ON A PROBLEM OF J. F. RITT

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In the Ritt algebra $R[u, v] = R(u_0, v_0, u_1, v_1, u_2, v_2, \ldots)$ where the derivation is such that $y'_i = y_{i+1}$ for $y = u$ or $v$, consider the differential ideal $\Omega = [uv] = ([uv], (uv)_1, (uv)_2, \ldots)$. Let $P = u_i \cdots u_m v_j \cdots v_n$ be a power product in $u, v$ and their derivatives. For sufficiently large $q$, it is known that $P^q \equiv 0 [uv]$. Power products of the form $u_i v_j$ are of particular interest; one of J. F. Ritt's unsolved problems is to find the smallest $q$ such that $(u_i v_j)^q \equiv 0 [uv]$. The purpose of this paper is to solve this problem in the special case $i = 1$. The main theorem is: The smallest $q$ such that $(u_1 v_j)^q \equiv 0 [uv]$ is $2 + j$. Part of the solution involves generalizing some results of D. G. Mead and part is an application of the well-known reduction process of H. Levi.

Basic Notions. H. Levi's reduction process [1] starts with replacing a factor $u_a v_b$ of $P = u_i \cdots u_m v_j \cdots v_n = u_a v_b Q$ by the other terms in $(uv)_{a+b}$. Thus we obtain the congruence

$$P \equiv -\sum_{i=0}^{a+b} \left( \frac{a+b}{i} \right) u_i v_{a+b-i} Q [uv].$$

Repeated substitutions of this kind eventually permit $P$ to be congruent to a linear combination of terms which are not in $[uv]$, unless all of the coefficients are zero. The unwieldy coefficients involved in this reduction process are simplified by $M$-congruences which may be introduced in the following way: Let $S$ be the set

$$(a_0, a_1, a_2, \ldots; b_0, b_1, b_2, \ldots; c_0, c_1, c_2, \ldots)$$

and $A$ the free algebra on $S$ over $R$. Define an (algebra) homomorphism $h: A \rightarrow R[u, v]$ by $h(r) = r$ if $r \in R$, $h(a_i) = u_i/i!$, $h(b_i) = v_i/i!$, and $h(c_i) = (uv)_i/i!$. There exists an isomorphism of $A/K$ into $R[u, v]$, where $K$ is the kernel of $h$ and this isomorphism becomes a differential isomorphism by defining $(y_i)_i = (i+1)y_{i+1}$ for $y_i = a_i$, $b_i$ or $c_i$ and leaving the derivative in $R$ unchanged. Now consider the algebra obtained by replacing $a_i$, $b_i$, and $c_i$ by $u_i$, $v_i$ and $(uv)_i$ respectively. The relation

Received June 19, 1964 and in revised form December 2, 1964. This paper was written while the author held a fellowship from the American Association of University Women.

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An \( M \)-congruence in the original algebra \( R\{u, v\} \) is an ordinary congruence in this isomorphic image of \( R\{u, v\} \).

Throughout this paper, the notation and terminology of [1] and [2] are used. All congruences are \( M \)-congruences and are taken modulo \([uv]\).

2. Minimum weight sequences. Let \( R\{u, v\} \) be a Ritt algebra in the indeterminates \( u, v \), \( \Omega = [uv] \) the differential ideal generated by the form \( X = uv \), and \( P = u_{i_1} \cdots u_{i_m} v_{j_1} \cdots v_{j_n} \) a power product of weight \( w = \sum_i i_p + \sum_j j_q \), and signature \((m, n)\). For each \((m_i, n_i)\), with \( 1 \leq m_i \leq m \) and \( 1 \leq n_i \leq n \), take the minimum weight of all factors of \( P \) of signature \((m_i, n_i)\). Subtract \( m_i n_i \) from that minimum. The result is one number for each pair \((m_i, n_i)\). The set of these excess weights is called the weight matrix of \( P \). A theorem of H. Levi [1] says that if there is a negative entry in the weight matrix of \( P \), then \( P \) is in \([uv]\). To reduce the number of excess weights that need to be computed, the idea of an ordered power product and the concept of a minimum weight sequence are introduced.

An ordered power product \( Q \) is of the form: \( Q = f_1 f_2 \cdots f_n \), where for each \( i, i = 1, 2, \cdots, n \), \( f_i = u_{d(i)} \) or \( f_i = v_{d(i)} \). For each \( m \leq n \), \( Q_m = f_1 \cdots f_m \) is an initial factor of \( Q \), of, say, signature \((k, l)\) and weight \( w \). The excess weight of \( Q_m \) is \( e_m = w - kl \). Thus \( Q \) has a sequence of excess weights of its initial factors, \( \{e_1, \cdots, e_n\} \). The sequence \( \{e_1, \cdots, e_n\} \) is called the initial factor weight sequence of \( Q \).

