U^2 \subset V$. It is easy to see (directly, or by 8.7 of [7]) that there is an M in $\mathscr{N}$ such that $M \subset U \cap N$. Now for any point y_0 in $x_0 U$ there is (again by hypothesis (b)) a point z in $(xx_0^{-1}y_0M) \cap G$. Since $z \in xx_0^{-1}y_0N \subset Ny_0N = y_0N$, we have $\bar{f}(y_0) = f(z)$. And since $z \in xx_0^{-1}y_0M \subset xUM \subset xU^2 \subset xV$, it follows that $|\bar{f}(x_0) - \bar{f}(y_0)| = |f(x) - f(z)| < \varepsilon$. Hence U is as desired and $\bar{f}$ is continuous at x_0.

(e) $\Rightarrow$ (a). Since every continuous real-valued function with domain $\bar{G}$ is bounded, this implication is obvious.

1.3. *Discussion.* If $(G_0, \mathscr{T})$ is a compact group and $\mathscr{N}$ denotes the family of subgroups of G_0 defined as in the hypothesis of Theorem 1.2, then the collection of translates of elements of $\mathscr{N}$ clearly constitutes a base for a P-space topology $\mathscr{P}$ on G_0. Since any G_δ set in G_0 that contains the identity must contain a member of $\mathscr{N}$, $\mathscr{P}$ is the smallest P-space topology containing $\mathscr{T}$. In fact,

$$\mathscr{P} = \{U : U \text{ is a countable intersection of } \mathscr{T}\text{-open subsets of } G_0\} .$$

Using these observations and 1.2, we have the following fact: A (dense) subgroup G of G_0 is pseudocompact if and only if G is $\mathscr{P}$-dense in G_0.

Gillman and Jerison present in 9.15 of [5] an example (due to Novák-Terasaka) of a pseudocompact space X for which $X \times X$ is not pseudocompact. In the positive direction, a number of authors (see especially [6] and [4]) have given various conditions on a family of pseudocompact spaces sufficient to ensure that the product be pseudocompact.

THEOREM 1.4. *The product of any set of pseudocompact groups is pseudocompact.*

Proof. Let the set A index the family $\{G_\alpha\}_{\alpha \in A}$ of pseudocompact groups, and let

$$G = \prod_{\alpha \in A} G_\alpha .$$

The uniqueness aspect of Weil's theorem assures us that the compact group $\prod_{\alpha \in A} \overline{G_\alpha}$ is (homeomorphic with) $\bar{G}$. According to 1.2, then, we need only show that each nonempty G_δ subset of $\prod_{\alpha \in A} \overline{G_\alpha}$ hits G.

Let U be such a set, say $U = \bigcap_{n=1}^{\infty} U_n$ where each U_n is a basic set of the form

$$U_n = \prod_{\alpha \in A} U_{n,\alpha} ;$$

here each $U_{n,\alpha}$ is open in $\overline{G_\alpha}$, and for each n we have $U_{n,\alpha} = \overline{G_\alpha}$ for all but finitely many α in A. Let

$$V_\alpha = \bigcap_{n=1}^{\infty} U_{n,\alpha} .$$

Then V_α is a nonempty G_δ set in $\overline{G_\alpha}$, and thus by 1.2 there is a point x_α in $V_\alpha \cap G_\alpha$. Evidently the point of $\bar{G}$ whose α coordinate is x_α lies in $U \cap G$.

In what follows we will consider at length Kister's question "Must a nondiscrete group with property U be pseudocompact?" We now quickly handle the converse question.

THEOREM 1.5. *Every pseudocompact group has property U.*

Proof. If f is a continuous real-valued function on the pseudocompact group G, then by 1.1 and (a) $\Rightarrow$ (e) of 1.2, f admits a continuous extension $\bar{f}$ on $\bar{G}$. Since $\bar{f}$ is uniformly continuous on $\bar{G}$, it follows that f is uniformly continuous on G.

2. **Property BU implies property U.** This theorem is proved in 2.8. Our key lemma is 2.2.

LEMMA 2.1. *If the topological group G is not a P-space, then some nonempty G_δ subset H of G has no interior. The set H may be chosen to be a closed subgroup.*

Proof. There is a sequence $\{V_k\}$ of neighborhoods of e for which $e \notin \operatorname{int} \bigcap_{k=1}^{\infty} V_k$. Selecting a sequence $\{U_k\}$ of symmetric neighborhoods of e such that $U_{k+1}^2 \subset U_k \cap V_k$ and defining $H = \bigcap_{k=1}^{\infty} U_k$, we see (directly, or from 5.6 of [7]) that the G_δ set H is a closed subgroup of G. Being a subgroup that is not open, H has no interior.

THEOREM 2.2. *If the topological group G has property BU, then G is totally bounded or G is a P-space.*

Proof. Suppose the conclusion fails. Since G is not a P-space, there is a sequence $\{U_k\}$ of neighborhoods of e for which $\operatorname{int} \bigcap_{k=1}^{\infty} U_k = \varnothing$. Since G is not totally bounded, there is, just as in the proof of 1.1, a neighborhood V of e and a sequence $\{x_k\}$ of points in G such that the sequence $\{x_k V\}$ is locally finite and pairwise disjoint.

For each integer k there is a continuous function f_k on G for which $f_k(x_k) = 1, f_k \equiv 0$ off $x_k(V \cap U_k)$, and $0 \leqq f_k \leqq 1$. The function $f = \sum_{k=1}^{\infty} f_k$ is bounded and continuous on G, and hence is (left) uniformly continuous. Thus there is a neighborhood W of e for which $|f(x) - f(y)| < 1$ whenever $x^{-1}y \in W$. We may take $W \subset V$. Since int $W \neq \emptyset$, we cannot have $W \subset \bigcap_{k=1}^{\infty} U_k$. Thus there is an integer m and a point p for which $p \in W \backslash U_m$. Now $x_m^{-1}(x_m p) \in W$, so that $|1 - f(x_m p)| = |f(x_m) - f(x_m p)| < 1$ and we have $f(x_m p) \neq 0$. Thus $x_m p \in \bigcup_k x_k(V \cap U_k)$. Since $x_m V \cap x_k V = \emptyset$ whenever $k \neq m$, we must have $x_m p \in x_m(V \cap U_m)$. But then $p \in U_m$, a contradiction completing the proof.

Our next result, used in the proof of 2.4, is given here in considerable generality because of its application in connection with Example 3.2.

THEOREM 2.3. *Let the topological group G be a P-space. Then the following are equivalent:*

(a) *G has property U;*

(b) *G has property BU;*

(c) *the characteristic function of every open-and-closed subset of G is uniformly continuous.*

Proof. Only the implication (c) $\Rightarrow$ (a) requires proof. Given a continuous real-valued function f on G, we note that for each rational pair (a, b), with $a \leqq b$, the set $f^{-1}([a, b])$ is closed; being a G_δ set in G, this set is also open. Since the characteristic function $\psi_{f^{-1}([a\ b])}$ is left uniformly continuous, there is a neighborhood $U_{a,b}$ of e such that $x^{-1}y \in U_{a,b}$ implies $|\psi_{f^{-1}([a\ b])}(x) - \psi_{f^{-1}([a\ b])}(y)| < 1$. That is, $x^{-1}y \in U_{a,b}$ implies that $x \in f^{-1}([a, b])$ if and only if $y \in f^{-1}([a, b])$. Let $U = \bigcap \{U_{a,b} : a, b \text{ rational and } a \leqq b\}$; then U is a neighborhood of e since G is a P-space. To establish the left uniform continuity of f it will clearly suffice to show that $f(x) = f(y)$ whenever $x^{-1}y \in U$. Suppose then that $x^{-1}y \in U$ and that $f(x) = p$. For appropriate sequences $\{a_k\}$ and $\{b_k\}$ of rational numbers, we have $\{p\} = \bigcap_k [a_k, b_k]$. Then $x \in f^{-1}([a_k, b_k])$ for all k. Since $x^{-1}y \in U_{a_k, b_k}$ for all k, we have $y \in \bigcap_k f^{-1}([a_k, b_k]) = f^{-1}(\{p\})$ and $f(y) = p = f(x)$.

COROLLARY 2.4. *If the topological group G has property BU and is not totally bounded, then G has property U.*

Proof. By Theorem 2.2, G is a P-space. The result now follows from 2.3.

Corollary 2.4 gives an affirmative answer to problem (a) of the introduction for groups which are not totally bounded. The trick which handles the totally bounded situation consists, roughly speaking, in reducing to the metrizable case (where the proof is easy).

LEMMA 2.5. *If a topological group G is metrizable and has property BU, then G is compact or discrete.*

Proof. This is immediate from Atsuji's Theorem 1 in [1]. For a direct proof (by contradiction), assume otherwise and note that by Theorem 2.2, G must be totally bounded. Since G is not compact, G is not complete. Hence there is a nonconvergent Cauchy sequence $\{x_k\}$ in G. By Tietze's theorem the function mapping x_k to $(-1)^k$ can be extended to a real-valued continuous function bounded on G, and this bounded function is obviously not uniformly continuous.

LEMMA 2.6. *If G is a topological group with property BU, and if H is a closed normal subgroup of G, then G/H has property BU.*

Proof. Let f be a bounded continuous real-valued function on G/H, and let $\varepsilon > 0$. Denoting by π the natural projection of G onto G/H, we note that $f \circ \pi$ is left uniformly continuous on G. Hence there is a neighborhood V of e for which

$$|f \circ \pi(x) - f \circ \pi(y)| < \varepsilon \text{ whenever } x^{-1}y \in V .$$

Of course $\pi(V)$ is a neighborhood of H in G/H. Now suppose that $(xH)^{-1}(yH) \in \pi(V)$. Then $x^{-1}yH = vH$ for some $v \in V$, so that $x^{-1}yh = v$ for some $h \in H$. Then $x^{-1}(yh) \in V$ and therefore

$$\begin{aligned}|f(xH) - f(yH)| &= |f \circ \pi(x) - f \circ \pi(y)| \\ &= |f \circ \pi(x) - f \circ \pi(yh)| < \varepsilon .\end{aligned}$$

That is, f is left uniformly continuous.

THEOREM 2.7. *Let G be a totally bounded group with property BU. Then G is pseudocompact.*

Proof. If G is not pseudocompact, then according to 1.2 there is a point p in $\bar{G}$ and a closed normal subgroup N of $\bar{G}$ such that $G \cap pN = \varnothing$ and $\bar{G}/N$ is metrizable. Since $pN \notin GN/N$ and GN/N is the continuous image of GN under the natural projection, GN/N is a dense proper subgroup of $\bar{G}/N$. Since a discrete subgroup of a topological group is closed (see 5.10 of [7]), it follows that GN/N is a nondiscrete, noncompact metrizable group.

It is clear that any group, one of whose dense subgroups has property BU, must itself have property BU. In particular the group GN, in which G is dense, has property BU. Hence GN/N has property BU by 2.6, and GN/N does not have property BU by 2.5. This contradiction completes the proof.

THEOREM 2.8. *A topological group has property BU if and only if it has property U.*

Proof. Use 2.4 and 2.7.

3. Kister's question. We first give a partial affirmative answer to the question posed by Kister in [8].

THEOREM 3.1. *If the topological group G has property BU and is not a P-space, then G is pseudocompact.*

Proof. The group G is totally bounded by 2.2, and hence is pseudocompact by 2.7.

EXAMPLE 3.2. We now give an example of a nondiscrete topological Abelian group that is a P-space and has property U. Such a group is clearly not pseudocompact: every pseudocompact P-space is finite. Hence this example shows that Kister's question mentioned in the summary has a negative answer.

Let A be an index set of cardinality $\aleph_1$ and let G consist of all elements x in $\prod_{\alpha \in A} \{1, -1\}_\alpha$ such that $x_\alpha = 1$ for all but finitely many coordinates α. Let Ω be the first uncountable ordinal and well-order A according to the order—type Ω: $A = \{\alpha : \alpha < \Omega\}$. For $\alpha \in A$, let

$$H_\alpha = \{x \in G : x_\beta = 1 \text{ for all } \beta < \alpha\} .$$

We decree that the subgroups H_α and each of their translates be open and thereby obtain a basis for a topology under which G is a topological group. Clearly G is a P-space and G is not discrete.

We shall show that G has property U. By Theorem 2.3 we need show only that the characteristic function ψ_W of an open-and-closed set W is uniformly continuous. For $\alpha \in A$, let $W_\alpha = \cup \{xH_\alpha : xH_\alpha \subset W\}$. Evidently $\{W_\alpha\}_{\alpha<\Omega}$ is a nondecreasing family of open-and-closed sets, and $\cup_{\alpha<\Omega} W_\alpha = W$. Since $\psi_{W_\alpha}(x) = \psi_{W_\alpha}(y)$ whenever $x^{-1}y \in H_\alpha$, the characteristic function of each W_α is uniformly continuous. Hence it suffices to show that $W = W_\alpha$ for some α.

Assume that $W \neq W_\alpha$ for all α, and let

$$V_\alpha = \cup \{xH_\alpha : xH_\alpha \cap W \neq \emptyset \text{ and } xH_\alpha \cap (G\backslash W) \neq \emptyset\} .$$

It is easy to see that each V_α is nonvoid and that $V_\alpha \supset V_\gamma$ whenever $\alpha < \gamma < \Omega$. It suffices now to prove that $\cap_{\alpha<\Omega} V_\alpha$ is nonvoid, since any element in this intersection belongs to the closures of both W and $G\backslash W$, contrary to the supposition that W is open-and-closed.

We prove that $\cap_{\alpha<\Omega} V_\alpha$ is nonvoid. For x in G and α in A, we define $N(x, \alpha)$ to be the number of elements in the finite set $\{\beta \in A : \beta < \alpha$ and $x_\beta = -1\}$. For $\alpha \in A$, we define

$$n_\alpha = \inf_{x \in V_\alpha} N(x, \alpha) \text{ and } J_\alpha = \{x \in V_\alpha : N(x, \alpha) = n_\alpha\} .$$

Clearly $\emptyset \neq J_\alpha \subset V_\alpha$ for all α. The integer-valued transfinite sequence $\{n_\alpha\}_{\alpha<\Omega}$ is nondecreasing because for $\alpha < \gamma$ we have

$$n_\alpha = \inf_{x \in V_\alpha} N(x, \alpha) \leqq \inf_{x \in V_\alpha} N(x, \gamma) \leqq \inf_{x \in V_\gamma} N(x, \gamma) = n_\gamma .$$

It follows that the sequence $\{n_\alpha\}_{\alpha<\Omega}$ is eventually constant. There is, then, an integer n_0 and an α_0 in A for which $n_\alpha = n_0$ whenever $\alpha \geqq \alpha_0$. We next show that $\{J_\alpha\}_{\alpha \geqq \alpha_0}$ is a nonincreasing family of sets. Suppose that $\alpha_0 \leqq \alpha < \gamma$ and $z \in J_\gamma$. Then $z \in J_\gamma \subset V_\gamma \subset V_\alpha$. Since also

$$n_\alpha \leqq N(z, \alpha) \leqq N(z, \gamma) = n_\gamma = n_0 = n_\alpha ,$$

we see that $z \in J_\alpha$. Now let Y consist of all elements y of G such that $y_\beta = 1$ for all $\beta \geqq \alpha_0$. Then Y is a countable set. Assume now that $\bigcap_{\alpha<\Omega} V_\alpha = \emptyset$, so that $\bigcap_{\alpha \geq \alpha_0} J_\alpha = \emptyset$. Then for each y in Y there is an $\alpha_y \geqq \alpha_0$ such that $y \notin J_{\alpha_y}$. Selecting γ_0 in A larger than each α_y, we find that $Y \cap J_{\gamma_0} = \emptyset$. Now choose z in J_{γ_0}. Then z also belongs to J_{α_0}, so that $N(z, \gamma_0) = N(z, \alpha_0) = n_0$. Hence $z_\beta = 1$ for $\alpha_0 \leqq \beta < \gamma_0$. Define w so that $w_\beta = z_\beta$ for $\beta < \gamma_0$ and $w_\beta = 1$ for $\beta \geqq \gamma_0$. Clearly w belongs to Y. Since $z \in J_{\gamma_0} \subset V_{\gamma_0}$, we have $w \in zH_{\gamma_0} \subset V_{\gamma_0}$. Also $N(w, \gamma_0) = N(z, \gamma_0) = n_0$, so that $w \in J_{\gamma_0}$. That is, w belongs to $Y \cap J_{\gamma_0}$, contrary to the relation $Y \cap J_{\gamma_0} = \emptyset$. Thus $\bigcap_{\alpha<\Omega} V_\alpha \neq \emptyset$, and we conclude that G has property U.

REMARK. It may be interesting to note that the group discussed above is Lindelöf (and hence normal). To see this, assume that $\mathscr{U}$ is a cover of G by basic open sets, and that $\mathscr{U}$ admits no countable subcover. For $\alpha \in A$, let $\mathscr{U}_\alpha$ consist of all elements of $\mathscr{U}$ which are translates of some H_β where $\beta \leqq \alpha$. Since each $\mathscr{U}_\alpha$ is countable, no $\mathscr{U}_\alpha$ is a cover for G. Let $U_\alpha = \bigcup \mathscr{U}_\alpha$. Then $\{U_\alpha\}_{\alpha<\Omega}$ is a nondecreasing sequence of proper subsets of G, and each U_α is a union of cosets of H_α. Let $V_\alpha = G \backslash U_\alpha$. As in Example 3.2 above, we have $\bigcap_{\alpha<\Omega} V_\alpha \neq \emptyset$: hence $\mathscr{U}$ does not cover G.

One may wonder whether Example 3.2 is typical of topological groups that are P-spaces: Do all topological groups that are P-spaces satisfy property U? The next theorem and the examples following it make Example 3.2 appear atypical.

THEOREM 3.3. *Let G be a nondiscrete topological group. If G admits a base $\mathscr{H}$ at the identity consisting of open subgroups such that card $(G/K) \geqq$ card $\mathscr{H}$ for some K in $\mathscr{H}$, then G does not have property U.*

Proof. We may clearly suppose that $H \subset K$ for all H in $\mathscr{H}$. By the cardinality hypothesis, there exists a subset $\{x_H\}_{H \in \mathscr{H}}$ of G, indexed by $\mathscr{H}$, such that $\{x_H K\}_{H\in\mathscr{H}}$ is a family of distinct cosets of K. Let $W = \bigcup_{H\in\mathscr{H}} x_H H$. Clearly W is open, and W is closed because

$$G\backslash W = \Big(G\backslash \bigcup_{H\in\mathscr{H}} x_H K\Big) \cup \bigcup_{H\in\mathscr{H}} (x_H K\backslash x_H H) .$$

Therefore ψ_W is continuous; we next show that ψ_W is not left uniformly continuous. Indeed, suppose that there exists an $H_0 \in \mathscr{H}$ such that $x^{-1}y \in H_0$ implies $x \in W$ if and only if $y \in W$. Since G is nondiscrete, there exists an H in $\mathscr{H}$ such that $H \subset H_0$ and $H \neq H_0$. If y is chosen so that $y \in x_H H_0 \backslash x_H H$, then $y \in x_H K\backslash x_H H \subset G\backslash W$. Since $x_H^{-1} y \in H_0$, we also have $x_H \in G\backslash W$. This contradicts the fact that $x_H \in x_H H \subset W$.

EXAMPLES 3.4. Let μ be a cardinal number less than the first strongly inaccessible cardinal[1]. Let G be the algebraic group $\{1, -1\}^\mu = \prod_{\alpha\in A}\{1, -1\}_\alpha$, where the index set A is ordered according to the least ordinal having cardinality μ. Let the subgroups

$$H_\alpha = \{x \in G : x_\beta = 1 \text{ for all } \beta < \alpha\}$$

and all their translates be a basis for a topology on G.

If ν denotes the smallest cardinal number which is the cardinal number of some cofinal subset of A, then evidently ν is the minimal cardinality of a base at the identity of G. If μ is chosen so that $\nu > \aleph_0$, then the nondiscrete topology imposed upon G is clearly a P-space topology, and under the condition $\nu > \aleph_0$ we can show that G does not have property U.

To do this, suppose first that $2^\kappa < \nu$ whenever $\kappa < \nu$. Then (from 12.4–12.6 of [5]) there is a set $\{\nu_\lambda\}_{\lambda\in\Lambda}$ of cardinal numbers such that card $\Lambda < \nu$, $\nu_\lambda < \nu$ for each λ in Λ, and $\sup \nu_\lambda = \nu$. Since there is then a cofinal set $\{\alpha_\lambda : \lambda \in \Lambda\}$ in A indexed by Λ, contrary to the minimality of ν, we conclude that $2^\kappa \geqq \nu$ for some $\kappa < \nu$. Now let $\mathscr{H}$ be a basis of open subgroups at the identity for which card $\mathscr{H} = \nu$, and choose $\beta \in A$ so that $H_\beta \in \mathscr{H}$ and card $\{\alpha \in A : \alpha < \beta\} \geqq \kappa$. Then

$$\operatorname{card}(G/H) \geqq 2^\kappa \geqq \nu ,$$

so that G does not have property U by 3.3.

4. Related concepts. Much of our earlier work is summarized in the following theorem. The symbol βG denotes the Stone-Čech compactification of the (completely regular) space G; it is, to within a homeomorphism leaving G fixed pointwise, the only compactification

[1] A cardinal number is said to be strongly inaccessible if it is an uncountable cardinal whose set of predecessors is closed under the standard operations of cardinal arithmetic. It is not known whether any strongly inaccessible cardinal number exists.

of G to which each bounded continuous real-valued function on G admits a continuous extension. The amusing suggestion that G might induce a topological group structure on βG is not original with us: The appearance of this phenomenon was explicitly pointed out in [6] by Glicksberg in connection with the Corson Σ-space mentioned earlier.

The implication (b) $\Rightarrow$ (g) of 4.1 below was given in [6], and Glicksberg asked whether or not the implication (a) $\Rightarrow$ (g) is valid. Our proof of its validity does not depend upon the results of [6].

If the identity in a topological group G admits a neighborhood U which is bounded (in the sense that for each nonempty open subset V of G there is a finite set F such that $U \subset FV$), then G is said to be locally bounded.

We remark finally that additional conditions equivalent to those listed below may be obtained by replacing the expression "G has property U" when it appears by the expression "G has property BU."

THEOREM 4.1. *For a topological group G, conditions* (a) *through* (g) *are equivalent, and each implies* (h). *If in addition G is nondiscrete, then all eight conditions are equivalent.*

(a) *G is pseudocompact;*
(b) *$G \times G$ is pseudocompact;*
(c) *G is pseudocompact and has property U;*
(d) *G is totally bounded and has property U;*
(e) *G is totally bounded and $\beta G = \bar{G}$;*
(f) *βG admits a topological group structure relative to which the inclusion mapping of G into βG is a topological isomorphism;*
(g) *every continuous real-valued function on G is almost periodic;*
(h) *G is locally bounded and has property U.*

Proof. Theorem 1.4 gives the implication (a) $\Rightarrow$ (b), and the converse follows from the fact that the continuous image of a pseudocompact space is pseudocompact. The implications (a) $\Rightarrow$ (c), (c) $\Rightarrow$ (d), and (d) $\Rightarrow$ (a) are 1.5, 1.1, and 2.7 respectively, while the implication (a) $\Rightarrow$ (e) follows from 1.1 and the implication (a) $\Rightarrow$ (e) of 1.2. That (e) $\Rightarrow$ (f) is obvious, and the implication (f) $\Rightarrow$ (d) follows from 2.8.

We have shown so far that the first six conditions listed are equivalent.

To deduce (g), suppose that (a) and (e) hold and let f be any continuous real-valued function on G. Being bounded, f admits a continuous real-valued extension to βG. A routine computation, based on the fact that every continuous real-valued function on the compact group βG is almost periodic on βG, shows that f is almost periodic on G.

To see that (g) implies (a), let f be any continuous real-valued function on G and let F be a finite subset of G with the property that for each x in G there exists y in F such that $|f(xz) - f(yz)| < 1$ whenever $z \in G$. Then for each x in G we have

$$|f(x)| = |f(xe)| < \max_{y \in F} |f(y)| + 1 .$$

Since the implication (d) $\Rightarrow$ (h) is obvious, we may complete the proof by supposing that G is nondiscrete and deducing (d) from (h). If (h) holds but (d) fails, then G is a P-space by 2.2. Let U be a bounded neighborhood of e and let $\{x_k\}$ be an infinite set of distinct points in U. For each pair (m, n) of distinct positive integers there is a neighborhood $V_{m,n}$ of the identity such that $x_m \notin x_n V_{m,n}$. Choosing a symmetric neighborhood V of the identity such that

$$V^2 \subset \bigcap_{(m,n)} V_{m,n} ;$$

we see easily that no set of the form xV can contain more than one of the points x_k. Thus there exists no finite subset F of G for which $U \subset FV$.

In the discussion and example which follow we will say that a uniform space on which each real-valued uniformly continuous function is bounded has property UB. Clearly any totally bounded uniform space has property UB, and Atsuji gives in [1] an example of a connected metric space that is not totally bounded but which has property UB. Further metric examples are given in exercises 15.D and 15.L of [5]. Although Atsuji in Theorem 7 of [2] characterizes uniform spaces with property UB by means of a chainability condition, the following question has not so far as we can determine been treated in the literature: Must a topological group with property UB be totally bounded? We now answer this question in the negative.

EXAMPLE 4.2. Let T denote the circle group and let G be the algebraic group $T^{\aleph_0} = \prod_{k=1}^{\infty} T_k$. Defining

$$d(x, y) = \sup_k |x_k - y_k|$$

for each pair of points x, y in G, we obtain a metric topology on G under which G is a topological group. To see that G has property UB, let f be a uniformly continuous real-valued function on G and find $\delta > 0$ such that $|f(x) - f(y)| < 1$ whenever $d(x, y) < \delta$. Choose an integer m so that, given any point t in T, there is a sequence $1 = t^0, t^1, \cdots, t^m = t$ in T such that $|t^{j+1} - t^j| < \delta/2$ for $0 \leqq j \leqq m-1$. We will show that $|f(x)| \leqq |f(e)| + m$ for all x in G. For a fixed x in G, select for each integer $k > 0$ a sequence $1 = x_k^0, x_k^1, \cdots, x_k^m = x_k$ in T such that $|x_k^{j+1} - x_k^j| < \delta/2$. The finite sequence $x^0, x^1, \cdots, x^m$ in G has the property that

$$d(x^{j+1}, x^j) \leqq \delta/2 < \delta \quad \text{for} \quad 0 \leqq j \leqq m-1 .$$

Hence $|f(x^{j+1}) - f(x^j)| < 1$ for $0 \leqq j \leqq m-1$, so that $|f(x) - f(e)| \leqq m$. Thus G has property UB.

To see that G is not totally bounded, let W be the open set $\{x \in G : d(x, e) < 1/2\}$. Regarding G as the usual compact topological group $T^{\aleph_0}$ with its Haar measure, we see that the G_δ set W has Haar measure 0. It follows that no finite number of translates of W can cover G.

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ALGEBRAS AND FIBER BUNDLES

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Let A be an associative algebra and $\hat{A}_n$ the family of all equivalence classes of irreducible representations of A of dimension exactly n. Topologizing $\hat{A}_n$ as in a paper about to appear in the Transactions of the American Mathematical Society, we show that for each n, A gives rise to a fiber bundle having $\hat{A}_n$ as its base space and the $n \times n$ total matrix algebra as its fiber.

Throughout this note A will be an arbitrary fixed associative algebra over the complex field C. By a *representation* of A we understand a homomorphism T of A into the algebra of all linear endomorphisms of some complex linear space $H(T)$, the *space of* T. We write $\dim(T)$ for the dimension of $H(T)$. Irreducibility and equivalence of representations are understood in the purely algebraic sense. If T is a representation, $r \cdot T$ will be the direct sum of r copies of T. Let $\hat{A}^{(f)}$ the family of all equivalence classes of finite-dimensional irreducible representations of A; and put

$$\hat{A}^{(n)} = \{T \in \hat{A}^{(f)} \mid \dim(T) \leqq n\},\ \hat{A}_n = \{T \in \hat{A}^{(f)} \mid \dim(T) = n\} .$$

We shall usually not distinguish between representations and the equivalence classes to which they belong.

Let T be a finite-dimensional representation of A. If for each a in A $\tau(a)$ is the matrix of T_a with respect to some fixed ordered basis of $H(T)$, then $\tau : a \to \tau(a)$ is a *matrix representation of A equivalent to T*.

By $A^{\#}$ we mean the space of all complex linear functionals on A, and by $\mathrm{Ker}(\varphi)$ the kernel of φ. If $T \in \hat{A}^{(f)}$, we put

$$\Phi(T) = \{\varphi \in A^{\#} \mid \mathrm{Ker}(T) \subset \mathrm{Ker}(\varphi)\} .$$

An element φ of $A^{\#}$ is *associated* with T if $\varphi \in \Phi(T)$. One element of $\Phi(T)$ is of course the character χ^T of $T(\chi^T(a) = \mathrm{Trace}(T_a)$ for a in A). An element T of $\hat{A}^{(f)}$ is uniquely determined by the knowledge of one nonzero functional in $\Phi(T)$ ([2], Proposition 2).

As in [2] we equip $\hat{A}^{(f)}$ with the *functional topology* as follows: If $T \in \hat{A}^{(f)}$ and $\mathscr{S} \subset \hat{A}^{(f)}$, T belongs to the functional closure of $\mathscr{S}$ if $\Phi(T) \subset (\bigcup_{S \in \mathscr{S}} \Phi(S))^-$ where $^-$ denotes closure in the topology of pointwise convergence on A.

Our main object in this note is to prove the following fact about

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the functional topology relativized to $\hat{A}_n$:

THEOREM 1. *Fix a positive integer n; and let T be any element of* $\hat{A}_n$. *Then there exists a neighborhood U of T in* $\hat{A}_n$, *and a function* τ *assigning to each S in U a matrix representation* τ_S *of A equivalent to S, such that for each a in A the matrix-valued function*

$$S \longrightarrow \tau_S(a) \quad (S \in U)$$

is continuous on U.

This asserts (see § 4) that, for each n, A gives rise to a fiber bundle with base space $\hat{A}_n$ whose fiber is the $n \times n$ total matrix algebra.

2. Preliminary results. The following Proposition 1 coincides with Proposition 7 of [2] (which was stated in [2] without proof). Proposition 1 is not required for what follows it; but its proof is related to later proofs.

PROPOSITION 1. *Let n be a positive integer; and suppose that* $\{T^{(i)}\}$ *is a net of elements of* $\hat{A}^{(n)}$ *converging to each of the p inequivalent elements* $V^1, \cdots, V^p$ *of* $\hat{A}^{(n)}$. *Then*

$$\sum_{s=1}^{p} (\dim (V^s))^2 \leqq n^2 . \tag{1}$$

Proof. Let $m_s = \dim (V^s)$, $q = \sum_{s=1}^{p} m_s^2$. Each $\Phi(V^s)$ has dimension m_s^2, and by the Extended Burnside Theorem ([1], Theorem 27.8) the $\Phi(V^s)$ $(s = 1, \cdots, p)$ are linearly independent subspaces of $A^{\#}$. Thus there are q linearly independent functionals $\varphi_1, \cdots, \varphi_q$ each of which is associated with some V^s. By the definition of the functional topology we can replace $\{T^{(i)}\}$ by a subnet, and choose for each $r = 1, \cdots, q$ and each i a functional φ_r^i in $\Phi(T^{(i)})$, such that

$$\varphi_r^i \xrightarrow[i]{} \varphi_r \; (r = 1, \cdots, q) . \tag{2}$$

Since the $\varphi_1, \cdots, \varphi_q$ are independent, (2) implies that for some i the $\varphi_1^i, \cdots, \varphi_q^i$ are independent. Since $\dim (\Phi(T^{(i)})) \leqq n^2$, it follows that $q \leqq n^2$. This proves (1).

REMARK. If A is a Banach algebra we have shown elsewhere ([2], Proposition 13) that a stronger inequality than (1) holds, namely

$$\sum_{s=1}^{p} \dim (V^s) \leqq n . \tag{3}$$

Probably (3) holds for arbitrary A, but we have not been able to prove it.

COROLLARY 1. *$\hat{A}_n$ is Hausdorff for each n.*

For each φ in $A^\sharp$ let us define S^φ to be the natural representation of A acting in A/J, where J is the left ideal of A consisting of those a such that $\varphi(ba) = 0$ for all b in A.

LEMMA 1. *Let $\{\varphi_i\}$ be a net of elements of $A^\sharp$, converging pointwise to an element φ of $A^\sharp$; and suppose the S^φ, S^{φ_i} are all finite-dimensional. Then*

$$\dim(S^\varphi) \leqq \liminf_i \dim(S^{\varphi_i}) . \tag{4}$$

Further, if σ is a matrix representation of A equivalent to S^φ, there exists for each i a matrix representation σ^i of A equivalent to S^{φ_i} such that

$$\lim_i (\sigma^i(a))_{jk} = (\sigma(a))_{jk} \tag{5}$$

for all a in A and all $j, k = 1, \cdots, \dim(S^\varphi)$.

Proof. Let π be the natural map of A onto A/J, where $J = \{a \in A \mid \varphi(ba) = 0 \text{ for all } b \text{ in } A\}$; and put $m = \dim(S^\varphi)$. Every element of $(A/J)^\sharp$ is of the form

$$\pi(a) \longrightarrow \varphi(ba) \qquad (a \in A)$$

for some b in A. Hence there are elements $a_1, \cdots, a_m b_1, \cdots, b_m$ of A satisfying

$$\varphi(b_j a_k) = \delta_{jk}(j, k = 1, \cdots, m) . \tag{6}$$

Since $\varphi_i \rightarrow \varphi$, (6) implies that

$$\det\{(\varphi_i(b_j a_k))_{j,k=1,\cdots,m}\} \neq 0 , \tag{7}$$

and hence $\dim(S^{\varphi_i}) \geqq m$, for all large i. This proves (4).

Now the a_k, b_j could have been chosen to satisfy not only (6) but also

$$(\sigma(x))_{jk} = \varphi(b_j x a_k) \tag{8}$$

$(x \in A; j, k = 1, \cdots, m)$; assume this done. By (7), for each large i there are unique complex numbers $c^i_{jk}(j, k = 1, \cdots, m)$ such that the elements $b^i_j = \sum_{k=1}^m c^i_{jk} b_k$ satisfy

$$(9) \qquad \varphi_i(b_j^i a_k) = \delta_{jk} \qquad (j, k = 1, \cdots, m) .$$

By (6) and (9)

$$(10) \qquad \lim_i c_{jk}^i = \delta_{jk} .$$

In view of (4) and (9), there are elements $a_{m+1}^i, \cdots, a_{p_i}^i, b_{m+1}^i, \cdots, b_{p_i}^i$ of A (where $p_i = \dim(S^{\varphi_i})$), such that

$$(11) \qquad \varphi_i(b_j^i a_k^i) = \delta_{jk}$$

for all large i and all $j, k = 1, \cdots, p_i$; (here we agree that $a_j^i = a_j$ for $j = 1, \cdots, m$). Now, if $j, k = 1, \cdots, p_i$ and $x \in A$, define

$$(\sigma^i(x))_{jk} = \varphi_i(b_j^i x a_k^i) .$$

From (8), (10), and (11), we verify that σ^i is a matrix representation equivalent to S^{φ_i} and that (5) holds. This completes the proof.

The following corollary was stated without proof as Proposition 8 of [2].

COROLLARY 2. *For each positive integer n, the map $T \to \chi^T (T \in \hat{A}_n)$ is a homeomorphism of $\hat{A}_n$ into $A^\#$ (the latter having the topology of pointwise convergence on A).*

Proof. Obviously $\chi^T \to T$ is continuous. To prove that $T \to \chi^T$ is continuous, we shall suppose that $T, \{T^i\}$ are elements of $\hat{A}_n$ and that $\varphi_i \xrightarrow[i]{} \chi^T$ pointwise on A, where for each i φ_i is associated with T^i; and we shall prove that $\chi^{T^i} \xrightarrow[i]{} \chi^T$ pointwise on A. Clearly this is sufficient.

By [2], Proposition 1, $S^{\chi^T} \cong n \cdot T$ and $S^{\varphi_i} \cong r_i \cdot T^i$, where $r_i \leqq n$. By (4) $r_i = n$ for all large i. Hence by (5) $\chi^T(a) = 1/n \text{ Trace } (S_a^{\varphi}) = \lim_i 1/n \text{ Trace } (S_a^{\varphi_i}) = \lim_i \chi^{T^i}(a)$ for all a in A. So $\chi^{T^i} \to \chi^T$, and the corollary is proved.

If M is any finite-dimensional complex linear space, the family $\mathscr{F}$ of all linear subspaces of M of fixed dimension r ($r \leqq \dim(M)$) has a natural compact topology. Indeed, if G is the unitary group on M (with respect to some fixed inner product), and G_0 is the subgroup of G which leaves stable some fixed L in $\mathscr{F}$, then $\mathscr{F}$ is in one-to-one correspondence with G/G_0, and the (compact) topology of $\mathscr{F}$ which makes this correspondence a homeomorphism is independent of the inner product and of L.

If p is any positive integer, M_p will be the $p \times p$ total matrix algebra over the complexes. Fix a positive integer n; and let $\mathscr{S}$ be the family of all those subalgebras A of M_{n^2} which contain 1 and are isomorphic with M_n. For each A in $\mathscr{L}$ let A' be the commuting algebra of A in M_{n^2}:

$$A' = \{a \in M_{n^2} \mid ab = ba \text{ for all } b \text{ in } A\} .$$

It is well known that $A' \in \mathscr{L}$ and that $A'' = A$ whenever $A \in \mathscr{L}$.

LEMMA 2. *The map $A \to A'$ is continuous on $\mathscr{L}$ to $\mathscr{L}$ (with the topology discussed above).*

Proof. If not, then, by the compactness of the space $\mathscr{M}$ of all n^2-dimensional subspaces of M_{n^2}, one can find a net $\{A_i\}$ of elements of $\mathscr{L}$ such that $A_i \to A$, $A_i' \to B$, where $A \in \mathscr{L}$, $B \in \mathscr{M}$, $A' \neq B$. Choose an element b of B which is not in A', and let a be any element of A. Then for each i we can choose an a_i in A_i and b_i in A_i' so that $a_i \to a$, $b_i \to b$. Since $a_i b_i = b_i a_i$, passing to the limit we obtain $ab = ba$, whence $b \in A'$, a contradiction.

LEMMA 3. *Let A be in $\mathscr{L}$, and let e be a minimal nonzero idempotent in A. Then there is a neighborhood U of A in $\mathscr{L}$, and a continuous function w on U to M_{n^2} such that*

(i) *$w(A) = e$, and*

(ii) *for each B in U $w(B)$ is a minimal nonzero idempotent in B.*

Proof. Choose an element a of A whose spectrum in A is $\{1, 2, \cdots, n\}$, and such that the spectral idempotent (in A) corresponding to the eigenvalue 1 of a is precisely e; that is,

$$e = ((n-1)!)^{-1}(2-a)(3-a)\cdots(n-a) . \tag{12}$$

Introducing a Hilbert space inner product into M_{n^2} in an arbitrary manner and projecting, we can construct a continuous function α on $\mathscr{L}$ to M_{n^2} such that $\alpha(A) = a$ and $\alpha(B) \in B$ for each B in $\mathscr{L}$. Let $\sigma(B)$ be the spectrum of $\alpha(B)$ (considered as an element either of B or of M_{n^2}). Since α is continuous, $\sigma(B)$ is continuous as a function of B. Thus there is a neighborhood U of A in $\mathscr{L}$, and n continuous complex functions $\lambda_1, \cdots, \lambda_n$ on U such that

(i) $\lambda_r(A) = r$ $(r = 1, \cdots, n)$,

(ii) for each B in U the $\lambda_1(B), \cdots, \lambda_n(B)$ are all distinct, and

(iii) $\sigma(B) = \{\lambda_1(B), \cdots, \lambda_n(B)\}$ for each B in U. Now, for B in U, put

$$w(B) = \prod_{j=2}^{n} (\lambda_j(B) - \lambda_1(B))^{-1}(\lambda_j(B)\cdot 1 - \alpha(B))) .$$

Clearly w is continuous on U, $w(B) \in B$ for each B in U, and $w(A) = e$.

Since $w(B)$ is the spectral idempotent corresponding to the eigenvalue $\lambda_1(B)$ of $\alpha(B)$ (which has multiplicity 1), $w(B)$ is a minimal idempotent of B for each B in U.

LEMMA 4. *If $A \in \mathscr{L}$, there is a neighborhood U of A in $\mathscr{L}$, and a continuous function w on U to M_{n^2}, such that, for each B in U, $w(B)$ is a minimal idempotent of the commuting algebra of B.*

Proof. This follows immediately from Lemmas 2 and 3.

3. Proof of Theorem 1. We have seen ([2], Proposition 1) that $S^{\chi^T} \cong n \cdot T$. Thus, putting $m = n^2$, we may choose elements $a_1, \cdots, a_m$, $b_1, \cdots, b_m$ of A as in the proof of Lemma 1 so that

$$\chi^T(b_j a_k) = \delta_{jk}(j, k = 1, \cdots, m) .$$

Since $S \rightarrow \chi^S$ is continuous on $\hat{A}_n$ (Corollary 2), there is a neighborhood U' of T in $\hat{A}_n$ such that $\det(\chi^S(b_j a_k))_{j,k} \neq 0$ for S in U'. Thus, as in the proof of Lemma 1, for each S in U' we find unique complex numbers $c_{jk}(S)$ such that the elements $b_j(S) = \sum_{k=1}^m c_{jk}(S) b_k$ satisfy

$$\chi^S(b_j(S) a_k) = \delta_{jk} \tag{13}$$

$(j, k = 1, \cdots, m; S \in U')$. We now set

$$(\sigma_S(x))_{jk} = \chi^S(b_j(S) x a_k)$$

$(j, k = 1, \cdots, m; S \in U'; x \in A)$, and verify as in the proof of Lemma 1 that, for S in U', σ_S is a matrix representation of A equivalent to $n \cdot S$. Since $S \rightarrow \chi^S$ is continuous (Corollary 2), the $c_{jk}(S)$ are continuous in S on U', and so

$$S \longrightarrow \sigma_S(x) \quad \text{is continuous on } U' \tag{14}$$

for each x in A.

Since $\sigma_S \cong n \cdot S$, Burnside's Theorem asserts that the range $\sigma_S(A)$ of σ_S belongs to $\mathscr{L}$. Further, it follows from (14) that $S \rightarrow \sigma_S(A)$ is continuous on U' (in the topology of n^2-dimensional subspaces discussed in § 2). Thus, by Lemma 4, there is a neighborhood U'' of T contained in U', and a function w on U'' to M_m such that, for each S in U'', $w(S)$ is a minimal idempotent of the commuting algebra of $\sigma_S(A)$.

We now consider M_m is acting on C^m (the space of complex m-tuples). Let $v_1, \cdots, v_m$ be a basis of C^m such that $v_1, \cdots, v_n$ is a basis of range $(w(T))$. By the continuity of w there will be a neighborhood U of T contained in U'' such that

$$w(S)v_1, \cdots, w(S)v_n, v_{n+1}, \cdots, v_m \tag{15}$$

is a basis of C^m for each S in U (the first n vectors of (15) being, of course, a basis of range $(w(S))$). Now for each S in U and x in A let $\rho_S(x)$ be the matrix of $\sigma_S(x)$ with respect to the ordered basis (15), and let $\tau_S(x)$ be the $n \times n$ matrix consisting of the first n rows and columns of $\rho_S(x)$. Since $w(S)$ is a minimal idempotent of the commuting algebra of $\sigma_S(A)$, σ_S restricted to range $(w(S))$ is an irreducible subrepresentation of σ_S and so is equivalent to S. Thus, for each S in U, τ_S is a matrix representation of A equivalent to S. Further, since $S \to w(S)$ is continuous on U, the basis (15) varies continuously with S on U; and therefore by (14) we conclude that $S \to \tau_S(x)$ is continuous on U for each x in A. This completes the proof of Theorem 1.

4. Fiber bundles associated with A. Fix a positive integer n, and let G_n be the group of all algebraic automorphisms of the total matrix algebra M_n. We are going to describe to within equivalence a fiber bundle B_n with base space $\hat{A}_n$, fiber M_n, and group G_n. To do so, it is sufficient to specify an open covering of $\hat{A}_n$, and to define on the overlap of any two sets in the covering the G_n-valued "coordinate transformation functions" ([3], §§ 2, 3). As our open covering we take the set of all the $U = U_T$ $(T \in \hat{A}_n)$ of Theorem 1. If $T, T' \in \hat{A}_n$, the coordinate transformation function $\Gamma_{T,T'}$ on $U_T \cap U_{T'}$ will assign to each S in $U_T \cap U_{T'}$ the following automorphism of M_n:

$$\Gamma_{T,T'}(S) : \tau_S^{(T)}(a) \longrightarrow \tau_S^{(T')}(a) \qquad (a \in A) .$$

(Here $\tau^{(T)}$ is the τ of Theorem 1). The property $\Gamma_{T,T''} = \Gamma_{T,T''} \circ \Gamma_{T,T'}$ (on $U_T \cap U_{T'} \cap U_{T''}$) obviously holds; and the continuity of the maps $S \to \tau_S^{(T)}(a)$ and $S \to \tau_S^{(T')}(a)$ assures us that $\Gamma_{T,T'}$ is continuous. Thus we have defined a fiber bundle of the required kind; its equivalence class clearly depends only on A.

Thus, if the algebra A has a large supply of finite-dimensional irreducible representations, the structure of the fiber bundles $B_n(n = 1, 2, \cdots)$ constitutes a significant feature of the structure of A. We hope in a later paper to discuss the structure of these bundles for certain special kinds of algebras associated with locally compact groups having "large" compact subgroups.

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A THEOREM OF LITTLEWOOD AND LACUNARY SERIES FOR COMPACT GROUPS

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Let G be a compact group and $f \in L^2(G)$. We prove that given $p < \infty$ there exists a unitary transformation U of $L^2(G)$ into $L^2(G)$, which commutes with left translations and such that $Uf \in L^p$. The proof is based on techniques developed by S. Helgason for a similar question. The result stated above, which is an extension of a theorem of Littlewood for the unit circle is then applied to the study of lacunary Fourier series.

The following two results concerning Fourier series of functions defined on the unit circle were proved by Littlewood [5]:

I. *Suppose that for any choice of complex numbers* α_n, *with* $|\alpha_n| = 1$, $\sum \alpha_n a_n e^{inx}$ *is the Fourier series of an integrable function* (*or a Fourier-Stieltjes series*) *then* $\sum |a_n|^2 < \infty$.

II. *Let* $\sum |a_n|^2 < \infty$. *Then given* $p < \infty$ *there exist complex numbers* α_n, *with* $|\alpha_n| = 1$, *such that* $\sum \alpha_n a_n e^{inx}$ *is the Fourier series of a function in* L^p.

Helgason [3] has generalized I to Fourier series on compact groups. Let G be a compact group with normalized Haar measure dx. If $f \in L^1(G)$ then f is uniquely represented by a Fourier series

$$f(x) \sim \sum_{\gamma \in \Gamma} d_\gamma Tr(A_\gamma D_\gamma(x))$$

where Tr denotes the usual trace, Γ is the set of equivalence classes of irreducible unitary representations of G, D_γ is a representative of the class γ, d_γ is the degree of γ, and A_γ is the linear transformation given by

$$A_\gamma = \int_G f(x) D_\gamma(x^{-1}) dx \, .$$

Helgason has proved

I′. *Suppose that, for any choice of unitary transformations* U_γ *on the Hilbert space of dimension* d_γ, $\sum_{\gamma \in \Gamma} d_\gamma Tr(U_\gamma A_\gamma D_\gamma(x))$ *is the Fourier series of an integrable function* (*or a Fourier-Stieltjes series*) *then*

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$$\sum_{\gamma \in \Gamma} d_\gamma Tr(A_\gamma A_\gamma^*) < \infty .$$

In view of the Schur-Peter-Weyl formula

$$\int_G |f(x)|^2 dx = \sum_{\gamma \in \Gamma} d_\gamma Tr(A_\gamma A_\gamma^*) ,$$

Helgason's result is an extension of I.

In this paper, using Helgason's techniques, we propose to extend II to compact groups in the same sense. That is we prove

II′. *Let* $\sum d_\gamma Tr(A_\gamma A_\gamma^*) < \infty$. *Given* $p < \infty$ *there exist unitary transformations* U_γ *such that* $\sum d_\gamma Tr(U_\gamma A_\gamma D_\gamma(x))$ *is the Fourier series of a function in* L^p.

This is accomplished as in [3] by proving and exploiting the "lacunarity" of a certain subset of the space of irreducible unitary representations of the product group $\prod_{i \in S} U(d_i)$ where $U(d_i)$ is the group of unitary transformations of the Hilbert space of dimension d_i and S is an arbitrary index set. In the last section we discuss in general lacunary properties of subsets of the space of irreducible representations of a compact group.

2. The main result.

For a positive integer n let $U(n)$ be the group of unitary transformations of the Hilbert space of dimension n. The normalized Haar measure on $U(n)$ will be denote by dV.

LEMMA 1. *Let* A *be an* $n \times n$ *matrix. Then for* $s = 1, 2, 3, \cdots$

$$\int_{U(n)} |Tr(AV)|^{2s} dV \leqq \frac{B(s)}{n^s} [Tr(AA^*)]^s \tag{1}$$

where $B(s)$ *is a constant depending only on* s.

Proof. Since dV is left and right invariant it is sufficient to prove the lemma when A is diagonal. Letting $e_1, e_2, \cdots, e_n$ be a basis for the Hilbert space on which A and V act and $a_i = \langle Ae_i, e_i \rangle$, $v_i = \langle Ve_i, e_i \rangle$ we have

$$\int_{U(n)} |Tr(AV)|^{2s} dV = \sum a_{i_1} \bar{a}_{i_2} a_{i_3} \bar{a}_{i_4} \cdots a_{i_{2s-1}} \bar{a}_{i_{2s}} \cdot \int_{U(n)} v_{i_1} \bar{v}_{i_2} \cdots v_{i_{2s-1}} \bar{v}_{i_{2s}} dV , \tag{2}$$

where the sum extends over all $i_1, i_2, \cdots, i_{2s}$ such that $1 \leqq i_j \leqq n$. Each integral in the sum is of the form

$$(3) \qquad \int_{U(n)} v_1^{j_1}\bar{v}_1^{k_1} \cdots v_n^{j_n}\bar{v}_n^{k_n} dV .$$

Now each such integral is zero unless $j_1 = k_1, \cdots, j_n = k_n$. For let W be a diagonal unitary matrix with elements α_i of modulus one on the main diagonal. Then by the invariance of dV, (3) becomes

$$\begin{aligned} \int_{U(n)} \prod_{i=1}^{n} \langle Ve_i, e_i\rangle^{j_i}\langle e_i, Ve_i\rangle^{k_i} dV &= \int_{U(n)} \prod_{i=1}^{n} \langle WVe_i, e_i\rangle^{j_i}\langle e_i, WVe_i\rangle^{k_i} dV \\ &= \prod_{i=1}^{n} \alpha_i^{j_i-k_i} \int_{U(n)} \prod_{i=1}^{n} \langle Ve_i, e_i\rangle^{j_i}\langle e_i, Ve_i\rangle^{k_i} dV . \end{aligned}$$

Thus if the integral is not zero, $\prod_{i=1}^{n} \alpha_i^{j_i-k_i} = 1$, for all choices of the α_i. Clearly this is possible only if $j_1 = k_1, \cdots, j_n = k_n$. Thus the sum (2) is equal to

$$(4) \qquad \sum |a_{i_1}|^2 |a_{i_2}|^2 \cdots |a_{i_s}|^2 \int_{U(n)} |v_{i_1}|^2 \cdots |v_{i_s}|^2 dV .$$

We shall see that for each integer s

$$(5) \qquad \int_{U(n)} |v_i|^{2s} dV \leqq \frac{B(s)}{n^s} \qquad (s = 1, 2, \cdots)$$

where $B(s)$ depends only on s. It then follows from Hölder's inequality that the integrals in (4) are bounded by $B(s)/n^s$ so that (4) is majorized by

$$\frac{B(s)}{n^s} \sum |a_{i_1}|^2 \cdots |a_{i_s}|^2 = \frac{B(s)}{n^s} [Tr(AA^*)]^s ,$$

and the lemma will be proved.

It is sufficient to calculate (5) for $i = 1$. Let $U_1(n-1)$ be the subgroup $\{T \in U(n) : Te_1 = e_1\}$. The space $U(n)/U_1(n-1)$ of left cosets $\{\tilde{V} = VU_1(n-1) : V \in U(n)\}$ can be identified with the unit sphere Σ_n in a complex n-dimensional Hilbert space. Since v_1 is constant on these cosets

$$\int_{U(n)} |v_1|^{2s} dV = \int_{\Sigma_n} |\langle Ve_1, e_1\rangle|^{2s} d\tilde{V} = \int_{|w_1|^2+\cdots+|w_n|^2=1} |w_1|^{2s} d\tilde{V}$$

where $d\tilde{V}$ is the unique normalized measure on Σ_n invariant with respect to $U(n)$ and

$$Ve_1 = w_1e_1 + \cdots + w_ne_n .$$

If we identify Σ_n with the real $(2n-1)$ dimensional sphere S^{2n-1} in real $2n$-dimensional space and $d\tilde{V}$ with dw, the normalized invariant measure on S^{2n-1}, then

$$\int_{U(n)} |v_1|^{2s} dV = \int_{x_1^2+\cdots+x_{2n}^2=1} (x_1^2 + x_2^2)^s dw \,. \tag{6}$$

By Minkowski's inequality and the invariance of dw (6) is bounded by

$$2^s \int_{S^{2n-1}} x_1^{2s} dw = 2^s \frac{\Omega(S^{2n-2})}{\Omega(S^{2n-1})} \int_{-1}^{1} x_1^{2s}(1 - x_1^2)^{n-1}(1 - x_1^2)^{-1/2} dx_1$$

where $\Omega(S^m) = \dfrac{2\pi^{(m+1)/2}}{\Gamma\left(\dfrac{m+1}{2}\right)}$ is the Euclidean surface area of the real m-dimensional unit sphere. Thus the integral in (6) is bounded by

$$2^s \frac{2\pi^{(2n-1)/2}}{\Gamma\left(n - \frac{1}{2}\right)} \cdot \frac{\Gamma(n)}{2\pi^{(2n)/2}} \cdot \frac{\Gamma\left(s + \frac{1}{2}\right)\Gamma\left(n - \frac{1}{2}\right)}{\Gamma(n + s)} \leqq \frac{B(s)}{n^s}$$

which proves (5).

Corollary 2. *Let J be the canonical representation $U \to U$ of $U(n)$ and $J_{s,t}$ be the tensor product of J, s times and $\tilde{J}$, the conjugate representation, t times. $J_{s,t}$ decomposes into at most $B(s + t)$ irreducible components. If $s \neq t$ then none of the components is the identity representation.*

Proof. If χ_T is the character of the representation T, then $\chi_{J_{s,t}}(V) = (\chi_J(V))^s\overline{(\chi_J(V))}^t = (Tr(V))^s\overline{(Tr(V))}^t$. Thus by the lemma

$$\int_{U(n)} |\chi_{J_{s,t}}(V)|^2 dV \leqq B(s + t) \,,$$

which proves the first statement.

The number of times the identity representation occurs in $J_{s,t}$ is

$$\int_{U(n)} \chi_{J_{s,t}}(V) dV = \int_{U(n)} (Tr(V))^s\overline{(Tr(V))}^t dV = 0$$

if $s \neq t$ by the statement following (3).

Lemma 3. *Let $G = \prod_{i\in S} U(d_i)$ be a product of unitary groups $U(d_i)$. Let $F(V)$ be a function on G of the form*

$$F(V) = \sum_{i \in S} d_i Tr(A_i V_i)$$

where A_i is a $d_i \times d_i$ matrix and V_i is the projection of V on $U(d_i)$. Then

$$\int_G |F(V)|^{2s} dV \leqq B(s)\left(\int |F(V)|^2 dV\right)^s$$

where dV is the normalized Haar measure on G.

Proof. It suffices to prove the lemma when

$$F(V) = \sum_{i=1}^{N} d_i Tr(A_i V_i) .$$

Then

$$(7) \quad \int_G |F(V)|^{2s} dV$$
$$= \sum \int_{U(d_i)\times\cdots\times U(d_n)} d_{i_1} Tr(A_{i_1} V_{i_1}) d_{i_2} \overline{Tr(A_{i_2} V_{i_2})} \cdots d_{i_{2s}} \overline{(A_{i_{2s}} V_{i_{2s}})} dV$$

where the sum extends over all $i_1, \cdots, i_{2s}$ such that $1 \leqq i_j \leqq N$. By the corollary the only terms in the sum which do not vanish are those of the form

$$(8) \qquad \int_{U(d_i)\times\cdots\times U(d_n)} d_{i_1}^2 |Tr(A_{i_1} V_{i_1})|^2 \cdots d_{i_s}^2 |Tr(A_{i_s} V_{i_s})|^2 dV .$$

By Hölder's inequality (8) is majorized by

$$d_{i_1}^2 \cdots d_{i_s}^2 \left[\int_{U(d_{i_1})} |Tr(A_{i_1} V_{i_1})|^{2s} dV_{i_1}\right]^{1/s} \cdots \left[\int_{U(d_{i_s})} |T_r(A_{i_s} V_{i_s})|^{2s} dV_{i_s}\right]^{1/s} .$$

which by Lemma 1 is majorized by

$$d_{i_1}^2 \cdots d_{i_s}^2 B(s) \frac{Tr(A_{i_1} A_{i_1}^*)}{d_{i_1}} \cdots \frac{Tr(A_{i_s} A_{i_s}^*)}{d_{i_s}} .$$

Hence the left side of (7) is bounded by

$$B(s)\left[\sum_1^N d_i Tr(A_i A_i^*)\right]^s = B(s)\left[\int_G |f(V)|^2 dV\right]^s ,$$

where the equality follows from the Peter-Weyl formula.

Now let G be an arbitrary compact group and Γ be the set of equivalence classes of irreducible representations of G. Let d_γ be the degree of the class γ. Then $G = \prod_{\gamma\in\Gamma} U(d_\gamma)$ is a compact group which can be thought of as the group of unitary transformations of $L^2(G)$ into $L^2(G)$ which commute with left translations. That is, if V is such a transformation then V corresponds to the element $\{V_\gamma\} \in G$ such that

$$Vf(x) \sim \sum_{\gamma\in\Gamma} d_\gamma Tr(V_\gamma A_\gamma D_\gamma(x))$$

whenever $f(x) \sim \sum_{\gamma\in\Gamma} Tr(A_\gamma D_\gamma(x)) \in L^2(G)$.

THEOREM 4. *Let $f \in L^2(G)$ and $p < \infty$, then for almost every $V \in G$, $Vf \in L^p(G)$.*

Proof. Let $Vf(x) = f(V, x) = \sum d_\gamma Tr(V_\gamma A_\gamma D_\gamma(x))$. Then $f(V, x)$ can be considered as a function on $G \times G$. For fixed $x \in G$ we have by Lemma 2 that

$$\int_G |f(V, x)|^{2s} dV \leqq B(s)\left[\int_G |f(V, x)|^2 dV\right]^s = B(s)\left[\sum_{\gamma \in \Gamma} d_\gamma Tr(A_\gamma A_\gamma^*)\right]^s$$
$$= B(s)\left[\int_G |f(x)|^2 dx\right]^s,$$

so that

$$\int_G\int_G |f(V, x)|^{2s} dx dV = \int_G\int_G |f(V, x)|^{2s} dV dx$$
$$\leqq B(s)\left[\int_G |f(x)|^2 dx\right]^s.$$

Therefore if $f \in L^2(G)$, then for almost every $V \in G$, $\int_G | Vf(x)|^{2s} dx < \infty$. Letting $s > p/2$ we obtain the theorem.

We remark for later use that for some V

$$\| Vf \|_p \leqq \| Vf \|_{2s} \leqq 2B(s) \| f \|_2 .$$

Indeed the set of V for which

$$\int_G | Vf(x)|^{2s} dx > 2B(s)\left[\int_G |f(x)|^2 dx\right]^s$$

cannot be of measure one.