Let \( P \) be any power product of signature \((m, n)\). Let \( Q_d \) be any factor of \( P \) of signature \((k, l)\), \( k + l = d \) and smallest excess weight, \( w_d \).

**Definition 2.1.** The sequence \( \{w_1, \cdots, w_{m+n}\} \) is the minimum weight sequence of \( P \).

A simple method of constructing \( \{w_1, \cdots, w_{m+n}\} \) is shown next.

**Lemma 2.2.** \( Q_d \mid Q_{d-1} \).
Proof. The \( u \) (and \( v \)) subscripts of \( Q_t \), \( t = 1, \ldots, m + n \), form a nondecreasing sequence of natural numbers; call them \( r(i) \) and \( s(i) \) respectively. Among all pairs \( (x, y) \), \( x + y = d \), by hypothesis,

\[
\sum_{i=1}^{x} r(i) + \sum_{i=1}^{y} s(i) - xy
\]

is a minimum for \( x = k \) and \( y = l \). We claim that the corresponding minimum for \( d + 1 \) is attained by either \( x = k + 1 \), \( y = l \) or \( x = k \), \( y = l + 1 \). Consider the two expressions:

\[
\begin{align*}
(1) & \quad \sum_{i=1}^{k+j+1} r(i) + \sum_{i=1}^{l-j} s(i) - (k+j+1)(l-j) \\
(2) & \quad \sum_{i=1}^{k+j} r(i) + \sum_{i=1}^{l-j+1} s(i) - (k+j)(l-j+1) .
\end{align*}
\]

Subtract (2) from (1). The difference

\[
\Delta = r(k+j+1) + k + 2j - s(l-j+1) - l
\]

should be positive for (1) to be larger for \( j > 0 \) than for \( j = 0 \).

Since \( w_d \) is a minimum,

\[
r(k+1) + (k+1) \geq s(l) + l .
\]

Therefore, if \( j > 0 \), \( 2j > 1 \), \( r(k+j+1) \geq r(k+1) \) and \( s(l-j) \leq s(l) \).

Thus \( \Delta > 0 \) and \( Q_{d+1} \) must be \( Q_d u_r \) or \( Q_d v_s \).

Lemma 2.3. \( Q_{d+1} \) is \( Q_d u_r \) or \( Q_d v_s \) according as

\[
s - r + l - k > 0
\]

or

\[
s - r + l - k < 0 .
\]

(For \( s - r + l - k = 0 \), either \( u_r \) or \( v_s \) yields a minimum.)

Proof. Let \( P = Q_d u_r \) and the excess weight of \( P \),

\[
e_{d+1} = w_d + r - (k+1)l .
\]

Replace \( u_r \) by \( v_s \) and call the new excess weight \( e_{d+1}^s \),

\[
e_{d+1}^s = w_d + s - (k)(l + 1) .
\]

The change gives a smaller excess weight if and only if

\[
e_{d+1}^s - e_{d+1} < 0 ,
\]

or

\[
s - r + l - k < 0 .
\]
The following theorems are simple consequences of Lemmas 2.2 and 2.3, and Levi's theorem.

**THEOREM 2.4.** For any power product $P$, there is an ordered power product $P^*$ whose initial factor weight sequence is the minimum weight sequence of $P$.

**THEOREM 2.5.** If an entry in the minimum weight sequence of $P$ is negative, then $P = 0 [uv]$.

3. Generalized $\alpha$-terms. If $P = u_{i_1} \cdots u_{i_m} v_{j_1} \cdots v_{j_n}$ and $j_q \geq m$ for $q = 1, 2, \ldots, n$, then $P$ is called an $\alpha$-term. For example, there is exactly one $\alpha$-term of signature $(m, n)$ and weight $mn$, namely $u^m v^n$. The initial factor weight sequence of $u^m v^n$ is its minimum weight sequence and consists entirely of zeros. In this section, a study is made of all power products with minimum weight sequences consisting entirely of zeros.

**DEFINITION 3.1.** A minimum weight sequence consisting entirely of zeros is called a zero-sequence.

**DEFINITION 3.2.** A power product whose minimum weight sequence is a zero-sequence is called a generalized $\alpha$-term.