We will also use the following

REMARK 5. Let $f \in C(G)$ be a continuous function such that for all self adjoint $V \in G$, $Vf \in C(G)$, then $f(x) \sim \sum d_\gamma Tr(A_\gamma D_\gamma(x))$ with $\sum d_\gamma Tr(|A_\gamma|) < \infty$ ($|A_\gamma|$ is the absolute value of the matrix A_γ). Indeed letting $\tilde{f}(x) = \overline{f(x^{-1})}$ we can write $f = (f + \tilde{f})/2 + i(f - \tilde{f})/2i = f_1 + if_2$. If $f_i(x) \sim \sum d_\gamma Tr(A_{\gamma,i} D_\gamma(x))$ $(i = 1, 2)$ then $A^*_{\gamma,i} = A_{\gamma,i}$. Therefore there exists a self adjoint $V = \{V_\gamma\} \in G$ such that $A_{\gamma,i} V_\gamma = |A_{\gamma,i}|$. Thus $\sum d_\gamma Tr(|A_{\gamma,i}| D_\gamma(x))$ is continuous so that applying a method of summation as in [4, 8.3] we obtain that the partial sums of $\sum d_\gamma Tr(|A_{\gamma,i}|) = \sum d_\gamma Tr(|A_{\gamma,i}| D_\gamma(e))$ are bounded. Thus $\sum d_\gamma Tr(|A_\gamma|) \leqq \sum d_\gamma Tr(|A_{\gamma,1}|) + \sum d_\gamma Tr(|A_{\gamma,2}|) < \infty$.

We shall call a series $\sum d_\gamma Tr(A_\gamma D_\gamma(x))$ satisfying $\sum d_\gamma Tr(|A_\gamma|) < \infty$ an *absolutely convergent* series. The space of such functions will be denoted by $A(G)$. It is easy to see that $A(G)$ consists of functions of the type $f * g$ with $f, g \in L^2(G)$. The space $A(G) = L^2(G) * L^2(G)$ has been studied in [1].

3. Lacunary Fourier series. Given a compact group G we shall say that a subset $E \subseteq \Gamma$ of the set of irreducible unitary representation of G is a *Sidon set* if it satisfies the following property:

A. $\sum d_\gamma Tr(|A_\gamma|) < \infty$ *whenever* $\sum_{\gamma \in E} d_\gamma Tr(A_\gamma D_\gamma(x))$ *is the Fourier series of a continuous function* (*cf.* [6, 5.7]).

A set $E \subset \Gamma$ will be called a set of type $\Lambda(p)$ (or $E \in \Lambda(p)$) for $p > 1$ if it satisfies

B. *If* $\sum_{\gamma \in E} d_\gamma Tr(A_\gamma D_\gamma(x))$ *is the Fourier series of an integrable function then it is the Fourier series of a function in* L^p (*cf.* [8]).

If B is a space of functions on G and $E \subseteq \Gamma$ we will denote by B_E those functions in B with a series of the form $\sum_{\gamma \in E} d_\gamma Tr(A_\gamma D_\gamma(x))$. It is seen as in [8, 1.4] that $E \in \Lambda(p)$ if, for some $r < p$, $L^r_E = L^p_E$. Clearly $\Lambda(p_1) \subseteq \Lambda(p_2)$ if $p_1 \geqq p_2$.

If $G = \prod_{i \in S} U(d_i)$ then S can be thought of as the set of irreducible representations of G consisting of the projections of G onto the $U(d_i)$. Lemma 2 shows that $S \in \Lambda(p)$ for every $p < \infty$. It is a simple matter to prove that S is also a Sidon set. Indeed, if $f(V) = \sum_{i \in S} d_i Tr(A_i V_i)$ is a continuous function belonging to $C_S(G)$ and if $U = \{U_i\} \in G$ then

$$Uf(V) = \sum_{i \in S} d_i Tr(A_i U_i V_i) = \text{left translation of } f \text{ by } U,$$

is also continuous. It suffices to pick the U_i so that $A_i U_i = |A_i|$ to obtain that $\sum_{i \in S} d_i Tr(|A_i|) < \infty$.

We shall now establish a characterization of sets of type $\Lambda(p)$ which will imply that every Sidon set is a $\Lambda(p)$ set for every $p < \infty$. For a group G denote by $\mathscr{R}_p = \mathscr{R}_p(G)$ the algebra of operators on $L^p(G)$ generated in the weak operator topology by the operators $\{R_y : y \in G\}$ where $R_y f(x) = f(xy)$. We shall use the fact [2, Th. 6] that $\mathscr{R}_p$ is (isometric and isomorphic to) the dual space of a Banach space A^p of continuous functions on G. $A^2 = A(G)$ the space of functions with absolutely convergent Fourier series [1].

The isomorphism between $\mathscr{R}_p$ and the dual space of A^p is given by $T \to \varphi_T$ where $\varphi_T(f) = Tf(e)$. This correspondence is well defined because every $T \in \mathscr{R}_p$ maps each element of A^p into a continuous function, indeed an element of A^p. We also have that $\mathscr{R}_p$ consists exactly of those bounded operators on L^p which commute with left translations.

Now if $T \in \mathscr{R}_p$, $p > 2$, then $T \in \mathscr{R}_2$ and $\| T \|_{\mathscr{R}_2} \leqq M \| T \|_{\mathscr{R}_p}$ where M is a constant depending only on p. For if $f \in L^2$ then by Theorem

3 there exists a unitary transformation U commuting with right translations (and therefore with elements of $\mathscr{R}_p$) such that $Uf \in L^p$. We can also choose U such that $\| Uf \|_p \leqq 2B(s) \| f \|_2$ where $B(s)$ is the constant appearing in Lemma 1 and $s > p/2$ (cf. the remarks following the proof of Theorem 4).

We then have that $TUf \in L^p$ and $U^*TUf = TU^*Uf = Tf \in L^2$. Also $\| Tf \|_2 = \| U^*TUf \|_2 \leqq \| TUf \|_2 \leqq \| TUf \|_p \leqq \| T \|_{\mathscr{R}_p} \| Uf \|_p \leqq \| T \|_{\mathscr{R}_p} M \| f \|_2$ where $M = 2B(s)$. Therefore $\| T \|_{\mathscr{R}_2} \leqq M \| T \|_{\mathscr{R}_p}$. This implies that $A^2 \subseteqq A^p$ and $\| \; \|_{A^p} \leqq M \| \; \|_{A^2}$. It is now a simple matter to prove:

THEOREM 6. *Let $E \subseteqq \Gamma$ be a set of irreducible unitary representations of G and $p > 2$. The following are equivalent:*

(a) *E is a set of type $\Lambda(p)$.*

(b) *If $T \in \mathscr{R}_2$ there exists $S \in \mathscr{R}_p$ such that $Tf = Sf$ for all $f \in L^p_E$.*

(c) *If $f \in A^p_E$ then $f \in A^2 = A(G)$.*

(d) *Every closed subspace of L^p_E which is invariant under left translations is the range of a projection P belonging to $\mathscr{R}_p$ which is self-adjoint in the sense that $P_\gamma = P^*_\gamma$ for each $\gamma \in \Gamma$.*

Proof. Let $E \in \Lambda(p)$. Then $L^p_E = L^2_E$ so that by the open mapping theorem there exists B such that $\| f \|_p \leqq B \| f \|_2$ for $f \in L^2_E$. As L^2_E is invariant under right and left translations there exists a projection P_E of L^2 onto L^2_E which commutes with right and left translations. If $T \in \mathscr{R}_2$ let $S = TP_E$, then $\| Sf \|_p \leqq \| T \| \| P_E f \|_p \leqq \| T \| B \| P_E f \|_2 \leqq B \| T \| \| f \|_p$. Thus $S \in \mathscr{R}_p$ and (a) implies (b).

Now assume (b) holds. If $f \in L^2_E$ then by Theorem 3 there exists $U \in \mathscr{R}_2$ such that $Uf \in L^p$; clearly $Uf \in L^p_E$. Let $S \in \mathscr{R}_p$ be such that $Sg = U^*g$ for all $g \in L^p_E$; then $SUf = U^*Uf = f \in L^p$. Hence $L^p_E = L^2_E$ so that (b) implies (a).

We now show that (a) and (b) imply (d). Indeed if (a) holds the projection P_E of L^2 onto L^2_E is bounded in L^p. Suppose the $Y \subseteqq L^p_E$ is invariant under left translations, let P_Y be the projection (belonging to $\mathscr{R}_2$) of L^2 onto the left invariant subspace of L^2 generated by Y. By (b) there exists $S \in \mathscr{R}_p$ with $S = P_Y$ on L^p_E. Then $P_E S = P_Y$ so that $P_Y \in \mathscr{R}_p$.

Suppose (d) holds and let U be a unitary self adjoint element of $\mathscr{R}_2$. Then $U^2 = I$ so that $P = (U + I)/2$ is a projection which commutes with left translations. Let Y be the subspace of L^p_E generated by $PL^2_E \cap L^p_E$. Then Y is invariant under left translations so that by (d) there is a self-adjoint projection of L^p onto Y commuting with left translations. Clearly this projection is PP_E so that $PP_E \in \mathscr{R}_p$. Hence $UP_E = (2P - I)P_E \in \mathscr{R}_p$. Therefore $UP_E f$ is continuous for every $f \in A^p$. In particular if $f(x) \sim \sum_{\gamma \in E} d_\gamma Tr(A_\gamma D_\gamma(x)) \in A^p_E$ then $UP_E f = Uf$ is continuous. Therefore by Remark 4, $A_E^p \subseteq A(G)$.

Finally since $A \subseteq A^p$ with $\|f\|_{A^p} \leq M\|f\|_A$, (c) implies that $A_E = A_E^p$, so that, by the closed graph theorem, $\|f\|_A \leq B\|f\|_{A^p}$ for each $f \in A_E^p$. Each $T \in \mathscr{R}_2$ defines therefore a continuous linear functional on A_E^p by $Tf(e)$. The Hahn-Banach extension of this functional determines, in view of the duality between A^p and $\mathscr{R}_p$, an element $S \in \mathscr{R}_p$ such that $Sf(x) = (SL_xf)(e) = (TL_xf)(e) = Tf(x)$ for $f \in A_E^p$; therefore $S = T$ on L_E^p. Thus (c) implies (b) and the theorem is proved.

REMARK 7. It suffices for condition (d) to be true that every closed left invariant subspace of L_E^p is the range of a projection. Indeed the argument used in [7, Th. 1] will show that such a projection can be chosen to be left invariant (and therefore belonging to $\mathscr{R}_p$).

THEOREM 8. *$E \subseteq \Gamma$ is a Sidon set if and only if for each $T \in \mathscr{R}_2$ there exists a finite measure μ on G such that $Tf = f*\mu$ for each $f \in L_E^2$.*

Proof. One applies the same duality argument used in the proof of Theorem 6 (cf. also [6, 5.7.3]. Assume first that E is a Sidon set. Then given $T \in \mathscr{R}_2$, define a linear functional F on C_E by $F(f) = Tf(e)$. Then F is well defined, since $f \in C_E \Rightarrow f \in A$; by the closed graph theorem F is continuous and has a Hahn-Banach extension to all of $C(G)$. That is, by the Riesz representation theorem there exists a bounded measure μ satisfying

$$F(f) = \int_G f(x^{-1})d\mu(x) \quad \text{for all} \quad f \in C_E .$$

Since T commutes with left translations $Tf = f*\mu$ for all $f \in C_E$. Conversely let E satisfy the hypothesis of the theorem, to prove that E is a Sidon set, let $f \in C_E$ and let T be an unitary element of $\mathscr{R}_2$. By hypothesis there exists a measure μ such that $Tf = f*\mu$. Hence $Tf \in C(G)$ and by Remark 5, $f \in A$.

COROLLARY 9. *Every Sidon set is a $\Lambda(p)$ set for every p.*

Proof. If μ is a bounded measure and $R_\mu f = f*\mu$, then $R_\mu \in \mathscr{R}_p$ for every p. Therefore, by Theorem 8 if E is a Sidon set condition (b) of Theorem 6 holds.

REMARK 10. In [4, 9.2] a sufficient condition for a set $E \subseteq \Gamma$ to be a Sidon set is given. This condition includes the requirements that the degrees of the representations of E be bounded. The fact that for $\prod_{i \in S} U(d_i)$, S is a Sidon set shows that this requirement is not necessary.

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TWO INEQUALITIES IN NONNEGATIVE SYMMETRIC MATRICES

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Marcus and Newman have made the following conjecture: Let $A = (a_{ij})$ be a $n \times n$ nonnegative symmetric matrix. Then

$$S(A)\,S(A^2) \leqq n\,S(A^3)\ ,$$

where

$$S(A) = \sum_{i,j=1}^{n} a_{ij}\ .$$

After reducing the conjecture to a standard maximum problem of linear programming we prove that it holds for $n \leqq 3$. A counter example shows that for $n \geqq 4$ the conjecture is wrong.

We also consider the following conjecture: Let $A = (a_{ij})$ be a $n \times n$ nonnegative symmetric matrix. Then

$$S(A^m) \leqq \sum_{i=1}^{n} s_i^m\ , \qquad m = 1, 2, \cdots\ ,$$

where

$$s_i = \sum_{j=1}^{n} a_{ij}\ , \qquad i = 1, \cdots, n\ .$$

The validity of this conjecture is established in two cases: (1) m up to 5 and any n, (2) n up to 3 and any m. The general case remains open. We conclude this paper with two generalizations of the second theorem.

NOTATION. Let $A = (a_{ij})$ be a $n \times n$ real matrix. A is called *nonnegative* if $a_{ij} \geqq 0, i, j = 1, \cdots, n$. The *quadratic form* corresponding to a symmetric A is denoted by $A(x, x)$, that is

$$A(x, x) = (Ax, x) = \sum_{i,j=1}^{n} a_{ij}x_i x_j\ .$$

Here (Ax, x) denotes, as usually, the *scalar product* of the real vectors x and Ax. Denote $e = (1, \cdots, 1)$ and $Ae = (s_1, \cdots, s_n) = s = s(A)$. $s_i = s_i(A)$ is thus the sum of the elements of the ith row of A. $s = s(A)$ is *the row sums vector* of A. A is *generalized stochastic* if A is nonnegative and if $s(A) = ce$, where c is a scalar. Further

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notations are:

$$S(A) = \sum_{i,j=1}^{n} a_{ij} = A(e, e) ,$$

$$S(x) = \sum_{i=1}^{n} x_i , \qquad x = (x_1, \cdots, x_n) ,$$

$$\left.\begin{aligned} A^m &= (a_{ij}^{(m)}) , \\ s_i^{(m)} &= s_i^{(m)}(A) = s_i(A^m) = \sum_{j=1}^{n} a_{ij}^{(m)} , \\ s^{(m)} &= s^{(m)}(A) = s(A^m) . \end{aligned}\right\} \quad m = 1, 2, \cdots .$$

1. The conjecture of Marcus and Newman.

1.1. The conjecture and its connection with linear programming. In [4, p. 634] the following conjecture is introduced: *Let $A = (a_{ij})$ be a $n \times n$ nonnegative symmetric matrix. Then*

$$S(A)\, S(A^2) \leqq nS(A^3) . \tag{1.1}$$

Using the notation introduced before, we have

$$\left\{\begin{aligned} S(A) &= \sum_{i=1}^{n} s_i , \\ S(A^2) &= \sum_{i=1}^{n} s_i^{(2)} = A^2(e, e) = (Ae, Ae) = \sum_{i=1}^{n} s_i^2 , \\ S(A^3) &= \sum_{i=1}^{n} s_i^{(3)} = A^3(e, e) = (Ae, A^2e) = \sum_{i=1}^{n} s_i s_i^{(2)} . \end{aligned}\right. \tag{1.2}$$

Hence, (1.1) can be written in the form

$$n \sum_{i=1}^{n} s_i s_i^{(2)} - \sum_{i=1}^{n} s_i \sum_{i=1}^{n} s_i^{(2)} \geqq 0 . \tag{1.3}$$

If the sets $s = (s_1, \cdots, s_n)$ and $s^{(2)} = (s_1^{(2)}, \cdots, s_n^{(2)})$ are *similarly ordered*, that is if $(s_i - s_j)(s_i^{(2)} - s_j^{(2)}) \geqq 0$ for every $1 \leqq i, j \leqq n$, then according to an inequality of Tchebychef [2, p. 43] the inequality (1.3) holds. However, the following example shows that for nonnegative symmetric matrices A, $s(A)$ and $s^{(2)}(A)$ need not be similarly ordered. Let

$$A = \begin{bmatrix} 6 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix} .$$

Then $s(A) = (8, 3, 4)$ and $s^{(2)}(A) = (54, 19, 16)$. $s(A)$ and $s^{(2)}(A)$ are therefore not similarly ordered.

Denote

$$\sum_{j=1}^{n}{}' a_{ij} = \sum_{j=1}^{n} a_{ij} - a_{ii} .$$

We have

$$a_{ii} = s_i - \sum_{j=1}^{n}{}' a_{ij} , \qquad i = 1, \cdots, n , \tag{1.4}$$

$$s_i^{(2)} = \sum_{j=1}^{n} a_{ij} s_j , \qquad i = 1, \cdots, n . \tag{1.5}$$

From (1.2), (1.4) and (1.5) follows

$$\begin{aligned} nS(A^3) - S(A)\,S(A^2) &= n \sum_{i=1}^{n} s_i^{(2)} s_i - \sum_{i=1}^{n} s_i \sum_{i=1}^{n} s_i^{(2)} \\ &= n \sum_{i=1}^{n} s_i \Big[\sum_{j=1}^{n}{}' a_{ij} s_j + s_i \Big(s_i - \sum_{j=1}^{n}{}' a_{ij} \Big) \Big] - \sum_{i=1}^{n} s_i \sum_{i=1}^{n} s_i^2 \\ &= n \sum_{i=1}^{n} s_i^3 - \sum_{i=1}^{n} s_i \sum_{i=1}^{n} s_i^2 - n \sum_{1 \leqq i < j}^{n} a_{ij} (s_i - s_j)^2 \\ &= \sum_{1 \leqq i < j}^{n} (s_i - s_j)(s_i^2 - s_j^2) - n \sum_{1 \leqq i < j}^{n} a_{ij} (s_i - s_j)^2 . \end{aligned}$$

Hence,

$$\begin{aligned} nS(A^3) &- S(A)\,S(A^2) \\ &= \sum_{1 \leqq i < j}^{n} (s_i + s_j)(s_i - s_j)^2 - n \sum_{1 \leqq i < j}^{n} a_{ij} (s_i - s_j)^2 . \end{aligned} \tag{1.6}$$

Using (1.6) we obtain a representation of the conjecture (1.1) by concepts of linear programming (see e.g. Gale [1]). Consider the following maximum problem: Let $s_1, \cdots, s_n$ be nonnegative numbers. Find numbers $a_{ij} = a_{ji}$, $i \neq j$; $i, j = 1, \cdots, n$, which satisfy the set of linear inequalities

$$\begin{cases} a_{ij} = a_{ji} \geqq 0, \; i \neq j ; & i, j = 1, \cdots, n , \\ \sum_{j=1}^{n}{}' a_{ij} \leqq s_i , & i = 1, \cdots, n , \end{cases} \tag{1.7}$$

and which maximize the linear function

$$\sum_{1 \leqq i < j}^{n} a_{ij} (s_i - s_j)^2 . \tag{1.8}$$

The problem (1.7), (1.8) is *a maximum standard problem of linear programming*. A set of numbers a_{ij} which satisfies the inequalities (1.7) is a *feasible solution of the problem*. A feasible solution which maximizes (1.8) is an *optimal solution*. The *dual* of the problem (1.7), (1.8) is the following *minimum standard problem*: Find numbers $y_1, \cdots, y_n$ which satisfy the set of inequalities

$$\begin{cases} y_i \geqq 0, \; i = 1, \cdots, n, \\ y_i + y_j \geqq (s_i - s_j)^2, \; i \neq j ; & i, j = 1, \cdots, n , \end{cases} \tag{1.7'}$$

and which minimize the function

$$\sum_{i=1}^{n} s_i y_i \; . \tag{1.8'}$$

It is obvious that the problem (1.7), (1.8) and its dual have optimal solutions.

From (1.6) it follows that the conjecture (1.1) can be represented in the following equivalent form: *Let* $\tilde{a}_{ij}$, $i \neq j$; $i, j = 1, \cdots, n$, *be an optimal solution of the maximum standard problem* (1.7), (1.8). *Then*

$$\sum_{1 \leqq i < j}^{n} \tilde{a}_{ij}(s_i - s_j)^2 \leqq \frac{1}{n} \sum_{1 \leqq i < j}^{n} (s_i + s_j)(s_i - s_j)^2 \; . \tag{1.9}$$

1.2. Proof for $n \leqq 3$. In this section we establish the validity of the conjecture for $n \leqq 3$.

THEOREM 1. *Let A be a $n \times n$ nonnegative symmetric matrix. Then for $n \leqq 3$ the inequality* (1.1) *holds. The equality sign holds in* (1.1) *if and only if A or A^2 is a generalized stochastic matrix.*

Proof. For $n = 1$ the inequality (1.1) holds trivially. For $n = 2, 3$ we use the representation of (1.1) by (1.9).

For $n = 2$ it is sufficient to prove that if

$$0 \leqq a_{12} \leqq \min (s_1, s_2) \; , \tag{1.10}$$

then

$$a_{12}(s_1 - s_2)^2 \leqq \frac{1}{2}(s_1 + s_2)(s_1 - s_2)^2 \; . \tag{1.11}$$

(1.10) implies

$$a_{12} \leqq \frac{s_1 + s_2}{2} \; , \tag{1.12}$$

and from (1.12) follows (1.11). Equality holds in (1.1) if and only if it holds in (1.11), and there it holds if and only if $s_1 = s_2$, that is if A is a generalized stochastic matrix. As by (1.3) we clearly have equality in (1.1) if A^2 is generalized stochastic, it follows that there are not nonnegative symmetric 2×2 matrices such that A^2 but not A is generalized stochastic. We remark that it is easily seen that for $n = 2$, s and $s^{(2)}$ are similarly ordered sets. (1.1) thus follows also from the inequality of Tchebychef.[1]

[1] As the referee suggests, the proof for $n = 2$ can be done directly by the methods in [4]. Using the notations in [4], we have

$$2S(A^3) - S(A)S(A^2) = w_1 w_2 (\lambda_1 - \lambda_2)^2 (\lambda_1 + \lambda_2) \quad \text{and} \quad \lambda_1 + \lambda_2 = tr(A) \geqq 0 \; .$$

The author wishes to thank the referee for this remark.

We prove now the theorem for $n = 3$. Without loss of generality we may assume that

$$0 < s_1 \leqq s_2 \leqq s_3 . \tag{1.13}$$

The assumption $0 < s_1$ does not restrict the generality. If $s_1 = 0$ then A is of the form

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & B & \\ 0 & & \end{bmatrix} .$$

Hence, using the validity of (1.1) for $n = 2$, we obtain

$$S(A)\, S(A^2) \leqq 2S(A^3) .$$

At first we treat the case

$$0 < s_1 < s_2 < s_3 . \tag{1.14}$$

Denote

$$a_{23} = a_{32} = x_1 , \qquad a_{13} = a_{31} = x_2 , \qquad a_{12} = a_{21} = x_3 .$$

The corresponding maximum problem is: Maximize

$$M(x_1, x_2, x_3) = x_1(s_2 - s_3)^2 + x_2(s_1 - s_3)^2 + x_3(s_1 - s_2)^2 , \tag{1.15}$$

where $x_i \geqq 0$, $i = 1, 2, 3$, satisfy the system of inequalities

$$\begin{cases} (1) \quad x_2 + x_3 \leqq s_1 \\ (2) \quad x_1 + x_3 \leqq s_2 \\ (3) \quad x_1 + x_2 \leqq s_3 . \end{cases} \tag{1.16}$$

The dual of the problem (1.15), (1.16) is the following problem: Minimize

$$y_1 s_1 + y_2 s_2 + y_3 s_3 , \tag{1.15'}$$

where $y_i \geqq 0$, $i = 1, 2, 3$, satisfy the system of inequalities

$$\begin{cases} y_2 + y_3 \geqq (s_2 - s_3)^2 \\ y_1 + y_3 \geqq (s_1 - s_3)^2 \\ y_1 + y_2 \geqq (s_1 - s_2)^2 . \end{cases} \tag{1.16'}$$

Let $\tilde{x}_1, \tilde{x}_2, \tilde{x}_3$ be an optimal solution of (1.15), (1.16) and $\tilde{y}_1, \tilde{y}_2, \tilde{y}_3$ an optimal solution of the dual problem. Let $\widetilde{(1.16)}$, $\widetilde{(1.16')}$ denote respectively the inequalities (1.16), (1.16′) after substituting $\tilde{x}_1, \tilde{x}_2, \tilde{x}_3$ and $\tilde{y}_1, \tilde{y}_2, \tilde{y}_3$ respectively.

According to our assumption (1.14) we have

$$(s_i - s_j)^2 > 0,\ i \neq j;\ i, j = 1, 2, 3,$$

and it follows therefore from $\widetilde{(1.16')}$ that at most one of the numbers $\tilde{y}_1, \tilde{y}_2, \tilde{y}_3$ is equal to zero. From the *equilibrium theorem* [1, p. 19] follows that in $\widetilde{(1.16)}$ equality holds at least in two of the inequalities. In $\widetilde{(1.16')}$ at least one strict inequality holds. For if three equalities hold then by solving the system of equations we get $\tilde{y}_2 < 0$, and so the solution is not feasible. Using again the equilibrium theorem we obtain that at least one of the numbers is equal to zero. As (1.14) holds, it follows that precisely one of those numbers is equal to zero. Summing up: In $\widetilde{(1.16)}$ the sign of equality holds at least twice and precisely one of the numbers $\tilde{x}_1, \tilde{x}_2, \tilde{x}_3$ vanishes.

We now consider all the sets x_1, x_2, x_3 for which the just obtained conditions hold. For every such set we decide whether it is a feasible solution (f.s) or whether it is not a feasible solution (n.f.s). For this decision we have to distinguish between the two following cases

$$(1.17) \qquad s_1 + s_2 \leqq s_3\ ,$$

$$(1.18) \qquad s_1 + s_2 \geqq s_3\ .$$

The result is given in the following table.

equality in (1.16) in the equations	x_1	x_2	x_3	case (1.17)	case (1.18)
(1), (2)	0	$s_1 - s_2$	s_2	n.f.s	n.f.s
(1), (2)	$s_2 - s_1$	0	s_1	f.s	f.s
(1), (2)	s_2	s_1	0	f.s	n.f.s
(1), (3)	0	s_3	$s_1 - s_3$	n.f.s	n.f.s
(1), (3)	s_3	0	s_1	n.f.s	n.f.s
(1), (3)	$s_3 - s_1$	s_1	0	n.f.s	f.s
(2), (3)	0	s_3	s_2	n.f.s	n.f.s
(2), (3)	s_3	0	$s_2 - s_3$	n.f.s	n.f.s
(2), (3)	s_2	$s_3 - s_2$	0	n.f.s	f.s

For any row of this table containing a f.s, the limit case $s_1 + s_2 = s_3$ is to be associated with this f.s.

When (1.17) holds, the optimal solution is one of the following feasible solutions

$$(x_1, x_2, x_3) = (s_2 - s_1, 0, s_1)\ ,$$
$$(x_1, x_2, x_3) = (s_2, s_1, 0)\ .$$

As

$$M(s_2 - s_1, 0, s_1) = (s_2 - s_1)(s_2 - s_3)^2 + s_1(s_1 - s_2)^2$$
$$< s_2(s_2 - s_3)^2 + s_1(s_1 - s_3)^2 = M(s_2, s_1, 0) ,$$

it follows that

$$(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3) = (s_2, s_1, 0) . \tag{1.19}$$

This optimal solution is unique.

When (1.18) holds, the optimal solution is one of the following feasible solutions

$$(x_1, x_2, x_3) = (s_2 - s_1, 0, s_1) ,$$
$$(x_1, x_2, x_3) = (s_3 - s_1, s_1, 0) ,$$
$$(x_1, x_2, x_3) = (s_2, s_3 - s_2, 0) .$$

As

$$M(s_3 - s_1, s_1, 0) - M(s_2, s_3 - s_2, 0) = (s_3 - s_1 - s_2)(s_2 - s_3)^2 + (s_1 + s_2 - s_3)(s_1 - s_3)^2 \geqq 0 \tag{1.20}$$

and

$$M(s_3 - s_1, s_1, 0) = (s_3 - s_1)(s_2 - s_3)^2 + s_1(s_1 - s_3)^2$$
$$> (s_2 - s_1)(s_2 - s_3)^2 + s_1(s_2 - s_1)^2 = M(s_2 - s_1, 0, s_1) ,$$

it follows that

$$(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3) = (s_3 - s_1, s_1, 0) . \tag{1.21}$$

As equality in (1.20) holds only if $s_1 + s_2 = s_3$, it follows that the optimal solution (1.21) is unique. We remark that the optimal solution can also be determined by the simplex method [1, ch. 4].

According to (1.9), (1.19) and (1.21) we have to prove that

$$M(s_2, s_1, 0) = s_2(s_2 - s_3)^2 + s_1(s_1 - s_3)^2 < \frac{1}{3} \sum_{1 \leqq i < j}^{3} (s_i + s_j)(s_i - s_j)^2 \tag{1.22}$$

when (1.17) holds, and that

$$M(s_3 - s_1, s_1, 0) = (s_3 - s_1)(s_2 - s_3)^2 + s_1(s_1 - s_3)^2 < \frac{1}{3} \sum_{1 \leqq i < j}^{3} (s_i + s_j)(s_i - s_j)^2 \tag{1.23}$$

when (1.18) holds.

Denote

$$s_1 = \alpha, \ s_2 = \alpha + \beta, \ s_3 = \alpha + \beta + \gamma . \tag{1.24}$$

The assumption (1.14) implies

$$\alpha, \beta, \gamma > 0 . \tag{1.25}$$

Assuming the validity of (1.17), we prove now that (1.22) holds. (1.17) gives

$$\alpha \leqq \gamma . \tag{1.26}$$

Denote

$$\begin{aligned} I_1 &= \sum_{1 \leqq i < j}^{3} (s_i + s_j)(s_i - s_j)^2 - 3M(s_2, s_1, 0) \\ &= (s_1 - s_2)^2(s_1 + s_2) + (s_2 - s_3)^2(s_3 - 2s_2) + (s_3 - s_1)^2(s_3 - 2s_1) . \end{aligned}$$

By the notation of (1.24) I_1 takes the form

$$I_1 = \beta^2(2\alpha + \beta) + \gamma^2(\gamma - \alpha - \beta) + (\beta + \gamma)^2(\beta + \gamma - \alpha) . \tag{1.27}$$

From (1.25), (1.26) and (1.27) follows

$$I_1 > \beta^2(2\alpha + \beta) + \gamma^2(2\gamma - 2\alpha) > 0 .$$

(1.22) is thus established.

Assuming the validity of (1.18), we prove that (1.23) holds. (1.18) gives

$$\alpha \geqq \gamma . \tag{1.28}$$

Denote

$$\begin{aligned} I_2 &= \sum_{1 \leqq i < j}^{3} (s_i + s_j)(s_i - s_j)^2 - 3M(s_3 - s_1, s_1, 0) \\ &= (s_1 - s_2)^2(s_1 + s_2) + (s_2 - s_3)^2(3s_1 + s_2 - 2s_3) + (s_1 - s_3)^2(s_3 - 2s_1) . \end{aligned}$$

By the notation of (1.24) I_2 takes the form

$$I_2(\alpha, \beta, \gamma) = \beta^2(2\alpha + \beta) + \gamma^2(2\alpha - \beta - 2\gamma) + (\beta + \gamma)^2(\beta + \gamma - \alpha) . \tag{1.29}$$

We distinguish between the following two cases

$$\beta + \gamma \geqq \alpha , \tag{1.30}$$

$$\beta + \gamma < \alpha . \tag{1.31}$$

At first assume that (1.30) holds. From (1.25), (1.28), (1.29) and (1.30) we obtain

$$\begin{aligned} I_2 &\geqq \beta^2(2\alpha + \beta) + \gamma^2(2\alpha - \beta - 2\gamma) + \gamma^2(\beta + \gamma - \alpha) \\ &= \beta^2(2\alpha + \beta) + \gamma^2(\alpha - \gamma) > 0 . \end{aligned}$$

(1.23) is thus established when (1.30) holds. Assume now that (1.31) holds. Write $I_2(\alpha, \beta, \gamma)$ in the following form

$$I_2(\alpha, \beta, \gamma) = \alpha(\beta - \gamma)^2 + \beta^3 + (\beta + \gamma)^3 - \gamma^2(\beta + 2\gamma) . \tag{1.32}$$

$I_2(\alpha, \beta, \gamma)$ is linear in α. Let β, γ be any constant positive numbers. As

$$I_2(\alpha, \gamma, \gamma) = 6\gamma^3 > 0 ,$$

we may assume that

$$(\beta - \gamma)^2 > 0 . \tag{1.33}$$

Using the validity of (1.23) when (1.30) holds, we obtain

$$I_2(\beta + \gamma, \beta, \gamma) > 0 . \tag{1.34}$$

From (1.32) and (1.33) it follows that

$$\lim_{\alpha \to +\infty} I_2(\alpha, \beta, \gamma) = +\infty . \tag{1.35}$$

As $I_2(\alpha, \beta, \gamma)$ is linear in α, it follows from (1.34) and (1.35) that $I_2(\alpha, \beta, \gamma) > 0$ when (1.31) holds. (1.23) is thus established also when (1.31) holds.

The proof of the theorem is completed in the case when (1.14) holds. We proved that in this case (1.1) holds strictly. From continuity considerations it follows that the theorem without the equality statement holds also if only (1.13) is assumed. (We have already mentioned that (1.13) can be considered as the general case). Hence, to complete our proof in the general case (1.13), we have to assume that (1.14) is invalidated and to check for possible cases of equality in (1.1). If (1.14) does not hold, there are three possibilities:

(1) $s_1 = s_2 = s_3$,
(2) $s_1 < s_2 = s_3$,
(3) $s_1 = s_2 < s_3$.

If (1) holds then the sign of equality in (1.1) holds for every A. In this case A is a generalized stochastic matrix.

In cases (2) and (3) we consider the corresponding maximum problems.

The maximum problem corresponding to (2) is: Maximize

$$M(x_1, x_2, x_3) = (s_1 - s_3)^2(x_2 + x_3) ,$$

where $x_i \geqq 0$, $i = 1, 2, 3$, satisfy the three inequalities

$$\begin{cases} x_2 + x_3 \leqq s_1 \\ x_1 + x_3 \leqq s_3 \\ x_1 + x_2 \leqq s_3 . \end{cases}$$

It is obvious that every feasible solution for which $x_2 + x_3 = s_1$ is an optimal solution. So there are infinitely many optimal solutions. If in this case the sign of equality holds in (1.1), then

$$s_1(s_1 - s_3)^2 = \frac{2}{3}(s_1 - s_3)^2(s_1 + s_3) ,$$

and therefore

$$s_1 = 2s_3 .$$

As the last equality contradicts (1.13), we conclude that in the case (2) strict inequality holds in (1.1).

The maximum problem corresponding to (3) is: Maximize

$$M(x_1, x_2, x_3) = (x_1 + x_2)(s_1 - s_3)^2 ,$$

where $x_i \geqq 0, i = 1, 2, 3,$ satisfy the three inequalities

$$\begin{cases} x_2 + x_3 \leqq s_1 \\ x_1 + x_3 \leqq s_1 \\ x_1 + x_2 \leqq s_3 \end{cases}$$

In order to determine optimal solutions of the problem, we have to distinguish between the following two cases

$$(3)_I \quad 2s_1 \leqq s_3 ,$$
$$(3)_{II} \quad 2s_1 > s_3 .$$

If $(3)_I$ holds then the only optimal solution is

$$(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3) = (s_1, s_1, 0) .$$

If $(3)_{II}$ holds then every feasible solution for which $x_1 + x_2 = s_3$ is an optimal solution. In this case there are infinitely many optimal solutions. If the sign of equality in (1.1) holds in the case $(3)_I$ then

$$2s_1(s_1 - s_3)^2 = \frac{2}{3}(s_1 + s_3)(s_1 - s_3)^2 ,$$

and therefore

$$(1.36) \qquad 2s_1 = s_3 .$$

If the sign of equality in (1.1) holds in the case $(3)_{II}$ then

$$s_3(s_1 - s_3)^2 = \frac{2}{3}(s_1 + s_3)(s_1 - s_3)^2 ,$$

and (1.36) is obtained again. As (1.36) contradicts $(3)_{II}$, it follows that in this case equality in (1.1) is excluded. Hence, in case (3) equality in (1.1) holds if and only if

$$s_1 = s_2,\ s_3 = 2s_1,\ (\tilde{x}_1, \tilde{x}_2, \tilde{x}_3) = (s_1, s_1, 0)\ ,$$

that is only for the matrix

$$A = \begin{bmatrix} 0 & 0 & s_1 \\ 0 & 0 & s_1 \\ s_1 & s_1 & 0 \end{bmatrix}. \tag{1.37}$$

A^2 is a generalized stochastic matrix (while A is not stochastic). It follows from (1.3) that if A or A^2 is a generalized stochastic matrix then equality in (1.1) holds. Hence, it follows that equality in (1.1) holds if and only if A or A^2 is a generalized stochastic matrix. This completes the proof of the theorem.

REMARK 1. The following example proves that the assumption of symmetry in Theorem 1 is essential. Let

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix}.$$

A is a positive nonsymmetric matrix. As

$$S(A) = 10,\ S(A^2) = 32,\ S(A^3) = 100\ ,$$

(1.1) does not hold.

It is obvious that (1.1) does not hold in general for real symmetric matrices with (some) negative elements. However, going over to the absolute values and denoting $|A| = (|a_{ij}|)$, one may think that for all $n \times n$, $n \leqq 3$, symmetric matrices

$$S(|A|)\, S(|A^2|) \leqq n\, S(|A^3|) \tag{1.1'}$$

holds. The following counter example shows that this is wrong. Let

$$A = \begin{bmatrix} 1 & -2 & 1 \\ -2 & 0 & 2 \\ 1 & 2 & -1 \end{bmatrix}.$$

As

$$S(|A|) = 12\ , \qquad S(|A^2|) = 36\ , \qquad S(|A^3|) = 128\ ,$$

(1.1′) does not hold.

REMARK 2. Let A be a 3×3 nonnegative symmetric matrix. Let r_1, r_2, r_3 be an orthonormal system of characteristic vectors of A corresponding respectively to the characteristic values $\alpha_1, \alpha_2, \alpha_3$. Let R be the orthogonal matrix with the columns r_1, r_2, r_3. As $A = RDR^T$, where D is the diagonal matrix $\{\alpha_1, \alpha_2, \alpha_3\}$ and R^T is the transposed of R, we have

$$S(A^m) = (A^m e, e) = (D^m R^T e, R^T e) = \sum_{i=1}^{3} \alpha_i^m [S(r_i)]^2 .$$

Hence, (1.1) for $n = 3$ is transformed to

$$\sum_{i=1}^{3} \alpha_i [S(r_i)]^2 \sum_{i=1}^{3} \alpha_i^2 [S(r_i)]^2 \leqq 3 \sum_{i=1}^{3} \alpha_i^3 [S(r_i)]^2 . \tag{1.38}$$

(1.38) is a necessary condition for a system of 3 orthonormal vectors r_1, r_2, r_3 and three real numbers $\alpha_1, \alpha_2, \alpha_3$ to be respectively a system of characteristic vectors and values of a 3×3 nonnegative symmetric matrix. It would be interesting to find similar necessary (or sufficient) conditions concerning $n \times n$ nonnegative symmetric matrices.

REMARK 3. From the considerations concerning the equality sign in the proof of Theorem 1 we conclude: *Let A be a 3×3 nonnegative symmetric matrix satisfying* (1.13). *A is not generalized stochastic while A^2 is generalized stochastic if and only if A is of the form* (1.37). In a recent paper [3] we characterize the matrices of this type for every n.

1.3. Counter example for $n \geqq 4$. In this section we bring a counter example which shows that for $n \geqq 4$ the conjecture of Marcus and Newman does not hold. Let

$$A_n = A_n(\alpha) = \begin{bmatrix} \alpha & 0 & - & - & 0 \\ 0 & 0 & & 0 & 1 \\ | & | & | & | & | \\ | & | & | & | & | \\ | & | & | & | & | \\ | & | & | & | & | \\ | & 0 & - & 0 & 1 \\ 0 & 1 & - & 1 & 0 \end{bmatrix} = \begin{bmatrix} \alpha & 0 \text{------} 0 \\ 0 & \\ | & \\ | & B_{n-1} \\ | & \\ 0 & \end{bmatrix}, n \geqq 4 . \tag{1.39}$$

$A_n(\alpha)$ is a $n \times n$ symmetric matrix depending on the real parameter α. For $\alpha \geqq 0$ $A_n(\alpha)$ is nonnegative. B_{n-1} is a $(n-1) \times (n-1)$ nonnegative symmetric matrix. B_{n-1}^2 is generalized stochastic (while B_{n-1} is not generalized stochastic). As

$$s(B_{n-1}^2) = (n-2)e \ , \qquad S(B_{n-1}^2) = (n-1)(n-2) \ ,$$
$$S(B_{n-1}) = 2(n-2) \ , \qquad S(B_{n-1}^3) = 2(n-2)^2 \ ,$$

we obtain

$$\begin{aligned} n\,S(A_n^3) - S(A_n)\,S(A_n^2) &= n[2(n-2)^2 + \alpha^3] \\ &\quad -[2(n-2)+\alpha][(n-1)(n-2)+\alpha^2] \\ &= [\alpha^2 - (n-2)][(n-1)\alpha - 2(n-2)] = f_n(\alpha) \ . \end{aligned}$$

The zeros of the polynomial $f_n(\alpha)$ are

$$\alpha_1 = -\sqrt{n-2} \ , \quad \alpha_2 = \frac{2(n-2)}{n-1} \ , \quad \alpha_3 = +\sqrt{n-2} \ ,$$

and therefore $f_n(\alpha) < 0$ for

$$\frac{2(n-2)}{n-1} < \alpha < \sqrt{n-2} \ . \tag{1.40}$$

Hence, for every α satisfying (1.40) the inequality (1.1) does not hold.

REMARK. Consider the following generalization of conjecture (1.1): *Let A be a $n \times n$ nonnegative symmetric matrix. Then*

$$S(A)\,S(A^m) \leqq n\,S(A^{m+1}) \ , \qquad m = 1, 2, \cdots \ . \tag{1.41}$$

For odd m (1.41) holds for every symmetric A [4, Th. 4]. For even m and $n \geqq 4$ a straightforward computation proves that (1.41) does not hold for the matrices (1.39), for α satisfying (1.40). For $m = 2$ and $n \leqq 3$ the validity of (1.41) is established in Theorem 1. For even $m > 2$ and $n = 3$ the problem remains open.

2. Upper bound for the sum of the elements of a power of a matrix.

2.1. A conjecture. In this section we state a conjecture which yields an upper bound for the sum of the elements of a power of a nonnegative symmetric matrix.

We first define a class of matrices: *Let $s = (s_1, \cdots, s_n)$ be a vector for which the condition*

$$0 < s_1 < s_2 < \cdots < s_n \tag{2.1}$$

holds. Denote by $\mathscr{A}_n(s)$ the class of all $n \times n$ nonnegative symmetric matrices for which $s(A) = s$.

By a straightforward computation, using (1.2), (1.4) and (1.5), we obtain

$$S(A^3) = \sum_{i=1}^{n} s_i^3 - \sum_{1 \leqq i < j}^{n} a_{ij}(s_i - s_j)^2 . \tag{2.2}$$

From (2.2) it follows that for every $A \in \mathscr{A}_n(s)$ the inequality

$$S(A^3) \leqq \sum_{i=1}^{n} s_i^3 \tag{2.3}$$

holds. Equality in (2.3) holds if and only if A is the diagonal matrix in $\mathscr{A}_n(s)$.

The following conjecture generalizes (2.3): *For every* $A \in \mathscr{A}_n(s)$ *the inequality*

$$S(A^m) \leqq \sum_{i=1}^{n} s_i^m , \qquad m = 3, 4, \cdots \tag{2.4}$$

holds. Equality in (2.4) *holds if and only if* A *is the diagonal matrix in* $\mathscr{A}_n(s)$.

REMARK 1. For $m = 1, 2$ (2.4) holds with equality sign for every $A \in \mathscr{A}_n(s)$. This is the reason why we did not include $m = 1, 2$ in our formulation of the conjecture.

REMARK 2. In the definition of the class $\mathscr{A}_n(s)$ we assumed that $s(A)$ satisfies (2.1). If we ommit this assumption only the equality statement of the conjecture is to be changed.

2.2. Proof for particular cases. In this section we prove some particular cases of the conjecture. The general case remains open.

THEOREM 2. *In the following two cases*
(1) $m = 3, 4, 5$ *and* $n = 1, 2, \cdots$
(2) $m = 3, 4, \cdots$ *and* $n = 1, 2, 3$
the inequality

$$S(A^m) \leqq \sum_{i=1}^{n} s_i^m \tag{2.4'}$$

holds for every $A \in \mathscr{A}_n(s)$. *The equality sign in these two cases holds only for the diagonal matrix in* $\mathscr{A}_n(s)$.

Proof. Let $A = (a_{ij}) \in \mathscr{A}_n(s)$. Assume that there exists an i, $1 \leqq i < n$, for which $a_{ni} > 0$. Define

$$A(\varepsilon) = \begin{bmatrix} a_{11} & \cdots & & \cdots & a_{1n} \\ \vdots & \ddots \text{ like} & \vdots & & \vdots \\ a_{i1} & \cdots & a_{ii} + \varepsilon & \cdots & a_{in} - \varepsilon \\ \vdots & & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{ni} - \varepsilon & \cdots & a_{nn} + \varepsilon \end{bmatrix} . \tag{2.5}$$

Here ε is a nonnegative parameter. For small $\varepsilon A(\varepsilon) \in \mathscr{A}_n(s)$. We have

$$\frac{dS[A^m(\varepsilon)]}{d\varepsilon}\bigg|_{\varepsilon=0} = S\left[\frac{dA^m(\varepsilon)}{d\varepsilon}\right]_{\varepsilon=0} = \sum_{k=0}^{m-1} S[A^k A'(0) A^{m-k-1}]$$
$$= \sum_{k=0}^{m-1} (A'(0)A^{m-k-1}e, A^k e) = \sum_{k=0}^{m-1} (A'(0)s^{(m-k-1)}(A), s^{(k)}(A))$$
$$= \sum_{k=0}^{m-1} [s_i^{(m-k-1)}(A) - s_n^{(m-k-1)}(A)][s_i^{(k)}(A) - s_n^{(k)}(A)] .$$

Hence,

$$\frac{d\, S[A^m(\varepsilon)]}{d\varepsilon}\bigg|_{\varepsilon=0} = \sum_{k=1}^{m-2} [s_i^{(m-k-1)}(A) - s_n^{(m-k-1)}(A)][s_i^{(k)}(A) - s_n^{(k)}(A)] . \tag{2.6}$$

Let us first bring the proof for the case (1). Let $\tilde{A} = \tilde{A}(m) = (\tilde{a}_{ij})$, $m = 3, 4, 5$, be an optimal matrix of the maximum problem

$$\operatorname*{Max}_{A \in \mathscr{A}_n(s)} S(A^m) .$$

For a fixed m, $m = 3, 4, 5$, we use induction on n. For $n = 1$ the theorem holds trivially. Suppose that the theorem holds for $n - 1$ (and the same fixed m). We prove shortly that the optimal $n \times n$ matrix $\tilde{A}$ has the following structure

$$\tilde{A} = \tilde{A}(m) = \begin{bmatrix} & & 0 \\ & B & \vdots \\ & & 0 \\ 0 & \cdots\ 0 & s_n \end{bmatrix} \tag{2.7}$$

B is a $(n-1) \times (n-1)$ nonnegative symmetric matrix and $s(B) = (s_1, \cdots, s_{n-1})$. Suppose that we have already proved that $\tilde{A}$ has the structure (2.7). By the induction assumption

$$S(B^m) \leqq \sum_{i=1}^{n-1} s_i^m$$

and equality holds only if B is diagonal. Hence,

$$S(\tilde{A}^m) = S(B^m) + s_n^m \leqq \sum_{i=1}^{n} s_i^m . \tag{2.8}$$

Equality in (2.8) holds if and only if $\tilde{A}$ is a diagonal matrix.

It remains to prove that $\tilde{A}$ has the structure (2.7). Assume that $\tilde{A}$ has not the above structure. There exists at least one i, $1 \leqq i \leqq n-1$, for which $\tilde{a}_{ni} > 0$. For this i the matrix $\tilde{A}(\varepsilon)$ is defined according to (2.5). As $\tilde{A}$ is an optimal matrix of the above defined maximum problem, and as for a small enough $\varepsilon > 0$ $\tilde{A}(\varepsilon) \in \mathscr{A}_n(s)$, the inequality

$$\left.\frac{Sd[\tilde{A}^m(\varepsilon)]}{d\varepsilon}\right|_{\varepsilon=0} \leqq 0 \tag{2.9}$$

must hold. From (2.6) and (2.9) we obtain

$$\sum_{k=1}^{m-2} [s_i^{(k)}(\tilde{A}) - s_n^{(k)}(\tilde{A})][s_i^{(m-k-1)}(\tilde{A}) - s_n^{(m-k-1)}(\tilde{A})] \leqq 0 . \tag{2.10}$$

We now consider separately the cases $m = 3, 4, 5$. By a suitable choice of i we obtain a contradiction to (2.10).

$m = 3$. For this case the theorem has already been proved by the representation (2.2). We give here an independent proof. Choose any $i, 1 \leqq i \leqq n-1$, for which $\tilde{a}_{ni} > 0$. According to our assumption there exists such an i. By (2.10) we obtain for this i

$$[s_i(\tilde{A}) - s_n(\tilde{A})]^2 = (s_i - s_n)^2 \leqq 0 . \tag{2.11}$$

(2.11) contradicts (2.1).

$m = 4$. Let $i, 1 \leqq i \leqq n-1$, be the smallest index for which $\tilde{a}_{ni} > 0$. According to our assumption there exists such an i. By (2.10) we obtain for this i

$$(s_n - s_i)[s_n^{(2)}(\tilde{A}) - s_i^{(2)}(\tilde{A})] \leqq 0 . \tag{2.12}$$

We have

$$\tilde{A}s = s^{(2)}(\tilde{A}) .$$

By (2.1) and by our choice of i we obtain

$$s_i^{(2)}(\tilde{A}) = \sum_{j=1}^{n} \tilde{a}_{ij}s_j \leqq s_i s_n , \tag{2.13}$$

$$s_n^{(2)}(\tilde{A}) = \sum_{j=1}^{n} \tilde{a}_{nj}s_j \geqq s_i s_n . \tag{2.14}$$

Hence,

(2.15) $$s_i^{(2)}(\tilde{A}) \leqq s_n^{(2)}(\tilde{A}) .$$

Equality in (2.15) implies equality in (2.13) and (2.14). Equality in (2.13) holds if and only if

$$\tilde{a}_{i1} = \cdots = \tilde{a}_{i,n-1} = 0,\ \tilde{a}_{in} = s_i .$$

Equality in (2.14) holds if and only if

$$\tilde{a}_{n,i+1} = \cdots = \tilde{a}_{nn} = 0, \tilde{a}_{ni} = s_n .$$

Hence,

(2.16) $$\tilde{a}_{in} = \tilde{a}_{ni} = s_i = s_n .$$

(2.16) contradicts (2.1) and therefore (2.15) holds strictly. (2.1) and the strict inequality in (2.15) contradict (2.12).

$m = 5$. From the set of all the indices $i, 1 \leqq i \leqq n,$ for which $\tilde{a}_{ni} > 0$ choose that i for which $s_i^{(2)}(\tilde{A})$ attains its minimum value. According to our assumption there exists an $i, 1 \leqq i < n,$ for which $\tilde{a}_{ni} > 0$. As we saw in the proof for $m = 4$, there exists an $i, 1 \leqq i < n$, which satisfies $\tilde{a}_{ni} > 0$ and for which strict inequality holds in (2.15). It follows that the i chosen now satisfies $i < n$. By (2.10) we obtain for this i

(2.17) $$2(s_n - s_i)[s_n^{(3)}(\tilde{A}) - s_i^{(3)}(\tilde{A})] + [s_n^{(2)}(\tilde{A}) - s_i^{(2)}(\tilde{A})]^2 \leqq 0 .$$

We have

$$s^{(3)}(\tilde{A}) = \tilde{A}^3 e = \tilde{A} s^{(2)}(\tilde{A}) = \tilde{A}^2 s(\tilde{A}) .$$

By (2.1) and by our choice of i we obtain

(2.18) $$s_n^{(3)}(\tilde{A}) = \sum_{j=1}^{n} \tilde{a}_{nj} s_j^{(2)} \geqq s_i^{(2)}(\tilde{A}) s_n ,$$

(2.19) $$s_i^{(3)}(\tilde{A}) = \sum_{j=1}^{n} \tilde{a}_{ij}^{(2)} s_j < s_i^{(2)}(\tilde{A}) s_n .$$

As $\tilde{a}_{ni} \neq 0$, it follows that $\tilde{a}_{ii}^{(2)} \neq 0$. As $\tilde{a}_{ii}^{(2)} \neq 0$ and as $i < n$, it follows that the strict inequality sign in (2.19) is justified. (2.18) and (2.19) imply

(2.20) $$s_i^{(3)}(\tilde{A}) < s_n^{(3)}(\tilde{A}) .$$

(2.1) and (2.20) contradict (2.17). The proof of the case (1) is thus completed.

We bring now the proof for the case (2). We give first the proof for $n = 3$. Let $\tilde{A} = \tilde{A}(m), m = 3, 4, \cdots,$ be an optimal matrix of the problem

$$\max_{A \in \mathscr{A}_3(s)} S(A^m) .$$

Assume that $\widetilde{A}(m)$, for a fixed m from $m = 3, 4, \cdots$, has not the structure (2.7). There are then two possibilities:

$$\tilde{a}_{31} \neq 0 , \tag{2.21}$$

$$\tilde{a}_{31} = 0, \tilde{a}_{32} \neq 0 . \tag{2.22}$$

If (2.21) holds then, according to (2.10), it is sufficient to prove that for every natural k the inequality

$$s_1^{(k)}(\widetilde{A}) < s_3^{(k)}(\widetilde{A}) \tag{2.23}$$

holds, while if (2.22) holds it is sufficient to prove that

$$s_2^{(k)}(\widetilde{A}) < s_3^{(k)}(\widetilde{A}) . \tag{2.24}$$

Assume that (2.21) holds. As

$$s_i^{(k)}(\widetilde{A}) = \sum_{j=1}^{3} \tilde{a}_{ij} s_j^{(k-1)}(\widetilde{A}) = \sum_{j=1}^{3} \tilde{a}_{ij}^{(k-1)} s_j,\ i = 1, 2, 3 ; \quad k = 2, 3, \cdots , \tag{2.25}$$

it follows that

$$s_1^{(k)}(\widetilde{A}) \leqq \min \left\{ s_3 s_1^{(k-1)}(\widetilde{A}),\ s_1 \max_j s_j^{(k-1)}(\widetilde{A}) \right\} , \tag{2.26}$$

$$s_3^{(k)}(\widetilde{A}) \geqq \max \left\{ s_1 s_3^{(k-1)}(\widetilde{A}),\ s_3 \min_j s_j^{(k-1)}(\widetilde{A}) \right\} . \tag{2.27}$$

We prove (2.23) by induction on k. For $k = 1$ (2.23) holds by (2.1). Assume that

$$s_1^{(k-1)}(\widetilde{A}) < s_3^{(k-1)}(\widetilde{A}) .$$

From this induction assumption follows that at least one of the two following equations holds:

$$s_1^{(k-1)}(\widetilde{A}) = \min_j s_j^{(k-1)}(\widetilde{A}) , \tag{2.28}$$

$$s_3^{(k-1)}(\widetilde{A}) = \max_j s_j^{(k-1)}(\widetilde{A}) . \tag{2.29}$$

The minimum and the maximum are strict. As (2.28) or (2.29) holds, it follows from (2.26) and (2.27) that

$$s_1^{(k)}(\widetilde{A}) \leqq s_3^{(k)}(\widetilde{A}) . \tag{2.23'}$$

To obtain (2.23) we have to show that equality cannot hold in (2.23′). Assume that (2.28) holds. Equality in (2.23′) implies

$$s_3^{(k)}(\widetilde{A}) = s_3 s_1^{(k-1)}(\widetilde{A}) .$$

From the last equation, using (2.25) and the fact that the minimum in (2.28) is strict, we obtain

$$\tilde{a}_{32} = \tilde{a}_{33} = 0,\ \tilde{a}_{31} = s_3 = \tilde{a}_{13} \leqq s_1 . \tag{2.30}$$

(2.30) contradicts (2.1). Assume that (2.29) holds. Similar to our last conclusion it follows now that equality in (2.23′) implies

$$\tilde{a}_{11} = \tilde{a}_{12} = 0,\ \tilde{a}_{13} = s_1,\quad \tilde{a}_{32}^{(k-1)} = \tilde{a}_{33}^{(k-1)} = 0 . \tag{2.31}$$

As from $\tilde{a}_{33} \neq 0$ follows $\tilde{a}_{33}^{(k-1)} \neq 0$, we obtain

$$\tilde{a}_{33} = 0 . \tag{2.32}$$

If $\tilde{a}_{32} \neq 0$, using (2.31) and (2.32), we obtain

$$\begin{cases} \tilde{a}_{33}^{(k-1)} \neq 0 , & k-1 \text{ even,} \\ \tilde{a}_{32}^{(k-1)} \neq 0 , & k-1 \text{ odd.} \end{cases} \tag{2.33}$$

(2.33) follows easily, e.g. from the directed graph corresponding to $\tilde{A}$. (2.33) contradicts (2.31) and therefore $\tilde{a}_{32} = 0$. We obtained

$$\tilde{a}_{32} = \tilde{a}_{33} = 0,\ \tilde{a}_{13} = \tilde{a}_{31} = s_1 = s_3 . \tag{2.34}$$

(2.34) contradicts (2.1). So (2.23) holds and the proof for this case is completed.

Assume that (2.22) holds. We prove (2.24) by induction on k. Assume that

$$s_2^{(k-1)}(\tilde{A}) < s_3^{(k-1)}(\tilde{A}) . \tag{2.35}$$

From (2.22), (2.25) and (2.35) follows

$$\begin{aligned} s_2^{(k)}(\tilde{A}) &\leqq s_3 s_2^{(k-1)}(\tilde{A}) , \\ s_3^{(k)}(\tilde{A}) &\geqq s_3 s_2^{(k-1)}(\tilde{A}) . \end{aligned} \tag{2.36}$$

Hence,

$$s_2^{(k)}(\tilde{A}) \leqq s_3^{(k)}(\tilde{A}) . \tag{2.24′}$$

To obtain (2.24) we have to show that equality cannot hold in (2.24′). Equality in (2.24′) implies equality in (2.36) and this implies $\tilde{a}_{33} = 0$. So we have

$$\tilde{a}_{31} = \tilde{a}_{33} = 0,\ \tilde{a}_{32} = s_3 = \tilde{a}_{23} \leqq s_2 . \tag{2.37}$$

(2.37) contradicts (2.1). So (2.24) holds and the proof for $n = 3$ is completed.

For $n = 2$ it is sufficient to prove that for every natural k

$$s_1^{(k)}(\tilde{A}) < s_2^{(k)}(\tilde{A}) .$$

This inequality can be easily proved by induction. Theorem 2 is thus established.

REMARK. It is easy to prove that if A is a nonnegative matrix with row sums $s_1, \cdots, s_n$; $s_1 \leqq s_2 \leqq \cdots \leqq s_n$, then

$$s_1^{m-1}S(A) \leqq S(A^m) \leqq s_n^{m-1}S(A) , \qquad m = 1, 2, \cdots ,$$

where the two bounds are sharp. As for $A \in \mathscr{A}_n(s)$

$$\sum_{i=1}^{n} s_i^m < s_n^{m-1}S(A) ,$$

and as the bound $s_n^{m-1}S(A)$ is sharp, it follows that the assumption of symmetry in Theorem 2 is essential.

2.3. Generalizations. Theorem 2 can be generalized to a larger class of matrices and also to a statement on minors of matrices.

Let $A = (a_{ij})$ be a $n \times n$ matrix, perhaps with complex elements. Denote $|A| = (|a_{ij}|)$. The row sums vector of $|A|$, $s(|A|)$, is denoted by $[s] = [s](A)$. The ith component of $[s]$ is denoted by $[s_i] = [s_i](A)$.

We bring now the first generalization of Theorem 2: *In the following two cases*

(1) $m = 3, 4, 5$ and $n = 1, 2, \cdots$

(2) $m = 3, 4, \cdots$ and $n = 1, 2, 3$

the inequality

$$S(|A^m|) \leqq \sum_{i=1}^{n} [s_i]^m \tag{2.38}$$

holds for every complex A such that $|A| \in \mathscr{A}_n([s])$. The equality sign in these two cases holds if and only if A is diagonal.

Proof. We have

$$\begin{aligned} S(|A^m|) &= \sum_{i,j=1}^{n} \left| \sum_{k_1,\cdots,k_{m-1}=1}^{n} a_{ik_1}a_{k_1k_2} \cdots a_{k_{m-1},j} \right| \\ &\leqq \sum_{i,j=1}^{n} \sum_{k_1,\cdots,k_{m-1}=1}^{n} |a_{ik_1}a_{k_1k_2} \cdots a_{k_{m-1},j}| = S(|A|^m) . \end{aligned} \tag{2.39}$$

As $|A| \in \mathscr{A}_n([s])$ it follows from Theorem 2 that

$$S(|A|^m) \leqq \sum_{i=1}^{n} [s_i]^m . \tag{2.40}$$

(2.39) and (2.40) imply (2.38). The equality statement follows from the equality statement in Theorem 2.