The power product $P = u_{i_1} \cdots u_{i_m} v_{j_1} \cdots v_{j_n}$ may be symbolically written $P = UV$. The degree of $P$, $d(P)$, is the sum of the degrees of $U$ and $V$, $d(P) = m + n$. The weight of $P$, $w(P)$, is the sum of the weights of $U$ and $V$, $w(P) = w(U) + w(V)$. $w(U)$ is called the $u$-weight of $P$.

**LEMMA 3.3.** Let $P = UV$ be a generalized $\alpha$-term of degree $d = m + n$. Then $P$ has either $u_n$ or $v_m$ as a factor, but not both $u_n$ and $v_m$ as factors.

**Proof.** By Theorem 2.4, let $Q$ be an initial factor of $P$ such that $d(Q) = d(P) - 1$ and the excess weight of $Q$ is zero. By Lemma 2.3, either

(1) $Q u_r = P$

or

(2) $Q v_s = P$.

If (1) holds, then $w(Q) = mn - r$. Also, $w(Q) = (m - 1)n$, hence
Suppose that $u_n$ and $v_m$ are both factors of $P$. Then let $T = P/u_nv_m$. The excess weight of $T$ is $-1$ and $P$ can not be a generalized $\alpha$-term.

**Theorem 3.4.** If $P$ and $P^*$ are generalized $\alpha$-terms of degree $d$ such that $P = UV$ and $P^* = U^*V^*$, then $V = V^*$.

**Proof.** The proof is by induction on $d$. For $d = 1$, $V = V^* = 1$ if $U = u_0$, and if $U = 1$, $V = V^* = v_0$. Assume that the theorem holds for power products whose degree is less than $d$. By Lemma 3.3, two cases may be distinguished:

**Case 1.** $u_n$ is a factor of $U$,

**Case 2.** $u_n$ is not a factor of $U$.

In Case 1, consider $Q = P/u_n$ and $Q^* = P^*/u_n$. The degree of $Q$ and $Q^*$ is less than $d$ and the induction hypothesis may be applied to $Q$ and $Q^*$ yielding the conclusion that $V = V^*$. In Case 2, $v_m$ must be a factor of $V$ and $V^*$. The induction hypothesis may then be applied to $R = P/v_m$ and $R^* = P^*/v_m$, yielding the conclusion that $V/v_m = V^*/v_m$; and hence, $V$ and $V^*$ are the same.

**Theorem 3.5.** If $A$ is a generalized $\alpha$-term of signature $(m, n)$, then

$$A = (-1)^J u_m^n v_m^n,$$

where $J$ is the $u$-weight of $A$.

**Proof.** The proof is by induction on $J$. If $J = 0$, $A$ is an $\alpha$-term and the theorem is true. By the theorem of [3], assume that $A$ involves $u_0$ and let

$$A = u_0^a v_a^b u_b^c \cdots.$$

By Lemma 1 of [3], with the roles of $u$ and $v$ interchanged, $i = a$, $j = b$, and $P = u_0^a v_a^{b-1}$,

$$A = -u_0^a v_a^{b-1} u_{a-1}^b v_{a-1}^c u_b^{c-1} \cdots.$$

The term on the right is a generalized $\alpha$-term and has $u$-weight one less than the $u$-weight of $A$. Using the induction hypothesis, the proof may be completed.
Since $u^m v^n$ is not in $[uv]$ by [1], the following remark summarizes this section.

**Theorem 3.6.** No generalized $\alpha$-term is in $[uv]$.

4. A reduction theorem. In ([2], p. 429), D. G. Mead has several theorems for $[y^2]$, which, for practical purposes, reduce the degree of the power product under consideration. Similar results may be developed for $[uv]$, with minor complications due to the asymmetrical definition of an $\alpha$-term. With the help of the symmetry theorems of [4], or [3], proper reduction theorems may be obtained.

For the rest of the paper, adopt the notation of [2]. Let $P$ be any power product of excess weight zero and $A$ the unique $\alpha$-term of the same weight and signature as $P$. The $M$-congruence $P \equiv cA$ is also written $m(P) = c$.

**Lemma 4.1.** Let $P = u_{i_1} \cdots u_{i_m} v_{j_1} \cdots v_{j_n}$. Then

\begin{align*}
(1) \quad m(P) &= m(u_{i_1} \cdots u_{i_m} v_{j_1+1} \cdots v_{j_n+1}) \\
(2) \quad m(P) &= (-1)^m m(u_{i_1} \cdots u_{i_m} v_{j_1} \cdots v_{j_n}).
\end{align*}

**Proof.** The proof of part 1 is the same as that of Lemma 1, p. 429, [2]. For part 2, the proof of part 1 with the $u_i$ and $v_j$ interchanged may be used, with the symmetry theorem of [4] or [3].