REMARK 1. For $A \in \mathscr{A}_n(s)$ (2.38) reduces to (2.4′). For $m = 1$ (2.38) holds with equality sign for every A. For $m = 2$ (2.38) holds, but the equality statement stated above does not fit this case.

REMARK 2. The only essential assumption about A is that $|A|$ is symmetric. $|A| \in \mathscr{A}_n([s])$ includes the additional assumption that the components of $[s]$ are positive and distinct. This assumption is needed only to obtain the equality statement.

The second generalization deals with minors of matrices. We introduce now several concepts and notations.

Let p and n be natural numbers, $1 \leqq p \leqq n$. Denote

$$Q_{pn} = \{(i_1, \cdots, i_p) \mid 1 \leqq i_1 < i_2 < \cdots < i_p \leqq n\}$$

($i_1, \cdots, i_p$ are natural numbers).

Let $i = (i_1, \cdots, i_p)$ and $j = (j_1, \cdots, j_p)$ be elements of Q_{pn}, and let A be a $n \times n$ matrix. The minor of A formed from the rows $(i_1, \cdots, i_p)$ and the columns $(j_1, \cdots, j_p)$ is denoted by

$$A\begin{pmatrix} i_1, \cdots, i_p \\ j_1, \cdots, j_p \end{pmatrix} = A\begin{pmatrix} i \\ j \end{pmatrix} .$$

The *pth compound matrix of A* is denoted by $C_p(A)$. $C_p(A)$ is a $\binom{n}{p} \times \binom{n}{p}$ matrix with elements $A\begin{pmatrix} i \\ j \end{pmatrix}$.

Let us now define the class of matrices $|\mathscr{A}_n([s])|$. *A matrix A belongs to the class $|\mathscr{A}_n([s])|$ if and only if A is symmetric and $|A|$ belongs to $\mathscr{A}_n([s])$.* Note that the definition includes the demand that all the components of $[s](A), A \in \mathscr{A}_n([s])|$, are positive and distinct. Note also that a matrix belonging to $|\mathscr{A}_n([s])|$ can be complex.

In [6, formula 12] Schneider obtained the following result: *Let A be a $n \times n$ matrix and p a natural number, $1 \leqq p \leqq n$. Then*

$$\sum_{j \in Q_{pn}} \left| A\begin{pmatrix} i \\ j \end{pmatrix} \right| \leqq [s_{i_1}] \cdot \cdots \cdot [s_{i_p}], \; i = (i_1, \cdots, i_p) . \tag{2.41}$$

In [5] Ostrowski obtained the following equality statement: *If* $[s_{i_1}] \cdot \cdots \cdot [s_{i_p}] \neq 0$ *then the equality sign in* (2.41) *holds if and only if in every column of the submatrix of A formed from the p rows $i_1, \cdots, i_p$, there exists at most one nonzero element.* From this statement follows: *If $A \in |\mathscr{A}_n([s])|$ and if $p \geqq 2$ then the equality sign in* (2.41) *holds for every $i \in Q_{pn}$ if and only if A is a diagonal matrix.*

We bring now the second generalization of Theorem 2: *Let p and n be natural numbers, $1 \leqq p \leqq n$. In the following two cases*

(1) $m = 3, 4, 5$ and $n = 1, 2, \cdots$

(2) $m = 3, 4, \cdots$ and $n = 1, 2, 3$

the inequality

$$\sum_{i,j\in Q_{pn}} \left| A^m\binom{i}{j} \right| \leqq \sum_{i\in Q_{pn}} ([s_{i_1}]\cdot \cdots \cdot[s_{i_p}])^m \tag{2.42}$$

holds for every A belonging to $|\mathscr{A}_n([s])|$. The equality sign in these two cases holds if and only if A is diagonal.

Proof. As A is symmetric, the compound matrix $C_p(A)$ is also symmetric. Applying (2.38) to $C_p(A)$ (see Remark 2 after (2.38)), we obtain

$$\begin{aligned} S\{|[C_p(A)]^m|\} = S[|C_p(A^m)|] &= \sum_{i\ j\in Q_{pn}} \left| A^m\binom{i}{j} \right| \\ &\leqq \sum_{i\in Q_{pn}} \left(\sum_{j\in Q_{pn}} \left| A\binom{i}{j} \right| \right)^m . \end{aligned} \tag{2.43}$$

(2.42) follows from (2.41) and (2.43). For $p=1$ the equality statement follows from the equality statement corresponding to (2.38). Equality in (2.42) for $p \geqq 2$ implies equality in (2.41) for every $i \in Q_{pn}$. As $A \in |\mathscr{A}_n([s])|$, it follows from the equality statement corresponding to (2.41) that A is diagonal. It is obvious that if A is diagonal then equality holds in (2.42).

REMARK 1. For $p=1$ and $A \in \mathscr{A}_n(s)$ (2.42) reduces to (2.4′). (2.42), including the equality statement, holds for $p \geqq 2$ also for $m = 1, 2$.

REMARK 2. If the conjecture (2.4) stated at the beginning of this chapter holds true, then the two generalizations given in this section hold also for all m and n.

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INFINITE PRODUCTS OF SUBSTOCHASTIC MATRICES

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This paper is about two types of infinite products of substochastic matrices $\{A_j\}$ namely: the left product defined by the sequence of left partial products $A_1, A_2A_1, A_3A_2A_1, \cdots$; and the right product defined by the sequence of right partial products $A_1, A_1A_2, A_1A_2A_3, \cdots$.

The basic theorem is that if the A_n are each ∞ by ∞ then:

a. There is a nonempty set E of substochastic sequences each of which (except possibly the zero sequence, 0) is the componentwise limit of a sequence of rows, one from each left partial product;

b. Any sequence $\{\rho_n\}$ of rows, one from each left partial product, can be approximated by a sequence of convex combinations $\{c_n\}$ of points of E (that is, $\{\rho_n - c_n\}$ converges componentwise to the zero sequence), and c. $E = \{0\}$ if and only if every sequence of rows, one from each left partial product, converges to 0.

Similar conclusions follow immediately for the right product of ∞ by ∞ doubly substochastic matrices.

The asymptotic behaviour of the right product of a special class of $\{A_n\}$ is also considered.

The finite case (that is, when all the A_n are r by r) for stochastic A_n is treated independently for convenience, even though the result in this case (Theorem 1) is actually a direct consequence of the basic Theorem 1′. Its conclusion is that there is an m by r stochastic matrix A with $1 \leqq m \leqq r$ and permutation matrices Q_n such that

a. if $m < r$ then for some stochastic $r - m$ by m matrices C_n:

$$\lim_{n\to\infty} \left\{A_nA_{n-1} \cdots A_1 - Q_n\begin{pmatrix} A \\ C_nA \end{pmatrix}\right\} = 0$$

and b. if $m = r$ then

$$\lim_{n\to\infty} \{A_nA_{n-1} \cdots A_1 - Q_nA\} = 0 .$$

Some results on fixed points are obtained in the finite case which carry over, in restricted form, to the infinite case.

A real matrix is said to be stochastic if none of its entries is negative and each of its row sums is 1. Two types of infinite products which arise naturally from a given sequence $\{A_n\}$ of stochastic matrices are those whose nth partial products are $R_n = A_1A_2 \cdots A_n$ and $L_n = A_nA_{n-1} \cdots A_1$ respectively. We'll call the sequence $\{R_n\}$ the right

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product and the sequence $\{L_n\}$ the left product of the A_n.

The right product is of interest in the theory of Markov chains with possibly nonstationary transition probabilities because if A_n is the matrix of probabilities $a_{ij}^{(n)}$ of transition from state i at time $n-1$ to state j at time n then the ijth entry $r_{ij}^{(n)}$ of R_n is the probability of transition to state j at time n from state i at time 0.

The left product has a similar interpretation: $l_{ij}^{(n)}$ is the probability of transition from state i at time $-n$ to state j at time 0.

We shall obtain theorems on the asymptotic behaviour of these partial products and on their fixed points. For example if the A_n are ∞ by ∞ stochastic matrices we can show that there is a sequence of rows, one from each L_n, which converges componentwise.

The finite and infinite cases are treated separately for clarity.

DEFINITION. A *permutation matrix* is a matrix of zeroes and ones which exactly one 1 in each row and each column.

THEOREM 1. *If $L_n = A_n A_{n-1} \cdots A_1$ and each A_n is an r by r stochastic matrix then there exists an m by r stochastic matrix A with $1 \leqq m \leqq r$, r by r permutation matrices Q_n and, if $m < r$, stochastic $r - m$ by m matrices C_n such that*:

$$\lim_{n\to\infty} \left\| L_n - Q_n \begin{pmatrix} A \\ C_n A \end{pmatrix} \right\| = 0 \quad \textit{if} \quad m < r \quad \textit{and}$$

$$\lim_{n\to\infty} \| L_n - Q_n A \| = 0 \quad \textit{if} \quad m = r\,.$$

Proof. Let S be the convex hull of the basis vectors $v_1 = (1, 0, 0, \cdots, 0)$, $v_2 = (0, 1, 0, \cdots, 0), \cdots,$ $v_r = (0, 0, 0, \cdots, 1)$. Each $(S)L_n$ is a convex polytope (that is, the convex hull of p points), these polytopes are nested (that is, $(S)L_{n+1} \subseteqq (S)L_n$ for all n) and none of them has more than r vertices (a point x of a polytope is a *vertex* if it is on no open line segment contained in the polytope). It can be shown that the intersection of such a family of convex polytopes is a convex polytope of r or fewer vertices. Let $K = \bigcap_{n \geqq 1} (S)L_n$ and denote its vertices by $k_1, \cdots, k_m$. Let A be the m by r matrix whose ith row is k_i. Let $v_i^{(n)}$ denote $(v_i)L_n$. For each n and each $t \leqq$ m there is a $v_{i_t}^{(n)}$ such that $k_t = \lim_{n\to\infty} v_{i_t}^{(n)}$. We can assume that for each n there are only m such $v_{i_t}^{(n)}$ so chosen. If $m < r$ extend the definition of i_t so that $\{v_{i_t}^{(n)} : m < t \leqq r\}$ is the set of $v_i^{(n)}$ not already chosen. Q_n is the matrix $(q_{ij}^{(n)})$ for which $q_{it}^{(n)}$ is 1 if $i = i_t$ and is 0 otherwise. If $m < r$ and $t > m$ let $k_t^{(n)}$ be the point of K closest to $v_{i_t}^{(n)}$. Since K is convex, $k_t^{(n)}$ is a convex combination, $\sum_{j=1}^{m} c_{tj}^{(n)} k_j$, of the vertices K. Therefore $C_n = (c_{ij}^{(n)})$ is an $r - m$ by m stochastic matrix and $k_t^{(n)} = (v_t)\begin{pmatrix} A \\ C_n A \end{pmatrix}$ for each $m < t \leqq r$. Consequently $(v_{i_t})Q_n\begin{pmatrix} A \\ C_n A \end{pmatrix} = (v_t)\begin{pmatrix} A \\ C_n A \end{pmatrix}$ if

$m < r$ and $(v_{i_t})Q_nA = (v_t)A$ if $m = r$. Theorem 1 then follows from the fact that $\lim_{n\to\infty} v_{i_t}^{(n)} = k_t$ if $1 \leqq t \leqq m$ and $\lim_{n\to\infty} v_{i_t}^{(n)} - k_t^{(n)} = 0$ if $t > m$.

Notice that $\lim_{n\to\infty} L_n = \begin{pmatrix} k_1 \\ k_1 \\ \vdots \\ k_1 \end{pmatrix}$ if m $= 1$ because K then consists of the one vertex k_1.

DEFINITION. A sequence $\{P_n\}$ of r by r matrices is *descending* if and only if $(S)P_{n+1} \subseteqq (S)P_n$ for all n sufficiently large. (S is as in the proof of Theorem 1). As a first corollary to Theorem 1 we have: m, Q_n, A and C_n (if $m < r$) such that $\lim_{n\to\infty}\left\| P_n - Q_n\begin{pmatrix} A \\ C_nA \end{pmatrix}\right\| = 0$ if $m < r$ and $\lim_{n\to\infty} \| P_n - Q_nA \| = 0$ if $m = r$, for all descending sequences because each such sequence (with the first N terms omitted) is the left product of some sequence of stochastic matrices. (All left products of stochastic matrices are, of course, descending sequences.) Another immediate corollary concerns doubly stochastic matrices (that is, stochastic matrices whose transposes are also stochastic). We shall state the corollary emphasizing the matrix entries for variety's sake.

COROLLARY 2. *If $\{A_n\}$ is a sequence of doubly stochastic r by r matrices and $R_n = A_1A_2 \cdots A_n$ then there exists an m by r stochastic matrix A with $1 \leqq m \leqq r$ and permutations q_n of the r indices such that for each $1 \leqq j \leqq r$:*

(a) *if $1 \leqq q_n(i) \leqq m$, $\lim_{n\to\infty} (r_{ji}^{(n)} - a_{q_n(i)j}) = 0$ and if $m < r$ there exist $r - m$ by m stochastic matrices C_n such that*

(b) *if $m < q_n(i) \leqq r$ then:*

$$\lim_{n\to\infty} (r_{ji}^{(n)} - \sum_{k=1}^{m} c^{(n)} q_n(i) k^a k_j) = 0 \; .$$

Some examples of $\{A_n\}$ with descending right products are provided by all those sequences of stochastic matrices $\{A_n\}$ which commute pairwise within a row permutation (i.e. $A_nA_{n'} = Q_{nn'}A_{n'}A_n$ for some permutation matrix $Q_{nn'}$). Because of their connection with Markov chains we shall investigate descending right products further. We'll impose further conditions on the A_n which are not too stringent but which give additional information about the C_n of Theorem 1. While doing so we acquire some information on the fixed points of A_n and R_n.

DEFINITION. B occurs frequently among the A_n if and only if $B = A_n$ for infinitely many n.

LEMMA. *If $\{A_n\}$ is a sequence of r by r stochastic matrices whose* *right partial products $R_n = A_1A_2 \cdots A_n$ are a descending sequence and B occurs frequently among the A_n then, in the notation of Theorem 1 there is an m by m permutation matrix D such that $AB = DA$.*

Proof. For some N, $\{R_{N+n}\}$ is the left product of some sequence of stochastic matrices A'_n. Let K be as in the proof of Theorem 1 applied to the A'_n. Then $K = \bigcap_{n>N}(S)R_n$. $(K)B \subseteqq K$ because $K = \bigcap\{(S)R_{n-1} : B = A_n \text{ and } n > N\}$. Suppose $x \in K$. Then, for infinitely many n, there are $x_n \in (S)R_{n-1}$ for which $x = (x_n)B$. A subsequence $\{x_{n_m}\}$ converges to some point $y \in S$. Therefore $(x_{n_m})B$ converges to $(y)B$ and hence $x = (y)B$. But $y \in K$ and hence $K \subseteqq (K)B$. Thus $K = (K)B$ and hence B permutes the vertices of K (rows of A). Let D be the m by m permutation matrix representing this row permutation then $AB = DA$.

B permutes all the vertices of K and fixes the barycentre, $1/m' \sum_{t=1}^{m'} k_{i_t}$, of each subset $\{k_{i_1}, k_{i_2}, \cdots, k_{i_{m'}}\}$ of m' vertices of K (rows of A) which it permutes. Therefore $(x)B = x$ for all x in the convex hull of these barycentres. There may be (left) fixed points of B outside the convex hull of the barycentres.

Let us enumerate all the matrices occurring frequently among the A_n so that A_{n_1} is the first such matrix and A_{n_p} is the pth such matrix distinct from $A_{n_{p-1}}$. Let D_{n_p} be the m by m permutation matrix corresponding to A_{n_p} (as in the lemma) and let $D_n = D_{n_p}$ if $A_n = A_{n_p}$. Applying the lemma to the first corollary to Theorem 1 we obtain:

THEOREM 2. *If $\{A_n\}$ is a sequence of r by r stochastic matrices each of which (except for finitely many n) occur frequently among the A_n and the n-th partial products $R_n = A_1A_2 \cdots A_n$ are descending then there exists an m by r stochastic matrix A (with $1 \leqq m \leqq r$), permutation matrices Q_n and, if $m < r$, $r - m$ by m stochastic matrices C_n such that given $\varepsilon > 0$ there is an N for which:*

$$\text{(a)} \quad \left\| R_n - Q_N \begin{pmatrix} D'_n A \\ C_N D'_n A \end{pmatrix} \right\| < \varepsilon \quad (\textit{if } m < r),$$

$$\text{(b)} \quad \| R_n - Q_N D'_n A \| < \varepsilon \quad (\textit{if } m = r),$$

for all $n > N$. D'_n is the permutation matrix which is the product $D_{N+1}D_{N+2} \cdots D_n$ of D_s defined in the previous paragraph. Moreover the barycentres of those sets of rows of A which are permuted by all the D_{n_p} is a (left) fixed point for all A_n (except perhaps the finitely many n for which A_n does not occur frequently). In particular the barycentre $b = 1/m \sum_{i=1}^{m} (a_{i1}, \cdots, a_{ir})$ of the rows of A is such a (left) fixed vector.

Let F be the convex hull of the barycentres mentioned in Theorem

2. F is fixed (pointwise) by each of those A_n which occur frequently. If all the A_n occur frequently then $(x)R_n = x$ for all n and all $x \in F$.

The fundamental theorems on the convergence of the powers of the transition matrix and the "classification of states" of a finite Markov chain with stationary transition probabilities (see for example [4] pp. 170–184) can be obtained from Theorem 1 by examination of the position of K in S. In the interest of brevity we shall not do so here but shall instead discuss two notions from the stationary case by way of sample applications of Theorems 1 and 2.

In the notation of the proof of Theorem 1 let T be the set of all i for which v_i is not in the set of basis vectors spanning K. Following the custom (see e.g. [2]) for the stationary case we'll say that *i leads to j* (written $i \leadsto j$) if and only if $r_{ij}^{(n)} > 0$ for some n. If the right product of the A_n is descending then for each i, $\lim_{n \to \infty} r_{ij}^{(n)} = 0$ for all $j \in T$ and; each $i \in T$ leads to some $j \notin T$ by the first corollary to Theorem 1. In the stationary case (i.e. when $A_n = A_1$ for all n):

$$T = \bigcup_{j \geq 1} \{i : i \leadsto j \text{ and } j \not\leadsto i\} .$$

This is precisely the definition of the set of *transient* (sometimes called *inessential*) *states* in the stationary case.

The notion of *regular* chain (in the terminology used in [6]) can be extended to the nonstationary case so as to obtain the same kind of basic result. Suppose the right product of the A_n is descending and that there is a product $P = A_{n_{p_1}} A_{n_{p_2}} \cdots A_{n_{p_q}}$ of frequently occuring $A_{n_{p_t}}$ (in the notation of Theorem 2) which is positive (i.e. $p_{ij} > 0$, all i, j). (The n_{p_t} are not necessarily distinct nor in increasing order). Call such $\{A_n\}$ *regular* sequences. It then follows that the right products R_n of regular sequences $\{A_n\}$ converge to a matrix all of whose rows are the vector k. No component of k is zero, $(k)R_n = k$ for all sufficiently large n (for *all* n, if $(S)R_{n+1} \subseteqq (S)R_n$ for all n) and k is the only vector in S with this property. Although this is equivalent to the corresponding result for the stationary case it is easy enough to obtain using the first corollary to Theorem 1 and the lemma preceding Theorem 2: All we need do is show that $m = 1$. To this end observe that according to the lemma, P permutes the vertices of K so that, for some n: $(x)P^n = x$ for all $x \in K$. If K had more than one vertex the line joining two of them would meet the boundary of S in a point x which is fixed by P^n. $(x)P^n$ can have no zero components because P is positive but x has zero components because it's in the boundary of S. This second application may also be found in a slightly less general form as Theorem 3 of [5].

DEFINITION. A real matrix is *substochastic* if and only if none of its entries is negative and 1 is an upper bound for its row sums.

Most of the foregoing results including Theorem 1 and its corollaries can be extended to infinite as well as finite substochastic matrices. To do so, consider the set S_0 of all substochastic sequences (i.e. the set of all real sequences of nonnegative terms whose sum is at most 1). S_0 is a compact, convex subset of the space of all real sequences under the product topology. The ∞ by ∞ substochastic matrices are associative and closed under matrix multiplication so that left and right product is defined for every sequence of such matrices.

THEOREM 1′. *If $\{L_n\}$ is the left product of a sequence of ∞ by ∞ substochastic matrices then there is a nonempty set, E, of substochastic sequences with the following properties:*

(a) *For each $k \in E$ (except possibly the zero sequence) and each n there is an integer $i_{n,k}$ such that for all j:*

$$\lim_{n\to\infty} l_{i_{n,k}j}^{(n)} = k_j .$$

(b) *For each sequence $\{i_n\}$ there is a convex combination $x^{(i,n)}$ of elements of E such that for all j:*

$$\lim_{n\to\infty} (l_{i_n j}^{(n)} - x_j^{(i,n)}) = 0 .$$

(c) *The zero sequence is the only element of E if and only if for all sequences $\{i_n\}$ and all j:*

$$\lim_{n\to\infty} l_{i_n j}^{(n)} = 0 .$$

Proof. For each subset F of S_0 let $co(F)$ be the set of convex combinations of elements of F and $\overline{co}(F)$ be the intersection of all closed convex sets containing F. Let W_n be the set consisting of 0 and all the rows of L_n, let $\bar{L}_n = \overline{co}(W_n)$ and $K = \bigcap_{n>1} \bar{L}_n$. K is convex and compact and $0 \in K$. Let E be the set of extremals of K (that is, $k \in E$ if and only if $k \in K$ and k is an interior point of no line segment in K) then $K = \overline{co}(E)$ by the Krein-Milman theorem. Part (a) of Theorem 1′ is proven by contradiction. Suppose $k \in E$ and a neighbourhood of k excludes 0 and all rows of L_n for all n in an infinite set Ω. Then, for a finite set Λ and some $\varepsilon > 0$, W_n is in the complement of $Z \equiv \bigcap_{j\in\Lambda} \{x \in S_0 : |x_j - k_j| < \varepsilon\}$ for each $n \in \Omega$.

Let $T_j^+ = \{x \in S_0 : x_j \geqq k_j + \varepsilon\}$, $T_j^- = \{x \in S_0 : x_j \leqq k_j - \varepsilon\}$ and $T_j = T_j^+ \cup T_j^-$. Then

$$K \subseteqq \bar{L}_n = \overline{co}\Big(\bigcup_{j\in\Lambda} (T_j \cap W_n)\Big)$$

$$= co\Big(\bigcup_{j\in\Lambda} \overline{co}(T_j \cap W_n)\Big) \qquad \text{(see [3] V 2.5)}$$

$$= co\left(\bigcup_{j\in A} \overline{co}((T_j^+ \cap W_n) \cup (T_j^- \cap W_n))\right)$$

$$= co\left(\bigcup_{j\in A} co(\overline{co}(T_j^+ \cap W_n) \cup \overline{co}(T_j^- \cap W_n))\right)$$

(again by [3] V 2.5)

$$\subseteqq co\left(\bigcup_{j\in A} co((T_j^+ \cap \bar{L}_n) \cup (T_j^- \cap \bar{L}_n))\right)$$

$$\subseteqq co\left(\bigcup_{j\in A} co(T_j \cap \bar{L}_n)\right).$$

If $U_{jn} = T_j \cap \bar{L}_n$ is empty for some $j \in A$, $n \in \Omega$ then $U_{jm} = \phi$ for all sufficiently large m because the U_{jn} are nested for fixed j. Rather than change notation, we can assume that $U_{jn} \neq \phi$ for all $n \in \Omega$ and all $j \in A$. Thus k is a convex combination, $\sum_{j\in A} \lambda_{jn} u_{jn}$, of elements u_{jn} of $co(U_{jn})$. U_{jn} is the union of $U_{jn}^+ = T_j^+ \cap \bar{L}_n$ and $U_{jn}^- = T_j^- \cap \bar{L}_n$. Assuming first that U_{jn}^+ and U_{jn}^- are nonempty for all $n \in \Omega$ we have $0 \leqq \mu_{jn} \leqq 1$ such that $u_{jn} = \mu_{jn} u_{jn}^+ + (1 - \mu_{jn}) u_{jn}^-$ for some $u_{jn}^+ \in U_{jn}^+$ and some $u_{jn}^- \in U_{jn}^-$. By successive extraction of subsequences we obtain u_j^+, u_j^-, μ_j and λ_j such that

$$\lim_{m\to\infty} u_{jn_m}^+ = u_j^+ , \quad \lim_{m\to\infty} u_{jn_m}^- = u_j^- , \quad \lim_{m\to\infty} \mu_{jn_m} = \mu_j ,$$

$$\lim_{m\to\infty} \lambda_{jn_m} = \lambda_j , \quad 1 \geqq \mu_j \geqq 0 , \quad 1 \geqq \lambda_j \geqq 0 \text{ and } \sum_{j\in A} \lambda_j = 1 .$$

Therefore $k = \sum_{j\in A} \lambda_j(\mu_j u_j^+ + (1 - \mu_j)u_j^-))$, and for all $j \in A$: u_j^+, $u_j^- \in K$ and u_j^+, $u_j^- \in T_j$. The extremality of k implies that $k = u_j^+$ or u_j^- for some j and hence that $k \in T_j$. Consequently $k \notin Z$, a contradiction. If, however, U_{jn}^+ or U_{jn}^- is ϕ for some (and hence all subsequent) n we can use a similar argument using the u_{jn} instead of the u_{jn}^+ and u_{jn}^-.

If $k \neq 0$ we can therefore assert that each sufficiently small neighbourhood of k excludes 0 but contains an element of W_n for all sufficiently large n. These elements must be rows of the L_n. Therefore k is the componentwise limit of a sequence of rows, one from each L_n.

To prove part (b) let d be the metric on S_0 which induces the product topology (see [1] II prop. 6, p. 97). Let $y_n \in L_n$ and z_n be a point of K closest to y_n in the metric. $d(z_n, y_n)$ is a null sequence because the $\bar{L}_n$ are nested. A sequence $\{x_n\}$ in $co(E)$ can be found for which $d(x_n, y_n)$ is a null sequence because $co(E)$ is dense in $\overline{co}(E)$ (see [3] V 2.4). Part (b) then follows if the i_nth row of L_n is used for y_n. Part (c) follows directly from parts (a) and (b). This completes the proof of Theorem 1′.

The conclusion of Theorem 1′ is valid if $\{L_n\}$ is replaced by any descending sequence $\{P_n\}$ of ∞ by ∞ substochastic matrices using the previous definition of "descending" with S replaced by S_0. Such sequences too are, except for finitely many terms, the left product of some sequence of substochastic matrices.

The statements about commutivity also carry over to the infinite case.

Corollary 2 extends to:

COROLLARY 2′. *If* $\{R_n\}$ *is the right product of* ∞ *by* ∞ *doubly stochastic matrices then there is a nonempty set, E, of substochastic sequences with the following properties:*

(a) *For each non-zero* $k \in E$ *and each n there is an integer* $i_{n,k}$ *such that for all j:*

$$\lim_{n\to\infty} r^{(n)}_{ji_{n.k}} = k_j \qquad \textit{and}$$

(b) *For each sequence* $\{i_n\}$ *there is a convex combination* $x^{(i,n)}$ *of elements of E such that for all j:*

$$\lim_{n\to\infty} r^{(n)}_{ji_n} - x^{(i,n)}_j = 0 ,$$

(c) *The zero sequence is the only element of E if and only if for all* $\{i_n\}$ *and for all j:*

$$\lim_{n\to\infty} r^{(n)}_{ji_n} = 0 .$$

A substochastic matrix is continuous on S_0 if and only if all of its columns are null sequences. If a continuous B occurs frequently among the A_n and their right product is descending then $(K)B = K$.

Theorem 2 and the remarks following it concerning fixed points also hold for ∞ by ∞ substochastic matrices A_n provided each A_n is continuous and K has only finitely many extremals.

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REFLECTION AND APPROXIMATION BY INTERPOLATION ALONG THE BOUNDARY FOR ANALYTIC FUNCTIONS

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Let there be given a function $f(z)$ analytic in an open connected set, not necessarily simply connected, which is bounded by simple closed analytic curves such that the function is continuous on the closure of the region and such that the real part of the function satisfies boundary conditions that are analytic in a neighborhood of the boundary. We want to interpolate $f(z)$ along the boundaries and find conditions that make the interpolants converge maximally to $f(z)$ throughout the closure of the region. The boundary condition on the real part of $f(z)$ permits the analytic continuation of $f(z)$ across the boundary curves and ensures that we are interpolating at points interior to the region of analyticity. In our error estimates (Theorem 1) maximal convergence depends in an essential way on how far we can reflect $f(z)$ and this in turn depends on the boundary values of the real part of $f(z)$ as well as on the geometry of the given region and its analytic boundaries. In Theorems 2 and 3, a simply connected region is considered. Special points of interpolation are given, these depend only on the parametric representation of the boundary curves and not a conformal map. These points are the image points of the Chebyshev polynomials.

Finally an example is given for a multiply connected region.

As is well known [2] Runge's beautiful theorem shows us that there exist certain "equidistributed" points on the analytic curves such that if we interpolate at these points the interpolants converge to the function. However, the proof depends on knowing the conformal map in order to know what the interpolation points are. Here we shall give conditions that do not require knowledge of the conformal map but for convergence depend on how far we can reflect. Along with these, we shall give simple error estimates. Moreover, we shall show that possible interpolation points are the images on the boundary of roots of the Chebyshev polynomials.

The aspects of this paper which are novel are

(i) the use of reflection

(ii) interpolation at boundary points which are gotten directly from the parametric representation of the boundary and do not depend on a conformal map

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(iii) the use of the images of the roots of the Chebyshev polynomials as possible interpolation points.

Notation. Let R be a connected set whose boundary is Γ. Let $\Gamma = \Gamma^1 \cup \Gamma^2 \cup \cdots \cup \Gamma^s$ where the Γ^j are bounded analytic contours in the $z = x + iy$ plane given by $F^j(x, y) = 0$ with $(F_x^j)^2 + (F_y^j)^2 \neq 0$ along Γ^j, $j = 1, 2, \cdots, s$, where the F^j are real-valued analytic functions of x and y. We assume further that the Γ^j are pairwise disjoint. Let Γ^1 contain in its interior $\Gamma^2, \Gamma^3, \cdots, \Gamma^s$ and contain in its exterior the point at infinity. Let Γ^j contain in its interior a_j $2 \leqq j \leqq s$. As shown in [3] there are "reflection" functions $G_j(z)$ defined on a neighborhood $D^j \cup \Gamma^j \cup \hat{D}^j$ of Γ^j. Assume $G_j(z)$ single-valved on $D^j \cup \Gamma^j \cup \hat{D}^j$ [3] shows.

(1) $z = \overline{G_j(z)}$ is Γ^j.

(2) $G_j(z)$ is analytic on $D^j \cup \Gamma^j \cup \hat{D}^j$, where D^j is contained in the connected R and D^j is contained in the complement of $\Gamma^j \cup D^j$ for $j = 1, 2, \cdots, s$.

(3) The transformation $\hat{z} = \overline{G(z)}$ is an involution; i.e. $\hat{\hat{z}} = z$.

(4) If z is in D^j then $\hat{z}$ is in $\hat{D}^j$ and if z is in $\hat{D}^j$ then $\hat{z}$ is in D^j.

(5) $\overline{G[D^j]} = \hat{D}^j$ and $\overline{G[\hat{D}^j]} = D^j$. We assume the boundary of $\hat{D}^j$, that is not Γ^j, is a contour C^j and $G_j(z)$ is continuous on $\hat{D}^j \cup C^j$.

THEOREM 1. (H 1) *Let $f(z)$ be an analytic single valued function on R whose boundary is Γ such that the real part of $f(z)$ solves the Dirichlet problem in R with real boundary values $B_j(z)$ on Γ^j where $B_j(z)$ are single-valued and continuous in $D^j \cup \Gamma^j \cup \hat{D}^j \cup C^j$ and analytic in $D^j \cup \Gamma^j \cup \hat{D}^j$. Let $f(z)$ be continuous and single-valued on $R \cup \Gamma$.*

(H. 2) *Let $z^j_{n_j 1}, z^j_{n_j 2}, \cdots, z^j_{n_j n_j+1}$, $n_j = 0, 1, 2, \cdots,$ be points of Γ^j, $j = 1, 2, \cdots, s$. Let $p^1_{n_1}(z)$ be the polynomial in z of degree n_1 that agrees with $B_1(z)$ at $z^1_{n_1 1}, z^1_{n_1 2}, \cdots, z^1_{n_1 n_1+1}$ and let $p^j_{n_j}(z)$ $(2 \leqq j \leqq s)$ be the polynomial in $1/(z - a_j)$ that agrees with $B_j(z)$ for $z - a_j = z^j_{n_j 1}$, $z^j_{n_j 2}, \cdots, z^j_{n_j n_j+1}$ where a_j is a point inside c^j.*

(H. 3) *Let*

$$\delta^1_{n_1} = \min_{t \text{ on } C^1} \prod_{k=1}^{n_1+1} |t - z^1_{n_1 k}|$$

$$\mu^1_{n_1} = \max_{z \text{ on } \Gamma^1} \prod_{k=1}^{n_1+1} |z - z^1_{n_1 k}|$$

and

$$\delta^j_{n_j} = \min_{t \text{ on } C^j} \prod_{k=1}^{n_j+1} \left| \frac{t - a_j - z^j_{n_j k}}{t - a_j} \right|, \quad 2 \leqq j \leqq s$$

$$\mu^j_{n_j} = \max_{z \text{ on } \Gamma^j} \prod_{k=1}^{n_j+1} \left| \frac{z - a_j - z^j_{n_j k}}{z - a_j} \right|, \quad 2 \leqq j \leqq s .$$

(H. 4) *Let* $\mu^j_{n_j}/\delta^j_{n_j} \to 0$ *as* $n_j \to \infty$, $j = 1, 2, \cdots, s$. *Then for*

$$\mu = \left(\frac{1}{n_1}, \frac{1}{n_2}, \cdots, \frac{1}{n_s}\right) \quad \textit{and} \quad |\mu| = \max\left\{\frac{1}{n_1}, \frac{1}{n_2}, \cdots, \frac{1}{n_s}\right\}.$$

(C. 1) $R_\mu(z) = \sum_{j=1}^s p^j_{n_j}(z)$ *converges uniformly to* $f(z)$ *in* $R \cup \Gamma$ *as* $|\mu| \to 0$ *and thus Re* $R_\mu(z)$ *converges uniformly to* $u(x, y)$ *in* R *and uniformly to* $B_j(z)$ *on* Γ^j.

(C. 2) *Moreover in* $R \cup \Gamma$:

$$|f(z) - R_\mu(z)| \leqq \frac{1}{2\pi} \sum_{j=1}^s \frac{L_j M_j}{\delta_j} \mu^j_{n_j}/\delta^j_{n_j}$$

where $L_j =$ *length of* C^j, $M_j = \max_{t \text{ on } C^j} |f(t)|$ *and* $\delta_j = \inf_{z \text{ on } \Gamma^j} \min_{t \text{ on } C^j} |t - z|$.

Proof. In order to avoid notation that only confuses, we shall prove the theorem for the case $s = 2$.

We first analytically continue $f(z)$ into $R \cup \Gamma^1 \cup \hat{D}^1 \cup \Gamma^2 \cup \hat{D}^2$. Let $f^*_j(z) = \overline{f[\overline{G_j(z)}]}$ for z in $\Gamma^j \cup \hat{D}^j$. $f^*_j(z)$ is defined and analytic for z in $\Gamma^j \cup \hat{D}^j$ since $\hat{z} = \overline{G_j(z)}$ is in D^j for z in $\hat{D}^j$ and $G_j(z)$ is analytic for z in $\Gamma^j \cup \hat{D}^j$. But $f^*_j(z) = \overline{f(z)}$ for z on Γ^j, thus on Γ^j

$$f(z) + f^*_j(z) = 2B_j(z).$$

Thus $f(z) = 2B_j(z) - f^*_j(z)$ analytically continues $f(z)$ into $\Gamma^j \cup \hat{D}^j$ since $f(z)$ is continuous up to and on Γ^j. Moreover, $f(z)$ is continuous on $\Gamma^j \cup \hat{D}^j \cup C^j$ since $G_j(z)$ and $B_j(z)$ are. Thus $f^*_j(z) \equiv \overline{f_j[\overline{G(z)}]}$ analytically continues $f^*(z)$ into $\Gamma^j \cup D^j$ since $f^*_j(z)$ is continuous up to and on Γ^j. Let $\alpha_{n+1}(z) = (z - z^1_{n1})(z - z^1_{n2}) \cdots (z - z^1_{nn+1})$

$$\beta_{m+1}(z) = \left(\frac{1}{z - a_2} - \frac{1}{z^2_{m1}}\right)\left(\frac{1}{z - a_2} - \frac{1}{z^2_{m2}}\right) \cdots \left(\frac{1}{z - a_2} - \frac{1}{z^2_{mm+1}}\right).$$

Then for z on Γ:

$$p_{nm}(z) = \frac{1}{2\pi i} \oint_{C^1} \frac{f(t)}{t - z} \frac{\alpha_{n+1}(t) - \alpha_{n+1}(z)}{\alpha_{n+1}(t)} dt - \frac{1}{2\pi i} \oint_{C^2} \frac{f(t)}{t - z} \frac{\beta_{m+1}(t) - \beta_{m+1}(z)}{\beta_{m+1}(t)} dt$$

where $p_{nm}(z)$ is a rational function of z, $p_{nm}(z) = p^1_n(z) + p^2_{m+1}(z)$ in which $p^1_n(z)$ is the polynomial in z of degree $\leqq n$ got by interpolating $f(z)$ along Γ^1 at $z^1_{n1}, z^1_{n2}, \cdots, z^1_{nn+1}$ and $p^2_{m+1}(z)$ is the polynomial of degree $m + 1$ in $1/(z - a_2)$ got by interpolating $f(z)$ along Γ^2 so that $p^2_{m+1}(a_2 + z^2_{mj}) = f(z^2_{mj})$. To see the latter let $x = 1/(z - a_2)$ and $y =$

$1/(t - a_2)$ then $\beta_{m+1}(t) = b_{m+1}(y)$ where $b_{m+1}(y)$ is a monic polynomial in y of degree $\leqq m + 1$, thus we have

$$\beta_{m+1}(t) - \beta_{m+1}(z) = b_{m+1}(x) - b_{m+1}(y) = (x - y) \sum_{i=1}^{m} a_i(x) y^i$$

where $a_i(x)$ are polynomials in x of degree $\leqq m$. But

$$x - y = \frac{1}{z - a_2} - \frac{1}{t - a_2} = (t - z)xy$$

thus

$$\int_{\sigma^2} \frac{f(t)}{t - z} \frac{\beta_{m+1}(t) - \beta_{m+1}(z)}{\beta_{m+1}(t)} dt$$

is a polynomial of degree $\leqq m + 1$ in $1/(z - a_2)$. The error for z on Γ is given by:

$$f(z) - p_{nm}(z) = \frac{1}{2\pi i} \int_{\sigma^1} \frac{f(t)}{t - z} \frac{\alpha_{n+1}(z)}{\alpha_{n+1}(t)} dt - \frac{1}{2\pi i} \int_{\sigma^2} \frac{f(t)}{t - z} \frac{\beta_{m+1}(z)}{\beta_{m+1}(t)} dt .$$

Note that:

$$|\alpha_{n+1}(z)| \leqq \mu_n^1 , \quad |\alpha_{n+1}(t)| \geqq \delta_n^1 \text{ for } z \text{ on } \Gamma^1 \text{ and } t \text{ on } C^1 \text{ and:}$$

$$\frac{\dfrac{1}{z - a_2} - \dfrac{1}{z_{n1}^2}}{\dfrac{1}{t - a_2} - \dfrac{1}{z_{n1}^2}} = \frac{z - a_2 - z_{n1}^2}{t - a_2 - z_{n1}^2} \frac{t - a_2}{z - a_2}$$

and thus

$$\left| \frac{\beta_{m+1}(z)}{\beta_{m+1}(t)} \right| \leqq \frac{\mu_m^2}{\delta_m^2} \text{ for } z \text{ on } \Gamma^2 \text{ and } t \text{ on } C^2.$$

From these it follows:

$$|f(z) - p_{nm}(z)| \leqq \frac{1}{2\pi} \left\{ \frac{L_1 M_1}{\delta_1} \frac{\mu_n^1}{\delta_1} + \frac{L_2 M_2}{\delta_2} \frac{\mu_m^2}{\delta_m^2} \right\} \qquad \text{for } z \text{ on } \Gamma,$$

where L_j is the length of C^j,

$$M_j = \max_{t \text{ on } C^j} |f(t)| , \quad \text{and} \quad \delta_j = \inf_{z \text{ on } \Gamma^j} \min_{t \text{ on } C^j} |t - z|$$

which is the result.

We next consider the case when Γ is a single analytic contour and ($C^j = C$) we write Γ in parametric form as $z(\sigma) = x(\sigma) + iy(\sigma)$ where $-1 \leqq \sigma \leqq 1$. Let $|z(\sigma_2) - z(\sigma_1)| \leqq A |\sigma_2 - \sigma_1|$, let Γ contain the origin and

THEOREM 2 (H. 1) *Let $f(z)$ be an analytic single-valued function on R whose boundary is Γ such that the real part of $f(z)$ solves the Dirichlet problem in R with real boundary values $B(z)$ on Γ where $B(z)$ is a single-valued analytic function on $D \cup \Gamma \cup \hat{D}$ continuous on $\Gamma \cup \hat{D} \cup C$. Let $f(z)$ be continuous and single-valued on $R \cup \Gamma$.*

(H. 2) *Let $z_j^n = z(\sigma_j^n)$ where*

$$\sigma_j^n = \cos\left[(2j-1)\pi/(2n+2)\right],\ j = 1, 2, \cdots, n+1$$

(H. 3) $\delta = \inf_{z \text{ on } \Gamma} \min_{t \text{ on } C} |t - z|$

(H. 4) $A < 2\delta$.

Then

(C. 1) $p_n(z) = \sum_{j=1}^{n+1} f(z_j^n) \dfrac{\omega_{n+1}(z)}{\omega'_{n+1}(z_j^n)(z - z_j^n)}$, *where*

$\omega_{n+1}(z) = \prod_{k=1}^{n+1}(z - z_k^n)$ *and prime denotes differentiation, converges uniformly to $f(z)$ on $R \cup \Gamma$ as $n \to \infty$*

(C. 2) $|f(z) - p_n(z)| \leqq \dfrac{LM}{\pi\delta}\left(\dfrac{A}{2\delta}\right)^{n+1}$

where M is a constant depending on f, L is length of Γ.

Proof. As in the proof of Theorem 1 we have for z on Γ

$$|f(z) - p_n(z)| \leqq \frac{L}{2\pi} M \max_{z \text{ on } \Gamma} |\omega_{n+1}(z)|/\delta^{n+2}.$$

But

$$|\omega_{n+1}(z)| = |(z - z_1^n)(z - z_2^n) \cdots (z - z_{n+1}^n)|$$
$$\leqq A^{n+1} |(\sigma - \sigma_1^n)(\sigma - \sigma_2^n) \cdots (\sigma - \sigma_{n+1}^n)|$$

where the σ_j^n are the roots of the Chebyshev polynomial

$$T_{n+1}(\sigma) = \cos\left[(n+1) \text{ arc } \cos\sigma\right]$$

of degree $n+1$. Thus since $(\sigma - \sigma_1^n)(\sigma - \sigma_2^n) \cdots (\sigma - \sigma_{n+1}^n)$ is monic

$$|\omega_{n+1}(z)| \leqq A^{n+1} T_{n+1}(\sigma)/2^n.$$

Thus

$$\max_{z \text{ on } \Gamma} |\omega_{n+1}(z)| \leqq A^{n+1}/2^n \quad \text{and} \quad |f(z) - p_n(z)| \leqq \frac{LM}{\delta\pi}\left(\frac{A}{2\delta}\right)^{n+1}.$$

Next let Γ: $z(s) = x(s) + iy(s)$ where s is arc length $0 \leqq s \leqq L$.

THEOREM 3. (H. 1) *Same as Theorem 2.*

(H. 2) *Same as Theorem 2 but $z_j^n = z(s_j^n)$ where*

$$s_j^n = \frac{L}{2}\cos[(2j-1)\pi/2(n+1)] + \frac{L}{2}, \; j = 1, 2, \cdots, n+1 .$$

(H 3) $\delta = \inf_{z \text{ on } \Gamma} \min_{t \text{ on } C} |t - z|$.

(H 4) $L < 4\delta$.

(C 1) *Same as Theorem* 2.

(C 2) $|f(z) - p_n(z)| \leqq \frac{LM}{\pi\delta}\left(\frac{L}{4\delta}\right)^{n+1}$

where M *is a constant depending on* f.

Proof. As in the proof of theorem for z on Γ:

$$|f(z) - p_n(z)| \leqq \frac{L}{2\pi} M \max |\omega_{n+1}(z)|/\delta^{n+2} .$$

But since $|z - z_j^n| \leqq |s - s_j^n|$ where $z = z(s)$ and $z_j^n = z(s_j^n)$ and since $|(s - s_1^n)(s - s_2^n) \cdots (s - s_{n+1}^n)| \leqq L^{n+1}/2^{2n+1}$ see e.g. [1] we have

$$|f(z) - p_n(z)| \leqq \frac{LM}{\pi\delta}\left(\frac{L}{4\delta}\right)^{n+1} .$$

EXAMPLE. We shall now apply the ideas of this paper to a particular geometrical configuration. Let

Γ^1 be a circle of radius 15 centered at the origin

Γ^2 be a circle of radius 1 centered at $(-13, 0)$

Γ^3 be an ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \; a = 1.075, \; b = 1 .$$

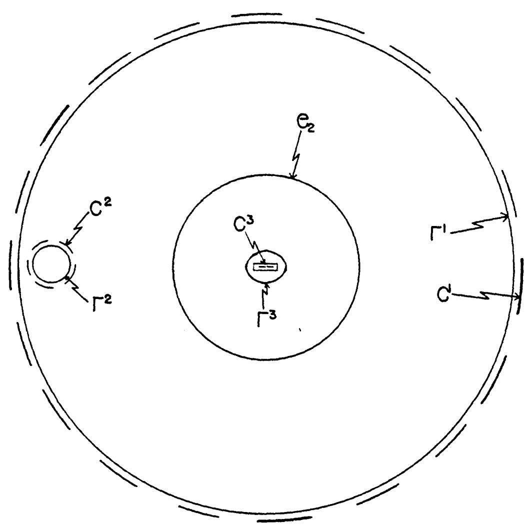

Fig. 1.

Let R be the interior of Γ^1 less Γ^2, Γ^3 and the interiors of Γ^2 and Γ^3. Let $f(z)$ be analytic on R and continuous on $R \cup \Gamma^1 \cup \Gamma^2 \cup \Gamma^3$. Moreover let the real part of $f(z)$ satisfy boundary conditions $B_1(z)$ on Γ^1, $B_2(z)$ on Γ^2 and $B_3(z)$ on Γ^3 where:

$B_1(z)$ is analytic on $|z| = 15$

$B_2(z)$ is analytic on $|z + 13| = 1$

and $B_3(z)$ is analytic in and on $\Gamma^3 - \{-.395 < x < .395, y = 0\}$ See figure.

For example we might have $Re\, f(z) = P_k(x, y)$ on Γ^k $(k = 1, 2, 3)$ where $P_k(x, y)$ is a polynomial.

Then since:

$$G_1(z) = (15)^2 z^{-1}$$

$$G_2(z) = (z + 13)^{-1} - 13$$

$$\text{(E. 1)} \qquad G_3(z) = \frac{z(a^2 + b^2) \pm 2ab\sqrt{z^2 + b^2 - a^2}}{a^2 - b^2} \quad a > b \text{ see [3]}$$

and $z = \hat{z} = \overline{G_k(z)}$ on Γ_k we have on Γ_k

$$P_k\left(\frac{z + \bar{z}}{2}, \frac{z - \bar{z}}{2i}\right) = P_k\left[\frac{z + G_k(z)}{2}, \frac{z - G_k(z)}{2i}\right] \equiv B_k(z)$$

which are meromorphic functions that fulfill the requirements of $B_1(z)$, $B_2(z)$ and $B_3(z)$ (in the case of $B_3(z)$ we make a cut between the foci $\pm\sqrt{a^2 - b^2})$.

Let

$$r_k = 15 \exp\left(\frac{2\pi i}{n + 1} k\right) \quad \text{and} \quad \alpha_{n+1}(z) = (z - r_1)(z - r_2) \cdots (z - r_{n+1})$$

and

$$p_n(z) = \sum_{k=1}^{n+1} f(r_k) \frac{\alpha_{n+1}(z)}{\alpha_{n+1}(r_k)(z - r_k)}$$

where the prime signifies differentiation. $p_n(z)$ is clearly the polynomial of degree $\leqq n$ that interpolates $f(z)$ at $z = r_k$ on Γ^1, $k = 1, \cdots, n + 1$. Next let

$$s_k = \exp\left(\frac{2\pi i}{m + 1} k\right)$$

and

$$\beta_{m+1}(z) = \left(\frac{1}{z + 13} - \frac{1}{s_1}\right)\left(\frac{1}{z + 13} - \frac{1}{s_2}\right) \cdots \left(\frac{1}{z + 13} - \frac{1}{s_{m+1}}\right)$$

and

$$q_m(z) = \sum_{k=1}^{m+1} f(s_k - 13) \frac{\beta_{m+1}(z)}{B'_{m+1}(s_k - 13)\left(\frac{1}{s_k} - \frac{1}{z+13}\right)(s_k)^2}$$

$q_m(z)$ is a polynomial of degree $\leqq m$ in $1/(z + 13)$ such that $q_m(s_k - 13) = f(s_k - 13)$ where $s_k - 13$ is on Γ^2, $k = 1, 2, \cdots, m + 1$.

Finally let l be the length of the ellipse Γ^3 and

$$\sigma_k = \cos [(2k - 1)\pi/2(j + 1)], k = 1, 2, \cdots, j + 1 .$$

Then the ellipse Γ^3 can be written

$$z(\sigma) = x(\sigma) + iy(\sigma) = a \cos (2\pi\sigma/l) + ib \sin (2\pi\sigma/l), \ -l/2 \leqq \sigma \leqq l/2$$

σ is are length parameter shifted. Let

$$t_k = z(\sigma_k l/2) \quad \text{and} \quad \kappa_{j+1}(z) = \left(\frac{1}{z} - \frac{1}{t_1}\right)\left(\frac{1}{z} - \frac{1}{t_2}\right) \cdots \left(\frac{1}{z} - \frac{1}{t_{j+1}}\right)$$

and

$$r_j(z) = \sum_{k=1}^{j+1} f(t_k) \frac{\kappa_{j+1}(z)}{\kappa'_{j+1}(t_k)\left(\frac{1}{t_k} - \frac{1}{z}\right)(t_k)^2}$$

$r_j(z)$ is clearly the polynomial in $1/z$ of degree $\leqq j$ such that $r_j(t_k) = f(t_k)$ $k = 1, 2, \cdots, j + 1$ where t_k is on Γ^3.

Then the assertion is

$$p_n(z) + q_m(z) + r_j(z)$$

converges uniformly to $f(z)$ on $R \cup \Gamma^1 \cup \Gamma^2 \cup \Gamma^3$ as

$$\frac{1}{n} + \frac{1}{m} + \frac{1}{j} \to 0 .$$

For Γ^1, we use Runge's theorem. Since $B_1(z)$ is analytic on Γ^1, then $f(z)$ can be continued across Γ^1, i.e. $f(z)$ is analytic for $15 \leqq |z| \leqq 15 + \varepsilon$ where ε is some positive number. Thus in the notation of the theorem

$$\begin{aligned}
\delta_n^1 &= \min_{|t|=15+\varepsilon} \left|\prod_{k=1}^{n+1} (t - r_k)\right| = \min_{|t|=15+\varepsilon} |t^{n+1} - 15^{n+1}| \\
&= \min_{|\tau|=1+\varepsilon/15} 15^{n+1} |\tau^{n+1} - 1| \geqq 15^{n+1} \min_{|\tau|=1+\varepsilon/15} \{|\tau|^{n+1} - 1\} \\
&= 15^{n+1}\{[1 + \varepsilon/15]^{n+1} - 1\} \\
\mu_n^1 &= \max_{|z|=15} \left|\prod_{k=1}^{n+1} (z - r_k)\right| = 15^{n+1} \max_{|\zeta|=1} |\zeta^{n+1} - 1| \leqq 2\cdot 15^{n+1}
\end{aligned}$$

and

(E 2) $u_n^1/\delta_n^1 \leqq 2/\{[1+\varepsilon/15]^{n+1}-1\} \to 0$ as $n \to \infty$.

For Γ^2, since $B_2(z)$ is analytic on Γ^2, then $f(z)$ can be continued across Γ^2, i.e. $f(z)$ is analytic for $1-\varepsilon \leqq |z+13| \leqq 1$ where ε is some positive number. Thus if $C^2 = \{z: |z+13| = 1-\varepsilon\}$ we have:

$$\max_{z \text{ on } \Gamma^2} |\beta_{m+1}(z)| = \max_{z \text{ on } \Gamma^2} \left| \prod_{k=1}^{m+1} \left(\frac{1}{z+13} - \frac{1}{s_k} \right) \right| = \max_{z \text{ on } \Gamma^2} \left| \left(\frac{1}{z+13} \right)^{m+1} - 1 \right|$$
$$= \max_{|\zeta|=1} \left| \left(\frac{1}{\zeta} \right)^{m+1} - 1 \right| \leqq 2$$

and

$$\min_{t \text{ on } C^2} |\beta_{m+1}(t)| = \min_{t \text{ on } C^2} \left| \left(\frac{1}{t+13} \right)^{m+1} - 1 \right|$$
$$= \min_{|\zeta|=1-\varepsilon} \left| \left(\frac{1}{\zeta} \right)^{m+1} - 1 \right| \geqq \left(\frac{1}{1-\varepsilon} \right)^{m+1} - 1 .$$

From these we see that

$$\text{(E. 3)} \qquad \frac{\max_{z \text{ on } \Gamma^2} |\beta_{m+1}(z)|}{\min_{t \text{ on } C^2} |\beta_{m+1}(t)|} \leqq \frac{2}{\left(\frac{1}{1-\varepsilon} \right)^{m+1} - 1} \to 0 \text{ as } m \to \infty ,$$

For Γ^3 we note from the reflection function $G(z)$ given by (E. 1) that the interior of the ellipse Γ^3 minus the line $-c \leqq x \leqq c$, $c^2 = a^2 - b^2$, is reflected exterior to the given ellipse but interior to the ellipse e_2

$$\frac{\hat{x}^2}{\hat{a}^2} + \frac{\hat{y}^2}{\hat{b}^2} = 1$$

where $\hat{a} = (a^2 + b^2)/c$, $\hat{b} = 2\,ab/c$.

In the case of our ellipse we have $a = 1.075$, $b = 1$ and $c = .395$, $\hat{a} = 5.46$, $\hat{b} = 5.44$, thus e_2 is contained in Γ^1, and does not intersect or contain points of Γ^2 and thus $f(z)$ can be extended to be analytic in $\Gamma^3 - \{z \mid -.395 < x < .395,\ y = 0\}$.

The length of the ellipse Γ^3 is given by:

$$l = 4\text{a} \int_0^{\pi/2} \sqrt{1 - k^2 \sin^2 \theta \, d\,\theta}$$

where

$k = c/a < 1$. In our case $k = .368$ and thus
$l = 4a\ (1.516)$

using a table for elliptic integrals. Let c^3 be the rectangular contour

$$(-.395 - \varepsilon \leqq x \leqq .395 + \varepsilon, y = -\varepsilon) , \ (x = .395 + \varepsilon, -\varepsilon \leqq y \leqq \varepsilon) ,$$
$$(-.395 - \varepsilon \leqq x \leqq .395 + \varepsilon, y = \varepsilon) , \ (x = -.395 - \varepsilon, -\varepsilon \leqq y \leqq \varepsilon)$$

where $\varepsilon > 0$ is arbitrarily small. Then consider

$$\frac{1}{2\pi i}\int_{\sigma^3} \frac{f(t)}{t-z} \frac{\kappa_{j+1}(z)}{\kappa_{j+1}(t)}\, dt .$$

But

$$| \kappa_{j+1}(z) | = \left|\prod_{k=1}^{j+1}\left(\frac{1}{z} - \frac{1}{t_k}\right)\right| = \left|\prod_{k=1}^{j+1}\frac{t_k - z}{t_k z}\right|$$
$$\leqq \prod_{k=1}^{j+1} \frac{\sigma - \sigma_k l/2}{| t_k | \, | z |} \leqq \left(\frac{l}{2}\right)^{j+1} \prod_{k=1}^{j+1} \frac{| \theta - \sigma_k |}{| t_k |} . \quad \sigma = \text{arc length},$$

where $-1 \leqq \theta \leqq 1$ since $|z| \geqq 1$ for z on Γ^3. Also for t on C^3

$$| \kappa_{j+1}(t) | = \left|\prod_{k=1}^{j+1}\frac{t_k - t}{t_k t}\right| \geqq (a - c - \eta\varepsilon)^{j+1} \prod_{k=1}^{j+1} \frac{1}{| t_k | \, | t |}$$

where η is some fixed constant. But since $|t| \leqq c + \varepsilon/2$ for t on c^3 we see that

$$| \kappa_{j+1}(t) | \geqq \left(\frac{a - c - \eta\varepsilon}{c + \varepsilon\sqrt{2}}\right)^{j+1} \prod_{k=1}^{j+1} \frac{1}{| t_k |} .$$

Combining the above results gives

$$A \equiv | \kappa_{j+1}(z)/\kappa_{j+1}(t) | \leqq \left(\frac{c + \varepsilon\sqrt{2}}{a - c - \eta\varepsilon}\right)^{j+1}\left(\frac{l}{2}\right)^{j+1} 2^{-j} \, | T_{j+1}(\theta) |$$

where we have utilized the fact that the σ_k are the roots of the Chebyshev polynomial $T_{j+1}(\theta) = \text{cox}\,[(j + 1) \text{ are} \cos\theta]$. Thus

$$A \leqq 2\left(\frac{c + \varepsilon\sqrt{2}}{a - c - \eta\varepsilon}\right)^{j+1}\left(\frac{l}{4}\right)^{j+1} .$$

But

$$\frac{l}{4a} = 1.516 < \frac{1}{a}\,\frac{a - c - \eta\varepsilon}{c + \varepsilon\sqrt{2}} = \frac{1}{1.075}\,\frac{.680 - \eta\varepsilon}{.395 + \sqrt{2\varepsilon}} , \ = 1.60 + g(\varepsilon)$$

where $g(\varepsilon) \to 0$ as $\varepsilon \to 0$. Thus for ε sufficiently small

$$\text{(E. 4)} \qquad l/4 \leqq \frac{a - c - \eta\varepsilon}{c + \varepsilon\sqrt{2}} .$$

Utilizing (E. 2), (E. 3) and (E. 4) we have from Theorem 1 that $p_n(z) + p_m(z) + r_j(z)$ converges uniformly to $f(z)$ in $R \cup \Gamma^1 \cup \Gamma^2 \cup \Gamma^3$ as $(1/n) + (1/m) + (1/j) \to 0$.

We remark finally that there would be no new difficulties if Γ had contained in addition $\Gamma^4 \cup \Gamma^5 \cup \Gamma^6$ where Γ^4 is the circle $|z - 10i| = 4$, Γ^5 the circle $|z + 10i| = 4$ and Γ^6 is the circle $|z - 12| = 2$.

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VISUALIZING THE WORD PROBLEM, WITH AN APPLICATION TO SIXTH GROUPS

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The word problem in certain groups is studied in algebraic terms with a geometric background. A relator is made to correspond to a plane complex so that generators are associated with 1-cells and defining relators are associated with 2-cells of the complex. In the case of less-than-one-sixth groups, the results obtained are essentially those found by Greendlinger.

Let $\mathscr{G} = \mathscr{F}/\mathscr{N}$ where $\mathscr{N}$ is a normal subgroup of a free group $\mathscr{F}$ with fixed free generators (understood to include inverses). Let $\mathscr{N}$ be the smallest normal subgroup containing a set $\mathscr{R}$ of cyclically reduced words (defining relators for $\mathscr{G}$). Nonempty words in $\mathscr{N}$ are relators for $\mathscr{G}$. Let $\mathscr{R}$ be closed under inverses and cyclic permutations. Assume each free generator appears in at least one defining relator.

In this paper we use complexes to study how relators depend upon defining relators. A complex is determined by a finite set E of elements (called edges), a partition of E into subsets (called boundaries), a partition of E into pairs of edges, and a cyclic order for the edges in each boundary; vertices and the property of connectedness can then be defined. After a free generator is assigned to each edge (with inverse free generators assigned to paired edges), the above-mentioned cyclic orders determine words (called values) for each boundary. More precisely, some word and all its cyclic permutations are the values of a boundary.

It is shown that each relator is a value of one of the boundaries of some spherical complex (a connected complex with Euler characteristic 2) whose other boundaries have defining relators for their values. The converse is also proved: if defining relators are the values of all but one of the boundaries of a spherical complex, then a value of the remaining boundary is a relator. Thus the question of recognizing the relators in $\mathscr{G}$—the word problem in $\mathscr{G}$—can be viewed as the question of determining the words which can correspond to one boundary of a spherical complex whose other boundaries correspond to defining relators.

These results are essentially a reformulation of the first two lemmas in a paper by Van Kampen who approached the problem geometrically. The proofs given here are combinatorial in nature.

In passing from a relator to a complex, we use a system (called a

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structure) which characterizes one construction of the relator from a collection of defining relators. Structures help to define certain basic relators.

The problem of recognizing relators is reduced to finding basic relators by showing that each freely reduced relator contains a subword which is a basic relator. When W is a cyclically reduced basic relator, some subword of W is a subword of a defining relator. The number of such subwords, contained disjointly in the cyclic word W, is estimated via simple calculations using a spherical complex associated with W. The calculations are given in § 8; they were suggested by the proof of the Five Color Theorem in [1] Courant and Robbins.

This estimate is applied when W is in a group $\mathscr{G}$ which is a less-than-one-sixth group or, briefly, a sixth group. A group $\mathscr{G}$ is called a sixth group if any subword common to 2 distinct defining relators has a length which is less than one sixth of the length of both of the defining relators. As a result, W is seen to contain a subword which is more than one half of a defining relator.

Thus a nonempty cyclically reduced word is a relator in a sixth group only if the word can be shortened by replacing one of its subwords X by a shorter word Y^{-1} where XY is a defining relator. This solves the word problem for sixth groups. Other proofs have been given by Tartakovskii and Greendlinger.

Our results are contained in the following

MAIN THEOREM. *In a presented group, each freely reduced relator contains a subword which is a certain kind of relator called a basic relator.*

If a cyclically reduced word W is a basic relator for a sixth group, then either W is a defining relator or the cyclic word W contains disjointly P_k subwords which are greater than $7\text{-}k/6$ of a defining relator $(k=2, 3, 4)$ and the integers P_k satisfy $3P_2+2P_3+P_4 \geqq 6$. Thus W contains a subword which is more than $1/2$ of a defining relator.

2. **Constructing relators.** Let $W \cong V_1XV_2$ and $V \cong V_1V_2$ be words in $\mathscr{F}$. Here "$\cong$" stands for "identically equal to". We write $W \rightarrow V$ (delete X) and $V \rightarrow W$ (insert X). If also $V \rightarrow U$ (delete Y), then $W \rightarrow U$ (delete X, Y). This leads to a definition of $W \rightarrow W'$ (delete $X_1, \cdots, X_n$) and $W' \rightarrow W$ (insert $X_n, \cdots, X_1$) for $n \geqq 1$.

A word W *splits* into one or more words $W_1, \cdots, W_n$ if the W_i can be put in a sequence $W_1', \cdots, W_n'$ so that $1 \rightarrow W$ (insert $W_1', \cdots, W_n'$) where 1 denotes the empty word. An *$\mathscr{R}$-word of type t* is any word which splits into t defining relators.

A product of a free generator and its inverse is a *null word.* If

W, W' are words such that either $W \cong W'$ or $W \to W'$ (delete $N_1, \cdots, N_k$) where the N_i are null words, then W *partially reduces* to W' and W' is a *partially reduced form* of W. If, in addition, no subword of W' is a null word, then W' is the freely reduced form of W. A *relator of type* t is a partially reduced $\mathscr{R}$-word of type t (i.e. a partially reduced form of an $\mathscr{R}$-word of type t).

The first lemma shows that each relator can be constructed from the empty word by insertions of defining relators, possibly followed by deletions of null words.

LEMMA 2.1. *Each relator is a partially reduced $\mathscr{R}$-word. In other words, each relator has at least one type.*

Proof. The collection of $\mathscr{R}$-words is closed under inverses and products. If W is an $\mathscr{R}$-word and x is a free generator, then it must be shown that xWx^{-1} is a partially reduced form of some $\mathscr{R}$-word W'. Suppose $1 \to W$ (insert $R_1, \cdots, R_n$) where the R_i are defining relators. Let x be the first letter in a defining relator $R \cong xY$. Put $W' \cong xWYY^{-1}x^{-1}$ so that W' partially reduces to xWx^{-1} and $1 \to W'$ (insert $R, R_1, \cdots, R_n, R^{-1}$). This completes the proof.

It can be shown if W'' is a cyclic permutation of a word W which splits into $W_1, \cdots, W_n$, then W'' splits into some cyclic permutations $W_1'', \cdots, W_n''$ of $W_1, \cdots, W_n$, respectively. Hence,

REMARK 2.1. The set of $\mathscr{R}$-words of type t is closed under cyclic permutations. The set of relators of type t is closed under cyclic permutations.

3. Structures for relators.

We need terminology for permutations of a finite set in order to define a structure. In this section, all sets are finite; 0 denotes the empty set.

Let θ be a cyclic permutation, acting on a set E. If $E \neq 0$, suppose $E = \{a_1, \cdots, a_m\}$ and either $m = 1$ with $a_1\theta = a_1$ or $m \geqq 2$ with $a_i\theta = a_{i+1}(1 \leqq i \leqq m-1)$ and $a_m\theta = a_1$. Then θ is represented by an *array* $H \cong a_1 \cdots a_m$ and by the m cyclic permutations of H. Any subword of H is said to partially represent θ. If $E = 0$, then θ is the empty permutation, represented by the empty array 1.

A set of words in $\mathscr{F}$ is associated with θ by assigning a free generator to each element in E. If x_i is assigned to a_i, then $V \cong x_1 \cdots x_m$ (a word in $\mathscr{F}$) is called the *value* of H or a value of θ. The values of θ are the cyclic permutations of V. If $E = 0$, the empty word is the only value of θ.

A cyclic permutation θ_B corresponds to each subset B of E. If

$B \neq 0$ and the elements of B form a subsequence $b_1, \cdots, b_k$ of $a_1, \cdots, a_m$, then θ_B is represented by the array $b_1 \cdots b_k$. If $B = 0$, then θ_B is the empty permutation.

A permutation β, acting on a nonempty set E, determines a partition of E into nonempty subsets $E_1, \cdots, E_n$, called *β-orbits*: two elements a, b are in the same β-orbit if $a\beta^i = b$ for some integer i. The *β-cycles* are the restrictions of β to the sets $E_1, \cdots, E_n$. The *length* of a β-cycle is the number of elements in the corresponding β-orbit. β is a *reflection* (*pure reflection*) if the length of each β-cycle is at most 2 (exactly 2).

A *structure* $S = (E, \beta, \rho, \theta)$ consists of a nonempty set E which is acted on by a permutation β, a reflection ρ, and a cyclic permutation θ. S has *carrier* E, *reduced carrier* $F = \{a : a \in E,\ a\rho = a\}$, *map* θ, and *reduced map* θ_F. It is required that there exist arrays H, H_ρ, representing θ, θ_F, respectively, such that

(I) There exist arrays $H_1, \cdots, H_n$, $n \geqq 1$, representing the β-cycles, such that $1 \rightarrow H$ (insert $H_1, \cdots, H_n$).

(II) Either ρ is the identity and $H_\rho \cong H$ or there exist arrays $I_1, \cdots, I_k$, $k \geqq 1$, representing the ρ-cycles of length 2, such that $H \rightarrow H_\rho$ (delete $I_1, \cdots, I_k$).

S is said to be of type n. The members of F are *fixed* elements; the members of $E - F$ are *cancelled* elements.

If H_ρ contains a subword I, of length 2, whose elements are a, b, then $S' = (E, \beta, \sigma, \theta)$ is also a structure where $a\sigma = b$, $b\sigma = a$ and $\sigma = \rho$ except on the set $\{a, b\}$. Indeed, if H_σ is defined by $H_\rho \rightarrow H_\sigma$ (delete I), then H_σ represents the reduced map of S'. We say that S *contracts* to S' in one step.

S is an *$\mathscr{R}$-structure* (*$\mathscr{N}$-structure*) if a free generator is assigned to each element in E in such a way that the values of the ρ-cycles of length 2 are null words and the values of the β-cycles are words in $\mathscr{R}$ (in $\mathscr{N}$). When S is an $\mathscr{R}$-structure, of type n, with map θ and reduced map θ_F, then the values of θ are $\mathscr{R}$-words of type n and the values of θ_F are relators of type n.

THEOREM 3.1. *Each relator is a value of the reduced map of some $\mathscr{R}$-structure.*

Proof. Use the definition of $\mathscr{R}$-structure and Lemma 2.1.

We now turn to some more definitions concerning a structure $S = (E, \beta, \rho, \theta)$. S is called *noncancelled* if there exist fixed elements in E. S is *cancelled* if E contains only cancelled elements. In the latter case, ρ is a pure reflection.

If A is a nonempty subset of E, then A is the carrier of a substructure T whenever A is closed under β and ρ. In this case, $T =$

$(A, \gamma, \sigma, \theta_A)$ where γ, σ are the restrictions of β, ρ, respectively, to the set A. T is a proper substructure if $A \neq E$. S is *minimal* if it has no proper substructures; S is *simple* if it has no proper cancelled substructure.

THEOREM 3.2. *Each relator is a value of the reduced map of some simple $\mathscr{R}$-structure.*

Proof. Use previous theorem and next lemma.

LEMMA 3.1. *Each structure has the same reduced map as some simple structure.*

Proof. Consider a nonsimple structure $S = (E, \beta, \rho, \theta)$ determined by the expressions $1 \rightarrow H$ (insert $H_1, \cdots, H_n$) and $H \rightarrow H_\rho$ (delete $I_1, \cdots, I_k$) as in the definition of a structure. Suppose $S_1 = (E_1, \beta_1, \rho_1, \theta_1)$ is the maximum cancelled proper substructure of S. Let H' denote the array that results from deleting all the elements in E_1 from H.

A sequence $H'_1, \cdots, H'_m$ remains after deleting from $H_1, \cdots, H_n$ the terms which represent the β_1-cycles.

A sequence $I'_1, \cdots, I'_l$ remains after deleting from $I_1, \cdots, I_k$ the terms which represent the ρ_1-cycles. Then the expressions $1 \rightarrow H'$ (insert $H'_1, \cdots, H'_m$) and $H' \rightarrow H_\rho$ (delete $I'_l, \cdots, I'_l$) determine a simple structure having the same reduced map as S.

4. Complexes. A *complex* $C = (E, \beta, \rho)$ consists of a finite, nonempty set E which is acted on by a permutation β and a pure reflection ρ. If α is the map β, followed by ρ (i.e. $\alpha = \beta\rho$), then the α-orbits, the elements in E, and the β-orbits are the *vertices*, *edges*, and *boundaries*, respectively, of C. Whenever a free generator is assigned to each edge, the values of the β-cycles are called the values of the boundaries of C.

C is a disjoint union of 2 complexes (E_i, β_i, ρ_i) for $i = 1, 2$ if E is a disjoint union of E_1, E_2 and β_i, ρ_i are the restrictions of β, ρ, respectively, to the set E_i $(i = 1, 2)$. If this is never the case, C is said to be *connected.*

Since E is a disjoint union of the ρ-orbits and each ρ-orbit contains exactly 2 edges, the number of edges is always even. Whenever a is an edge, $a\rho$ is called the *inverse* of a. If v, $2e$, n denote the numbers of vertices, edges and boundaries of C, then $v - e + n$ is the Euler characteristic. A *spherical* complex is a connected complex with Euler characteristic 2.

Note that when $S_1 = (E_1, \beta_1, \rho_1, \theta_1)$ is a cancelled structure, then $C_1 = (E_1, \beta_1, \rho_1)$ is a complex. Furthermore, S_1 is minimal if and only if C_1 is connected.

5. From structures to complexes. We now describe a transition from a noncancelled structure S to a cancelled structure S_1; with S_1 there is associated a complex C_1.

Suppose $S = (E, \beta, \rho, \theta)$ is a noncancelled $\mathscr{R}$-structure, of type $n \geqq 1$, with $H, H_\rho, H_1, \cdots, H_n$ as in § 3. A cancelled $\mathscr{N}$-structure $S_1 = (E_1, \beta_1, \rho_1, \theta_1)$, of type $n + 1$, is defined as follows.

Let $H_\rho \cong a_1 \cdots a_m$. Since S is noncancelled, H_ρ is nonempty and $m \geqq 1$. Choose m new elements $b_1, \cdots, b_m$; put $E_1 = E \cup \{b_1, \cdots, b_m\}$. θ_1 is represented by HH_{n+1} where $H_{n+1} \cong b_m \cdots b_1$. Then $H_\rho H_{n+1} \rightarrow 1$ (delete $J_1, \cdots, J_m$) where $J_i \cong a_i b_i$ $(1 \leqq i \leqq m)$. The β_1-cycles are represented by $H_1, \cdots, H_n, H_{n+1}$. The β_1-cycle represented by H_{n+1} is called the *distinguished* β_1-cycle of S_1.

If ρ is the identity, then the ρ_1-cycles are represented by $J_1, \cdots, J_m$. If ρ is not the identity, then we have $H \rightarrow H_\rho$ (delete $I_1, \cdots, I_k$) where the I_i represent the ρ-cycles of length 2. In this case, $HH_{n+1} \rightarrow 1$ (delete $I_1, \cdots, I_k, J_1, \cdots, J_m$) and the I_i, J_i represent the ρ_1-cycles.

A free generator is assigned to each b_i so that the values of H_ρ and H_{n+1} are inverse words. This insures that the values of the J_i are null words and the value of H_{n+1} is a relator. The $\mathscr{N}$-structure S_1 is now complete and $C_1 = (E_1, \beta_1, \rho_1)$.

With reference to the construction of S_1, we have:

REMARK 5.1. If ab is a subword, of length 2, of some cyclic permutation of H_ρ and if $a\rho_1 = c$, $b\rho_1 = d$, then dc is a subword of some cyclic permutation of H_{n+1}. In other words, if a, b are distinct fixed elements of S and $a\theta_F = b$ where θ_F is the reduced map of S, then $b\rho_1\beta_1 = a\rho_1$.

LEMMA 5.1. *If S is simple or minimal, then S_1 is minimal.*

Proof. Since minimal implies simple for structures, we assume S is simple. Suppose a nonempty proper subset A_1 (of E_1) is closed under β_1 and ρ_1. Then $A_2 = E_1 - A_1$ also has this property; A_1, A_2 are carriers of substructures of S_1. Thus all the elements b_i are in the same A_j, say in A_2. Therefore all the elements in A_1 are cancelled elements in S. But then A_1 is the carrier of a proper cancelled substructure of S, contary to the assumption that S is simple.

THEOREM 5.1. *For each relator W there is a cancelled, minimal $\mathscr{N}$-structure $S_1 = (E_1, \beta_1, \rho_1, \theta_1)$, of type $t \geqq 2$ and a connected complex $C_1 = (E_1, \beta_1, \rho_1)$ such that the β_1-cycles can be represented by t arrays whose values are W^{-1} and $t - 1$ defining relators.*

Proof. By Theorem 3.2 there is a simple $\mathscr{R}$-structure S, of type $n \geqq 1$, where W is one of the values of the reduced map of S. Earlier we constructed a cancelled $\mathscr{N}$-structure $S_1 = (E_1, \beta_1, \rho_1, \theta_1)$, of type $n + 1$, whose β_1-cycles satisfy the desired condition. By Lemma 5.1 S_1 is minimal; hence, $C_1 = (E_1, \beta_1, \rho_1)$ is connected.

6. Spherical complexes.

The relationship between relators and spherical complexes is given in Theorem 6.2 and in Theorem 6.4. Their proofs depend on Theorem 6.1 and Theorem 6.3, which are converses. Three preliminary lemmas are needed.

LEMMA 6.1. *Let $H_1, \cdots, H_n$ be arrays with disjoint sets of elements satisfying $1 \to H$ (insert $H_1, \cdots, H_n$) where H is an array and $n \geqq 2$. Suppose H has a subword I, of length 2, whose letters a, b are in H_i, H_j, respectively, for $i < j$. Then $1 \to H$ (insert $H_1, \cdots, H_{i-1}, K, H_{i+1}, \cdots, H_{j-1}, H_{j+1}, \cdots, H_n$) for some array K, having subword I, such that $1 \to K$ (insert H_i, H_j).*

Proof. Let H' be the array such that $1 \to H'$ (insert $H_1, \cdots, H_i, \cdots, H_j$) and $H' \to H$ (insert $H_{j+1}, \cdots, H_n$). Then I is a subword of H'. We also have $1 \to H'$ (insert $H_1, \cdots, H_{i-1}, H_i, H_j, H_{i+1}, \cdots, H_{j-1}$). Let $H_i \cong A_1aA_2$ and $H_j = B_2bB_1$.

If $I \cong ab$, then B_2 is the empty array and we put $K \cong A_1abB_1A_2$. If $I \cong ba$, then B_1 is the empty array and we put $K \cong A_1B_2baA_2$.

LEMMA 6.2. *Let the array $abc_1 \cdots c_r$ $(r \geqq 1)$ represent a β-cycle μ corresponding to a β-orbit B of a connected complex $C = (E, \beta, \rho)$. Assume $a\rho = b$. Then C has the same Euler characteristic as some connected complex $C' = (E', \beta', \rho')$ having 2 fewer edges than C.*

Proof. Put $B' = \{c_1, \cdots, c_r\}$ and $E' = E - \{a, b\}$. Let μ' be the cyclic permutation represented by the array $c_1 \cdots c_r$. Define ρ' to be the restriction of ρ to the set E'. Define β' by putting $\beta' = \mu'$ on B' and $\beta' = \beta$ on $E' - B'$. The connectedness of C' follows from the connectedness of C. Thus, it suffices to show that C has one more vertex than C'.

Since $a\beta\rho = b\rho = a$, $\{a\}$ is a vertex of C. c_r is the only edge in E' having different images under $\beta\rho$ and $\beta'\rho'$. In fact, $c_r\beta\rho = a\rho = b$ and $c_r\beta'\rho' = c_1\rho' = c_1\rho$. Furthermore $b \neq c_1\rho$ since $a \neq c_1$ and $a\rho = b$.

Let $d = c_1\rho$, $\alpha = \beta\rho$, $\alpha' = \beta'\rho'$. There is an α-orbit V whose α-cycle is represented by an array of the form c_rbdD and V is a disjoint union of $\{b\}$ and an α'-orbit V' whose α'-cycle is represented by c_rdD. Thus C, C' have the same vertices, except that $\{a\}$ and V in C are replaced by V' in C'. This completes the proof.

LEMMA 6.3. *Let $C = (E, \beta, \rho)$ be a connected complex with $n \geqq 2$ boundaries. Let $A = \{a_1, \cdots, a_r\}$, $B = \{b_1, \cdots, b_s\}$ be β-orbits whose β-cycles μ, ν are represented by arrays $a_1 \cdots a_r$ and $b_1 \cdots b_s$, respectively. Assume $b_1\rho = a_r$. Then C has the same Euler characteristic as some connected complex $C' = (E, \beta', \rho)$ having $n - 1$ boundaries.*

Proof. Let μ' be the cyclic permutation represented by the array $a_1 \cdots a_r b_1 \cdots b_s$. Define β' by putting $\beta' = \mu'$ on the set $A \cup B$ and $\beta' = \beta$ otherwise. Then C has one more boundary than C' since 2 β-orbits A, B are replaced by one β'-orbit $A \cup B$. We must show that C' has one more vertex than C (i.e. that $\beta'\rho$ has one more orbit than $\beta\rho$).

Only $b_s = b_1\beta^{-1}$ and a_r have different images under $\beta\rho$ than under $\beta'\rho$. In fact $b_s\beta\rho = b_1\rho = a_r$ and $b_s\beta'\rho = a_1\rho$; $a_r\beta\rho = a_1\rho$ and $a_r\beta'\rho = b_1\rho = a_r$. Furthermore $a_1\rho \neq a_r$ since $a_1 \neq b_1$ and $b_1\rho = a_r$.

Let $c = a_1\rho$, $\alpha = \beta\rho$, and $\alpha' = \beta'\rho$. There is an α-orbit V whose β-cycle is represented by an array of the form $b_s a_r c D$ and V is a disjoint union of 2 α'-orbits V', V'' whose α'-cycles are represented by the arrays a_r and $b_s c D$. Thus C, C' have the same vertices, except that V is replaced by V' and V''. Therefore C' has one more vertex than C.

The connectedness of C' follows from the connectedness of C.

THEOREM 6.1. *Let $S = (E, \beta, \rho, \theta)$ be a minimal, cancelled structure of type $n \geqq 1$. Then $C = (E, \beta, \rho)$ is a spherical complex.*

Proof. Use induction on the number $2e$ of edges of C. Suppose $2e = 2$. Then $E = \{a, b\}$ and $a\rho = b$, $b\rho = a$; hence C is connected. If the β-orbits are $\{a\}$ and $\{b\}$ so that $n = 2$, then $a\beta\rho = a\rho = b$, $b\beta\rho = b\rho = a$ and $\{a, b\}$ is the only vertex. Thus $v - e + n = 1 - 1 + 2 = 2$. If $\{a, b\}$ is the only β-orbit so that $n = 1$, then $a\beta\rho = b\rho = a$, $b\beta\rho = a\rho = b$ and $\{a\}$, $\{b\}$ are the only vertices. Thus, $v - e + n = 2 - 1 + 1 = 2$.

Now assume that $2e \geqq 4$ and that the theorem holds for complexes with fewer than $2e$ edges. Let $H, H_1, \cdots, H_n$ represent θ and the β-cycles and let $I_1 \cong ab, I_2, \cdots, I_e$ represent the ρ-cycles. Assume that

$$1 \rightarrow H \text{ (insert } H_1, \cdots, H_n)$$
$$H \rightarrow 1 \text{ (delete } I_1, \cdots, I_e) .$$

Suppose a is in H_i, b is in H_j.

Case 1. $(i = j)$ Then I_1 is a subword of H_i. Let β_1 be the β-cycle represented by H_i. It cannot happen that $H_i \cong I_1$ since then $C_1 = (E_1, \beta_1, \rho_1)$ is a subcomplex of C where $E_1 = \{a, b\}$ and I_1 represents the only ρ_1-cycle. Also S is minimal so C is connected; hence $C = C_1$. This is contrary to $2e \geqq 4$. Therefore, some cyclic permutation of H_i is of the form $abc_1 \cdots c_r$ $(r \geqq 1)$.

A minimal, cancelled structure $S' = (E', \beta', \rho', \theta')$ is determined as follows. Let $H', H_1, \cdots, H_{i-1}, H'_i, H_{i+1}, \cdots, H_n$ represent θ' and the β'-cycles, and $I_2, \cdots, I_e$ represent the ρ'-cycles, where

$$\begin{aligned}
H &\to H' \text{ (delete } I_1) \\
H_i &\to H'_i \text{ (delete } I_1) \\
1 &\to H' \text{ (insert } H_1, \cdots, H_{i-1}, H'_i, H_{i+1}, \cdots, H_n) \\
H' &\to 1 \quad \text{(delete } I_2, \cdots, I_e) .
\end{aligned}$$

The complexes $C' = (E', \beta', \rho')$ and C have the same Euler characteristic and C' is connected by Lemma 6.2.

Case 2. $(i \neq j)$ Suppose $i < j$ (Treatment of $j < i$ is similar.) A minimal, cancelled structure $S' = (E, \beta', \rho, \theta)$ is determined as follows. Let $H_1, \cdots, H_{i-1}, K, H_{i+1}, \cdots, H_{j-1}, H_{j+1}, \cdots, H_n)$ represent the β'-cycles where K has the subword I_1 and

$$\begin{aligned}
1 &\to H \text{ (insert } H_1, \cdots, H_{i-1}, K, H_{i+1}, \cdots, H_{j-1}, H_{j+1}, \cdots, H_n) \\
1 &\to K \text{ (insert } H_i, H_j) .
\end{aligned}$$

This is possible by Lemma 6.3.

The complexes $C' = (E, \beta', \rho)$ and C have the same Euler characteristic and C' is connected (by Lemma 6.1). In fact, some cyclic permutation of K, H_i, and H_j are of the forms $a_1 \cdots a_r b_1 \cdots b_s$, $a_1 \cdots a_r$, and $b_1 \cdots b_s$, respectively, where $I_1 \cong a_r b_1$. Now S' and C' can be treated as in Case 1, since I_1 is a subword of K.

Thus, in either one or two steps, we can always find a new minimal, cancelled structure whose associated connected complex has $2(e-1)$ edges such that the original and new complexes have the same Euler characteristic. By the induction assumption, the new complex has Euler characteristic 2; hence, so does the original complex. This completes the proof.

THEOREM 6.2. *For each relator W there is some spherical complex C with $n \geqq 2$ boundaries such that a free generator is assigned to each edge (with inverse free generators assigned to inverse edges), W^{-1} is a value of one of the boundaries, and defining relators are the values of the remaining boundaries.*

Proof. Use Theorem 5.1 and Theorem 6.1.

THEOREM 6.3. *Let $C = (E, \beta, \rho)$ be a spherical complex with $n \geqq 1$ boundaries. Then there exists some minimal, cancelled structure $S = (E, \beta, \rho, \theta)$.*

Proof. Use induction on n. We first prove the case $n = 1$. Here β itself is the only β-cycle. This case will be proved by induction on the number $2e$ of edges. When $2e = 2$, we have $\beta = \rho$ and we take $\theta = \beta$.

Now assume $n = 1$, $2e \geqq 4$ and the theorem holds for complexes having one boundary and fewer than $2e$ edges. There must be a vertex containing just one edge since, if not, we have $2e \geqq 2v$ and $v - e + 1 = 2$ (where v is the number of vertices). But this implies $e \geqq v$ and $v = 1 + e$ which is impossible. If $\{a\}$ is a vertex, let $b = a\rho$. Then $a\beta = b$ since $a\beta\rho = a$. Thus β is represented by some array $H \cong I_1H'$ where $I_1 \cong ab$.

A connected complex $C' = (E', \beta', \rho')$ with $2e - 2$ edges and 1 boundary is defined by $E' = E - \{ab\}$ if we take ρ' to be the restriction of ρ to E' and put $\beta' = \beta_A$ with $A = E'$. Now apply the induction assumption to C'. There exists a minimal, cancelled structure $S' = (E', \beta', \rho', \theta')$. There exist an array X representing $\theta' = \beta'$ and arrays $I'_2, \cdots, I'_e$ representing the ρ'-cycles such that $X \to 1$ (delete $I'_2, \cdots, I'_e$). But since H' is a cyclic permutation of X, there exist arrays $I_2, \cdots, I_e$ representing the ρ'-cycles such that $H' \to 1$ (delete $I_2, \cdots, I_e$).

Since $H \to H'$ (delete I_1), we have that $\theta = \beta$ is represented by an array H satisfying $H \to 1$ (delete $I_1, I_2, \cdots, I_e$). Thus $S = (E, \beta, \rho, \theta)$ is a cancelled structure which is minimal since C is connected.

Now suppose $n \geqq 2$. Assume that the theorem holds for complexes having fewer than n boundaries. We need only consider the case that there exist two edges a, b, in different boundaries, such that $a\rho = b$. For if an edge and its image under ρ are always in the same boundary, then one boundary E_1 consists of the edges in some subcomplex which must be the whole complex C, by the connectedness of C. But then $n = 1$.

Thus, we can choose two β-cycles μ, ν represented by arrays $a_1 \cdots a_r$ and $b_1 \cdots b_s$, respectively, such that $a_r\rho = b_1$. Form a connected complex $C' = (E, \beta', \rho)$, having $n - 1$ boundaries, as in Lemma 6.3. The induction assumption implies that there is a minimal, cancelled structure $S' = (E, \beta', \rho, \theta)$. Here one of the β'-cycles μ' is represented by the array $a_1 \cdots a_rb_1 \cdots b_s$. There exist arrays $H, H_1, \cdots, H_{n-1}$ representing θ and the $n - 1$ β'-cycles such that $1 \to H$ (insert $H_1, \cdots, H_{n-1}$). Then $a_1 \cdots a_rb_1 \cdots b_s$ is a cyclic permutation of H_i, for some i. Thus $1 \to H_i$ (insert A, B) or $1 \to H_i$ (insert B, A) for some arrays A, B which are cyclic permutations of $a_1 \cdots a_r$ and $b_1 \cdots b_s$, respectively. In either case, H splits into $H_1, \cdots, H_{i-1}, A, B, H_{i+1}, \cdots, H_{n-1}$ which represent the β-cycles. Thus $S = (E, \beta, \rho, \theta)$ is a cancelled structure which is minimal since C is connected.

THEOREM 6.4. *Let $C = (E, \beta, \rho)$ be a spherical complex with $n \geqq 2$*

boundaries such that a free generator is assigned to each edge (with inverse free generators assigned to inverse edges). If all but one of the boundaries have values which are defining relators, then each value of the remaining boundary is a relator.

Proof. A minimal, cancelled structure $S = (E, \beta, \rho, \theta)$ exists by Theorem 6.3. Suppose an array H represents θ. Since H splits into arrays representing the ρ-cycles, W splits into null words so that W is a relator. Since H splits into arrays representing the β-cycles, W splits into $n-1$ defining relators and a word K (a value of the "remaining" β-cycle). Since W is a relator, K must be a relator.

7. Sides of nontrivial complexes. In this section each complex $C = (E, \beta, \rho)$ is nontrivial (i.e. has $n \geqq 3$ boundaries). When C is also spherical, we show that each β-cycle can be represented by an array which is broken up into a product $X_1 \cdots X_t$ ($t \geqq 1$) where each X_i has certain properties. The X_i will be called sides. In order to define sides, we classify the edges of C. Let a be an edge.

If either $a\rho\beta = a$ or $a\rho\beta\rho\beta \neq a$, then a is *initial.* If either $a\beta\rho = a$ or $a\beta\rho\beta\rho \neq a$, then a is *final.* Thus, if a is initial, final, or neither, then $a\rho$ is final, initial, or neither, respectively. Also, if a is initial, then $a\beta^{-1}$ is final; if a is final, then $a\beta$ is initial.

An array $X \cong a_1 \cdots a_r$ ($r \geqq 1$), which partially represents a β-cycle, is a *side* if a_1 is the only initial edge in X and a_r is the only final edge in X. If $X \cong a_1 \cdots a_r$ is a side, then the array $Y \cong b_r \cdots b_1$, where $a_i\rho = b_i$ ($1 \leqq i \leqq r$), is called the *inverse* of X.

LEMMA 7.1. *If $X \cong a_1 \cdots a_r$ is a side, so is its inverse $Y \cong b_r \cdots b_1$.*

Proof. It suffices to check that Y partially represents a β-cycle when $r \geqq 2$. i.e. $b_{i+1}\beta = b_i$ for $1 \leqq i \leqq r-1$. Indeed, $b_{i+1}\beta = a_{i+1}\rho\beta = a_i\beta\rho\beta = b_i\rho\beta\rho\beta = b_i$. The last equality holds since b_i is not initial for $1 \leqq i \leqq r-1$.

LEMMA 7.2. *Let $C = (E, \beta, \rho)$ be a connected complex with $n \geqq 3$ boundaries. Then each boundary contains at least one initial edge and at least one final edge (possibly the same edge).*

Proof. Suppose the array $A \cong a_1 \cdots a_r$, $r \geqq 1$, represents a β-cycle so that $\{a_1, \cdots, a_r\}$ is a boundary. Let $B \cong b_r \cdots b_1$ be the inverse of A. Suppose all the a_i are not final. Then all the b_i are not initial.

When $r \geqq 2$, $b_{i+1}\beta = b_i$ for $1 \leqq i \leqq r-1$ as in the proof of the previous lemma. $b_1\beta = a_1\rho\beta = a_r\beta\rho\beta = b_r\rho\beta\rho\beta = b_r$. When $r = 1$,

$a_1\beta = a_1$ and $b_1\beta = b_1\rho\beta\rho\beta = b_1$. In either case, $E_1 = \{a_1, \cdots, a_r, b_1, \cdots, b_r\}$ is closed under β and ρ. Hence $C_1 = (E_1, \beta_1, \rho_1)$ is a subcomplex where β_1, ρ_1 are the restrictions of β, ρ to E_1. C_1 must be the whole complex by the connectedness of C. But C_1 has just two boundaries: $\{a_1, \cdots, a_r\}$ and $\{b_1, \cdots, b_r\}$. This contradicts $n \geqq 3$. Thus some a_i is final and then $a_i\beta$ is initial.

LEMMA 7.3. *Let $C = (E, \beta, \rho)$ be a connected complex with $n \geqq 3$ boundaries. Then each β-cycle can be represented by a product $X_1 \cdots X_t$ $(t \geqq 1)$ where each X_i is a side. This representation is unique to within a cyclic permutation of these sides.*

Proof. Let μ be a β-cycle. Choose an array M, representing μ, so that the first letter of M is an initial edge. (Then the last letter of M is a final edge.) Therefore $M \cong X_1 \cdots X_t$, $t \geqq 1$, where an edge in M is initial (final) if and only if it is the first (last) letter in some X_i. The essential uniqueness of this representation follows from the fact that each edge can be placed uniquely in one of four classes: initial but not final, neither initial nor final, final but not initial and both initial and final. This completes the proof.

Vertices containing exactly 2 edges are called *nonessential*; all other vertices are *essential*. If the inverse arrays $X \cong a_1 \cdots a_r$ and $Y \cong b_r \cdots b_1$ $(r \geqq 2)$ are sides, then $\{a_i, b_{i+1}\}$ are nonessential vertices for $1 \leqq i \leqq r-1$ since $a_i\beta\rho = a_{i+1}\rho = b_{i+1}$ and $b_{i+1}\beta\rho = b_i\rho = a_i$. The next lemma shows that all nonessential vertices arise in this way.

LEMMA 7.4. *If $\{a_1, b_2\}$ is a nonessential vertex of a complex $C = (E, \beta, \rho)$ and if $a_2 = a_1\beta$, $b_1 = b_2\beta$, then a_1a_2 and b_2b_1 are subwords of sides.*

Proof. $a_2\rho = a_1\beta\rho = b_2$; $b_1\rho = b_2\beta\rho = a_1$. We must show that a_1 is not final and a_2 is not initial. Indeed, $a_1\beta\rho \neq a_1$ since $b_2 \neq a_1$; $a_1\beta\rho\beta\rho = a_2\rho\beta\rho = b_2\beta\rho = a_1$. Also, $a_2\rho\beta \neq a_2$ since $a_2\rho\beta = b_2\beta = b_1$ and $b_1\rho = a_1 \neq b_2 = a_2\rho$. Similarly, b_2 is not final and b_1 is not initial. This completes the proof.

The relationships between essential vertices, final edges, and sides can now be given.

LEMMA 7.5. *Let $C = (E, \beta, \rho)$ be a nontrival complex. An edge is in an essential vertex if and only if the edge is final. An edge is final if and only if it is the last letter in some side.*

Proof. Let a be an edge. Suppose a is in an essential vertex V. If $V = \{a\}$, then $a\beta\rho = a$ and a is final. If V contains at least 3 edges, then $a, b = a\beta\rho$ and $c = b\beta\rho$ are distinct edges. Thus, $a\beta\rho\beta\rho = c \neq a$; hence, a is final.

Now suppose a is final. If $a\beta\rho = a$, then $\{a\}$ is a vertex. If $a\beta\rho\beta\rho \neq a$, then $b = a\beta\rho \neq a$ and $c = b\beta\rho \neq a$. Also $a \neq b$ and the fact that $\beta\rho$ is a one-to-one map imply that $b = a\beta\rho \neq b\beta\rho = c$. Therefore there is an essential vertex containing a, b, c among its edges. The second statement of Lemma 7.5 follows from the proof of Lemma 7.3.

THEOREM 7.1. *Let $C = (E, \beta, \rho)$ be the connected complex associated with a cancelled, minimal $\mathscr{N}$-structure $S = (E, \beta, \rho, \theta)$, of type $n \geqq 3$. Assume that the values of the β-cycles are cyclically reduced words. Let $2s$, w denote the number of sides and the number of essential vertices of C. Then there is no vertex containing just one edge, $2s \geqq 3w$, and $w - s + n = 2$.*

Proof. If $\{a\}$ were a vertex, then $a\beta\rho = a$; hence $a\beta = a\rho$. Let $b = a\beta$. Then ab partially represents some β-cycle μ. Since $a\rho = b$, the value of ab is a null word which is a subword of a value of μ. This contradicts the assumption that the values of the β-cycles are cyclically reduced words. Hence, there is no vertex $\{a\}$.

Therefore each essential vertex contains at least 3 edges. Using Lemma 7.5 and the resulting fact that there is a one-to-one correspondence between final edges and sides, we get $2s \geqq 3w$.

We know that $v - e + n = 2$ where v, $2e$ are the numbers of vertices and edges of C. We show that $v - e = w - s$ by letting each pair of inverse sides (of length $m \geqq 2$) replace $2m$ edges and $m - 1$ nonessential vertices. In fact, if $X \cong a_1 \cdots a_m$, $Y \cong b_m \cdots b_1$ are inverse sides ($m \geqq 2$), then the letters in X, Y are the discarded edges and $\{a_i, b_{i+1}\}$ for $1 \leqq i \leqq m - 1$ are the discarded vertices. Thus each step reduces both v and e by $m - 1$. Lemma 7.4 assures us that each nonessential vertex (if any) will be discarded in this process. After a finite number of steps, we have discarded all edges which are not sides and all nonessential vertices. Thus $v - e = w - s$ and $w - s + n = 2$.

8. Calculations. Let $S = (E, \beta, \rho, \theta)$ be a noncancelled, minimal $\mathscr{R}$-structure, of type $n \geqq 2$, with reduced map θ_F. Assume that the values of θ_F are cyclically reduced words. Let $S_1 = (E_1, \beta_1, \rho_1, \theta_1)$ be a cancelled, minimal $\mathscr{N}$-structure, of type $n + 1$, associated with S. (Thus the values of the β_1-cycles are cyclically reduced words.) Suppose that the distinguished β_1-cycle has m sides in the complex $C_1 = (E_1, \beta_1, \rho_1)$.

Consider a side X of a nondistinguished β_1-cycle of C_1. X will be called a *fixed side* whenever the inverse of X is a side of the distinguished β_1-cycle. In such a case, the letters in X are all fixed elements in E.

Let B_k^i denote the number of nondistinguished boundaries having k sides, i sides of which are fixed; put $B_k = \sum_i B_k^i$. Then we have

$$(1)\qquad n = \sum_{k=1}^{\infty} B_k; \qquad 2s = m + \sum_{k=1}^{\infty} kB_k$$

$$m = \sum_{k=1}^{\infty} \sum_{1 \leqq i \leqq k} iB_k^i .$$

From Theorem 7.1 applied to C_1 we get $6w - 6s + 6(n + 1) = 12$ and $4s \geqq 6w$. Therefore

$$(2)\qquad 6n - 2s \geqq 6 .$$

From (1) and (2) we get

$$\sum_{k=1}^{5} (6 - k)B_k \geqq m + 6 + \sum_{k=7}^{\infty} (k - 6)B_k$$

and

$$(3)\qquad \sum_{k=1}^{5} (6 - k)B_k \geqq m + 6 .$$

Now expand the left hand side of (3):

$$(4)\qquad \sum_{k=1}^{5} (6 - k)B_k = \sum_{k=1}^{4} (5 - k)B_k^1 + \sum_{k=1}^{5} B_k^1 + \sum_{k=1}^{5} (6 - k)B_k^0 + \sum_{k=2}^{5} \sum_{i=2}^{k} (6 - k)B_k^i .$$

Further,

$$(5)\qquad \sum_{k=2}^{5} \sum_{i=2}^{k} (6 - k)B_k^i \leqq 2B_2^2 + B_3^2 + \sum_{k=2}^{5} \sum_{i=2}^{k} iB_k^i .$$

This can be seen as follows:
When (i, k) is neither $(2, 2)$ nor $(2, 3)$, we have $(6 - k) \leqq i$.
When $i = k = 2$, $(6 - k)B_k^i = 2B_2^2 + iB_k^i$.
When $i = 2$, $k = 3$, $(6 - k)B_k^i = B_3^2 + iB_k^i$.
Now use (3), (4), and (5) to get:

$$\sum_{k=1}^{4} (5 - k)B_k^1 + \sum_{k=1}^{5} (6 - k)B_k^0 + \sum_{k=1}^{5} B_k^1 + 2B_2^2 + B_3^2 + \sum_{k=2}^{5} \sum_{i=2}^{k} iB_k^i \geqq m + 6 .$$

But

$$m \geqq \sum_{k=1}^{5} B_k^1 + \sum_{k=2}^{5} \sum_{i=2}^{k} iB_k^i .$$

Therefore,

$$\sum_{k=1}^{4}(5-k)B_k^1 + \sum_{k=1}^{5}(6-k)B_k^0 + 2B_2^2 + B_3^2 \geqq 6 . \tag{6}$$

9. Minimal relators. A *minimal* relator of type n is a value of the reduced map of a minimal $\mathscr{R}$-structure of type n (i.e. a minimal structure, of type n, which is also an $\mathscr{R}$-structure). Similarly a nonminimal relator corresponds to a nonminimal $\mathscr{R}$-structure.

We aim to show that each relator splits into minimal relators. We prove this by showing that an analogous situation holds for the reduced map of a structure S and the reduced maps $\theta_1, \cdots, \theta_r$ of the minimal substructures of S. This requires the following.

DEFINITION. Let $\theta, \theta_1, \cdots, \theta_r$ be cyclic permutations acting on sets $E, E_1, \cdots, E_r$, respectively, such that $E = E_1 \cup \cdots \cup E_r$ is a disjoint union ($r \geqq 1$). θ *splits* into $\theta_1, \cdots, \theta_r$ if the θ_i can be put in a sequence $\theta'_1, \cdots, \theta'_r$ and if arrays $H, H_1, \cdots, H_r$, representing $\theta, \theta'_1, \cdots, \theta'_r$, respectively, can be chosen so that $1 \rightarrow H$ (insert $H_1, \cdots, H_r$).

THEOREM 9.1. *The reduced map of any structure S splits into the reduced maps of the minimal substructures of S.*

The proof of Theorem 9.1 requires a lemma.

LEMMA 9.1. *Suppose the structure $S = (E, \beta, \rho, \theta)$ contracts to the structure $S' = (E, \beta, \sigma, \theta)$ in one step. If S satisfies Theorem* 9.1, *so does S'.*

Proof. By assumption there exist arrays H_ρ, H_σ representing the reduced maps of S, S', respectively, such that $H_\rho \rightarrow H_\sigma$ (delete I) for some array I, of length 2, whose elements are a, b. $\sigma = \rho$ except on the set $\{a, b\}$; $a\sigma = b$, $b\sigma = a$. Let $H_\rho \cong XIY$ and $H_\sigma \cong XY$.

If $S_i = (E_i, \beta_i, \rho_i, \theta_i)$ are the minimal substructures of S ($1 \leqq i \leqq r$), then there exist arrays $M_1, \cdots, M_r$ representing the reduced maps of $S_1, \cdots, S_r$, respectively, such that $1 \rightarrow H_\rho$ (insert $M_1, \cdots, M_r$). Suppose $a \in E_i$, $b \in E_j$.

Case 1. ($i = j$) Since E_i is closed under β and σ, E_i is the carrier of a substructure S'_i of S'. The fact that S_i is minimal implies that each nonempty proper subset A (of E_i) is not closed under both β and ρ; hence A is not closed under both β and σ. Thus S'_i is a minimal substructure of S'.

Let $M_i \cong PIQ$. Then the possibly empty array $M'_i \cong PQ$ represents the reduced map of S'_i. Finally, $1 \rightarrow H_\sigma$ (insert $M_1, \cdots, M_{i-1}, M'_i, M_{i+1}, \cdots, M_r$).

Case 2. ($i < j$) By Lemma 6.1, there is an array K such that $1 \to H_\rho$ (insert $M_1, \cdots, M_{i-1}, K, M_{i+1}, \cdots, M_{j-1}, M_{j+1}, \cdots, M_r$), $1 \to K$ (insert M_i, M_j), and K is of the form $K \cong PIQ$.

E_i, E_j, and hence $E_i \cup E_j$ are closed under β and ρ. Then $E_i \cup E_j$ is closed under σ and is the carrier of a substructure S_i' of S'. Since E_i, E_j, and each nonempty proper subset of either E_i or E_j are not closed under both β and σ, we have that S_i' is a minimal substructure of S'.

The possibly empty array $K' \cong PQ$ represents the reduced map of S_i'; $1 \to H_\sigma$ (insert $M_1, \cdots, M_{i-1}, K', M_{i+1}, \cdots, M_{j-1}, M_{j+1}, \cdots, M_r$). This completes the proof of Lemma 9.1.

Now Theorem 9.1 can be proved. Let $S = (E, \beta, \rho, \theta)$ be a structure with k ρ-cycles of length 2. If $k = 0$, then ρ is the identity, the β-orbits are the carriers of the minimal substructures of S, and θ is the reduced map of S. Theorem 9.1 holds in this case since θ splits into the β-cycles (by the definition of a structure).

If $k \geqq 1$, then there exist structures $T_0 = (E, \beta, \rho_0, \theta), \cdots, T_k = (E, \beta, \rho_k, \theta)$ where ρ_0 is the identity and $\rho_k = \rho$, $T_k = S$ such that T_i contracts to T_{i+1} in one step ($0 \leqq i \leqq k-1$). Use Lemma 9.1 and the fact that T_0 satisfies Theorem 9.1 to get that $T_k = S$ satisfies Theorem 9.1. This completes the proof.

Since each relator is a value of the reduced map of some $\mathscr{R}$-structure, we have

COROLLARY 9.1. *Each relator splits into minimal relators.*

The next 3 lemmas will be useful later.

LEMMA 9.2. *A nonminimal relator, of type $n \geqq 2$, splits into relators having types smaller than n.*

Proof. Observe that a relator of type 1 is necessarily minimal. Use Theorem 9.1 and the fact that a nonminimal structure, of type $n \geqq 2$, has minimal substructures whose types have sum n.

LEMMA 9.3. *Let $S = (E, \beta, \rho, \theta)$ be a structure. If the array $H \cong ac_1 \cdots c_r bD$, $r \geqq 1$, represents θ and if the fixed elements a, b satisfy $a\beta = b$, then $\{c_1, \cdots, c_r\}$ is closed under β and ρ.*

Proof. There exist arrays $H_1, \cdots, H_n$ representing the β-cycles $\mu_1, \cdots, \mu_n$, respectively, such that $1 \to H$ (insert $H_1, \cdots, H_n$). Since $a\beta = b$, ab is a subword of H_i for some i, $1 \leqq i \leqq n$. Since ab is not a subword of H, we have $i < n$. The set $\{c_1, \cdots, c_r\}$ must be the union of the β-orbits corresponding to some subsequence of $\mu_{i+1}, \cdots,$

μ_n. Hence, $\{c_1, \cdots, c_r\}$ is closed under β.

Since a, b are fixed elements, we have that $\{c_1, \cdots, c_r\}$ is closed under ρ.

LEMMA 9.4. *Let a, b be fixed elements of a minimal structure $S = (E, \beta, \rho, \theta)$ with reduced map θ_F. If $a\beta = b$, then $a\theta = b$ and $a\theta_F = b$.*

Proof. If $a\theta \neq b$, then there is an array $ac_1 \cdots c_r b$, $r \geqq 1$, which partially represents θ. Lemma 9.3 implies that $\{c_1, \cdots, c_r\}$ is the carrier of a proper substructure of S. This is impossible since S is minimal. Thus, $a\theta = b$. But then $a\theta_F = b$ since a, b are fixed elements.

10. **Asymmetric relators.** Let W be an $\mathscr{R}$-word with $1 \rightarrow W$ (insert $R_1, \cdots, R_n$) where the R_i are defining relators. We always consider just one mode of performing the insertions (if there is more than one). Since each letter of W originates from a letter of one of the R_i, there is a one-to-one correspondence between the letters in W and the letters in $R_1, \cdots, R_n$.

Let $X \cong X_1xX_2$ and $Y \cong Y_2yY_1$ be any two of the R_i. Suppose that x, y correspond to the letters u, v in W; that u, v can cancel with each other during free reduction of W; and that the words X_2X_1x and yY_1Y_2 are inverses. Then we say that u, v can cancel *symmetrically* or that W is a *symmetric* $\mathscr{R}$-word.

In this situation, either u, v are adjacent in W or u, v are separated by a nonempty subword (of W) which freely reduces to 1. We indicate this by saying that u, v can cancel either *immediately* or *eventually*; W is either immediately or eventually symmetric. If no two letters of W can cancel symmetrically during free reduction of W, then W is an *asymmetric* $\mathscr{R}$-word. Finally, an asymmetric (symmetric) relator of type t is a partially reduced asymmetric (symmetric) $\mathscr{R}$-word of type t.

LEMMA 10.1. *If a word W splits into $t \geqq 2$ defining relators, two of which are X, Y, then W splits into two words U, V such that*
U splits into $p \geqq 1$ defining relators, one of which is X,
V splits into $q \geqq 1$ defining relators, one of which is Y,
and $p + q = t$.

Proof. Use induction on t. The lemma holds for $t = 2$ with $U \cong X$, $V \cong Y$. Let $t \geqq 3$ and assume the lemma is true for smaller t. Suppose $1 \rightarrow W$ (insert $R_1, \cdots, R_t$) and $X \cong R_i$, $Y \cong R_j$ for $i < j$. Let W' be the word such that $1 \rightarrow W'$ (insert $R_1, \cdots, R_{t-1}$) and $W' \rightarrow W$ (insert R_t).

If $j = t$, choose $U \cong W'$, $V \cong R_t$. If $j < t$, then by the induction assumption W' splits into two words U', V' which split into p' defining relators and q' defining relators among which are X and Y, respectively, where $p' + q' = t - 1$. We can choose U, V so that either $U \cong U'$ and $V' \to V$ (insert R_t) or $V \cong V'$ and $U' \to U$ (insert R_t).

LEMMA 10.2. *An eventually symmetric $\mathscr{R}$-word W, of type $t \geqq 2$, is freely equal to some immediately symmetric $\mathscr{R}$-word W', of type t.*

Proof. Suppose $1 \to W$ (insert $R_1, \cdots, R_t$) where the R_k are defining relators. Let W contain the letters u, v which can eventually cancel symmetrically during free reduction of W. Suppose that u, v correspond to the letters x, y in $R_i \cong X_1xX_2$, $R_j \cong Y_2yY_1$. Apply the previous lemma with $X \cong R_i$, $Y \cong R_j$ to find the words U, V. Then U, V have cyclic permutations U', V', respectively, such that the product U', V' is a cyclic permutation of W.

Let $U' \cong M_1mM_2$ and $V' \cong N_2 \cong N_2nN_1$ where m, n correspond to x, y, respectively. Since u, v can cancel in W, either N_1M_1 or M_2N_2 freely reduces to 1. Thus W has a cyclic permutation $mM_2N_2nN_1M_1$ which partially reduces to either M_2N_2 or N_1M_1.

Put $W'' \cong M_2M_1mnN_1N_2$ which is an $\mathscr{R}$-word of type t. In fact, M_2M_1m is a cyclic permutation of U and is an $\mathscr{R}$-word of the same type as U by Remark 2.1. Similarly, nN_1N_2 and V are $\mathscr{R}$-words of the same type. Thus W'' is a product of $\mathscr{R}$-words whose types have sum t.

Either W'' partially reduces to M_2M_1 or W'' has a cyclic permutation which partially reduces to N_1M_1. Thus W'' has a cyclic permutation W' which is freely equal to W.

LEMMA 10.3. *Let W be a word which splits into $t \geqq 2$ defining relators $R_1, \cdots, R_t$. If two letters u, v in W can immediately cancel symmetrically, then W also splits into $t - 2$ defining relators and one or more null words.*

Proof. Let $R_i \cong X_1xX_2$ and $R_j \cong Y_2yY_1$ where x, y correspond to u, v, respectively. By assumption, X_2X_1x and yY_1Y_2 are inverses so that $X_1xyY_1Y_2X_2$ and $X_1Y_1Y_2yxX_2$ freely reduce to 1.

The proof of Lemma 6.1 shows that W splits into $t - 2$ defining relators and a word U. Either $U \cong X_1xyY_1X_2$ (with $Y_2 \cong 1$) or $U \cong X_1Y_2yxX_2$ (with $Y_1 \cong 1$). In either case, U freely reduces to 1 so that U splits into one or more null words. Thus, W splits into $t - 2$ defining relators and one or more null words.

LEMMA 10.4. *Suppose* $1 \to U$ (insert X, Y) *where* X, Y *are relators of types* $p, q \geqq 0$ *with the understanding that a relator of type* 0 *is a null word. Let* U *have a subword* N *which is a null word whose letters* u, v *correspond to a letter in* X *and a letter in* Y, *respectively. Let* V *be defined by* $U \to V$ (delete N). *Then* V *is a relator of type* $p + q$.

Proof. If $p = q = 0$, then X, Y and hence V are null words. If $p > 0$, $q = 0$, then $V \cong X$. If $p = 0$, $q > 0$, then either $V \cong Y$ or V is a cyclic permutation of Y.

Finally, if $p > 0$, $q > 0$, then X, Y are partially reduced forms of $\mathscr{R}$-words P, Q of types p, q, respectively. Then U is a partially reduced form of an $\mathscr{R}$-word M, of type $p + q$, such that $1 \to M$ (insert P, Q). Thus U is a relator of type $p + q$; hence so is V.

LEMMA 10.5. *If a word* W *splits into null words and/or relators having types whose sum is* $t \geqq 1$, *then this is also true for each word* W' *which is freely equal to* W.

Proof. It suffices to check the cases when W' is obtained from W by a single insertion or deletion of a null word N. If $W \to W'$ (insert N), then W' satisfies the lemma.

Now suppose $W \to W'$ (delete N). By assumption $1 \to W$ (insert $W_1, \cdots, W_r$) where $W_1, \cdots, W_r$ are null words and/or relators having types whose sum is t. Let W_k have type t_k with $t_k = 0$ if W_k is a null word. The lemma holds when each W_k is a null word since then W' also splits into null words. Therefore, assume some W_k is not a null word so that $t_1 + \cdots + t_r = t$.

One possibility is that the letters in N correspond to letters in the same W_i so that $W_i \to W'_i$ (delete N) for some word W'_i. If $t_i = 0$, W'_i is the empty word. If $t_i \geqq 1$, W'_i is either empty or a relator of type t_i. In any case, $1 \to W'$ (insert $W_1, \cdots, W_{i-1}, W'_i, W_{i+1}, \cdots, W_r$).

The other possibility is that the letters in N correspond to letters in two words W_i, W_j so that $r \geqq 2$. Lemma 6.1 implies that W splits into $r - 2$ W_k's, having types whose sum is $t - t_i - t_j$, and a word U which splits into W_i, W_j. Then W' splits into the same $r - 2$ W_k's and a word V such that $U \to V$ (delete N). By the previous lemma, V is a relator of type $t_i + t_j$. This completes the proof.

LEMMA 10.6. *A symmetric relator* W, *of type* $t \geqq 2$, *splits into null words and/or relators having types smaller than* t.

Proof. Let W be a partially reduced form of a symmetric $\mathscr{R}$-word V of type t. By Lemma 10.2 V is freely equal to an immediately symmetric $\mathscr{R}$-word V' of type t. By Lemma 10.3 either $t = 2$ and V' splits into null words or $t \geqq 3$ and V' splits into null words and relators having types whose sum is $t - 2$, (since a defining relator is a relator of type 1). By Lemma 10.5, W splits into null words and/or relators having types whose sum is $t - 2$. This implies Lemma 10.6.

THEOREM 10.1. *Each relator splits into null words and/or asymmetric relators.*

Proof. Let W be a relator of type $t \geqq 1$. When $t = 1$, W is a defining relator which is an asymmetric relator. Use induction on t. Let $t \geqq 2$ and assume the theorem for relators of type smaller than t. Theorem 10.1 then follows from Lemma 10.6.

11. Proof of Main Theorem. In order to solve the word problem in the presented group $\mathscr{G}$, it suffices to be able to recognize the asymmetric, minimal relators which we call *basic* relators.

THEOREM 11.1. *Each relator splits into null words and/or basic relators.*

Proof. Use Lemma 9.2, Lemma 10.6 and the fact that a relator of type 1 (a defining relator) is a basic relator. This completes proof.

We now consider a basic relator in a sixth group. More specifically, consider a cyclically reduced relator W which is a value of the reduced map of a minimal, noncancelled $\mathscr{R}$-structure $S = (E, \beta, \rho, \theta)$, of type $n \geqq 2$. Then some cyclic permutation of W is the freely reduced form of an $\mathscr{R}$-word V of type n, where V is a value of θ. We assume that V is an asymmetric $\mathscr{R}$-ward so that W is an asymmetric relator. The structure S characterizes one method of freely reducing V to a word which is a cyclic permutation of W. As usual, let $S_1 = (E_1, \beta_1, \rho_1, \theta_1)$ be the cancelled $\mathscr{N}$-structure associated with S; $C_1 = (E_1, \beta_1, \rho_1)$. Note that C_1 has no vertex containing just one edge (by Theorem 7.1).

In this situation, consider the B_k^i of § 8. The following lemma implies that $B_1^1 = B_2^2 = B_3^2 = 0$.

LEMMA 11.1. *Let $S = (E, \beta, \rho, \theta)$ be a noncancelled, minimal structure with associated cancelled structure $S_1 = (E_1, \beta_1, \rho_1, \theta_1)$. Let $C_1 = (E_1, \beta_1, \rho_1)$ and assume that C_1 has no vertex containing just one edge. Suppose the product XY of nonempty arrays partially represents a nondistinguished β_1-cycle and X, Y are both sides in C_1.*

Then X, Y are not both fixed sides. Also, there is no nondistinguished β_1-cycle which is represented by one fixed side.

Proof. Suppose X, Y are fixed sides. This assumption together with the fact that XY partially represents a nondistinguished β_1-cycle imply that XY partially represents θ_F, the reduced map of S. Let a be the last letter in X; let b be the first letter in Y. Since Y is a side of C_1, b is an initial edge. Also, since $\{b\}$ cannot be a vertex of C_1, we have $b\rho_1\beta_1 \neq b$.

Since a, b are fixed elements of S and $a\beta = a\beta_1 = b$, we have $a\theta_F = b$ by Lemma 9.4. By Remark 5.1 $b\rho_1\beta_1 = a\rho_1$. Hence $b\rho_1\beta_1\rho_1\beta_1 = a\rho_1\rho_1\beta_1 = a\beta_1 = b$. This contradicts the fact that b is an initial edge of C_1. Thus, both X and Y cannot be fixed sides.

Now let Z be a fixed side, representing a nondistinguished β_1-cycle. If Z is of length $\geqq 2$, let a, b be the last and first letters of Z, respectively, so that $a \neq b$. We get a contradiction as before.

If Z is of length 1 and $Z \cong a$, then $a\beta = a\beta_1 = a$ and $a\rho = a$. Hence, $\{a\}$ is the carrier of a proper substructure of S, which is again a contradiction. This completes the proof.

Let the arrays MX and YN represent nondistinguished β_1-cycles μ, ν. respectively. Assume that the values of MX, YN are the defining relators R_1, R_2, respectively, and that X, Y are inverse sides.

If $\mu \neq \nu$, then R_1, R_2 are not inverses since V is asymmetric. Hence, R_1 and R_2^{-1} are distinct defining relators with a common subword (the value of X). The less-than-one-sixth property implies that

$$(*) \qquad l(X) < \frac{1}{6}\, l(MX) \quad \text{and} \quad l(Y) < \frac{1}{6}\, l(YN)\,.$$

It is also possible that $\mu = \nu$. In this case R_1, R_2 are cyclic permutations of one another. Once again (*) will hold provided that R_1, R_2 are not inverses. But this proviso holds.

LEMMA 11.2. *If T is a nonempty cyclically reduced word, then no cyclic permutation of T is the word T^{-1}.*

Proof. Let $U \cong T_2T_1$ be a cyclic permutation of $T \cong T_1T_2$. If $U \cong T^{-1}$, then $T_1 \cong T_1^{-1}$, $T_2 \cong T_2^{-1}$; hence T_1, T_2 are empty words, contradiction.

Thus, for C_1, we also have $B_k^0 = 0$ for $1 \leqq k \leqq 6$. From (6) in § 8, we get $3B_2^1 + 2B_3^1 + B_4^1 \geqq 6$. This implies the Main Theorem with $P_k = B_k^1$, $k = 2, 3, 4$.

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