5. The main theorem. By Levi's theorem $(u_i v_j)^t \equiv 0$ for $t = i + j + 1$. For $i = 0$, $u_i v_j$ is an $\alpha$-term and not congruent to zero, so that $1 + j$ is the smallest power of $u_i v_j$ in $[uv]$. We conjecture that, in general, the smallest power of $u_i v_j$ in $[uv]$ is $i + j + 1$. Theorem 5.1 implies that the conjecture is true for $i = 1$.

**Theorem 5.1.**

$$(u_i v_j)^{i+j} \equiv (-1)^{i+j} (1 + j)! \ (u_i v_{i+j})^{i+j},$$

for $j \geq 0$.

**Proof.** Equivalently we may show that

$$m(u_i v_j)^{i+j} = (-1)^{i+j} (1 + j)!,$$

for $j \geq 0$.

The proof is by induction on $j$; since $u_i v_{i} \equiv - u_i v_{i+1}$, $m(u_i v_{i+1}) = -1$ and the theorem is true for $j = 0$. Assume that

$$m(u_i v_j)^{i+j} = (-1)^{i+j} (1 + t)!$$

for $t < j$. 

By Levi's theorem, \( Q = u_0 u_i v_j^{i+1} \equiv 0 \), and hence, \( Q' \equiv 0 \). Thus
\[
(u_i v_j)^{i+j} + 2j (u_0 u_i^{-1} u_0 v_j^{j+1}) + (j + 1)^2 u_0 u_i v_j^{j+1} = 0.
\]
Applying Lemma 4.1 results in the equation,
\[
(1) \quad m(u_i v_j)^{i+j} = -2j m(u_i v_j^{j+1}) - (j + 1)^2 m(u_i v_j \ldots v_j).
\]
Since \((u_i v_j^{j+1})' = 0\), we also have
\[
(2) \quad m(u_i^{-1} u_0 v_j^{j+1}) = -\frac{j(j+1)}{2j} m(u_i v_j \ldots v_j).
\]
Substituting (2) into (1) yields,
\[
m(u_i v_j)^{i+j} = -(1 + j) m(u_i v_j \ldots v_j).
\]
Apply the induction hypothesis to the term on the right, to complete the proof.

6. More general results. Using the same methods, more general results may be obtained.

**Theorem 6.1.** For \( j \geq 0, \ k > j - 1 \geq 0, \ l \geq j - 1 \geq 0 \)
\[
m(v_j u^k v_k u_{l+j}) = (-1)^{k+l-j-1} j \binom{k}{j} \binom{l}{j} + (j + 1) j \binom{k}{j} \binom{l}{j-1}.
\]

**Proof.** The proof is by induction on the triple \((j, k, l)\), and a useful equation is derived first. Differentiate each of the following congruences, for \( j \geq 1, \ k \geq j, \ l \geq j - 1 \):
\[
(1) \quad v_{j-1} u^k v_k u_{l+j} \equiv 0
\]
\[
(2) \quad u_0 v_j u^k v_k u_{l+j} \equiv 0.
\]
The resulting congruences yield, after applying Lemma 4.1, the equations:
\[
(3) \quad m(v_j u^k v_k u_{l+j}) = -\frac{j}{2k} m(v_j u^k v_k u_{l+j})
\]
\[
\quad - \frac{k + 1}{2k} m(v_j u^k v_k^{-1} u_{l+j})
\]
\[
\quad - \frac{l + 2 - j}{2k} m(v_j u^k v_k u_{l+j});
\]
\[
(4) \quad m(v_j u^k v_k u_{l+j}) = -\frac{1}{2k} m(v_j u^k v_k u_{l+j});
\]

Combining (3) and (4), and representing \( v_j u_i^k v_i^1 u_{i+1-j} \) by \((j, k, l)\), the conclusion is:

\[
(5) \quad m(j, k + 1, l) = -m(j, k, l) - lm(j - 1, k, l - 1) .
\]

If \( m(p, q, r) = (-1)^{q+r-p+1}\left\{p! \left(\frac{q}{p}\right)\left(\frac{r}{p}\right) + (p-1)! \left(\frac{q}{p-1}\right)\left(\frac{r}{p-1}\right)\right\} . \)

then (5) implies

\[
m(p, q + 1, r) = (-1)^{(q+1)+r-p+1}\left\{p! \left(\frac{q+1}{p}\right)\left(\frac{r}{p}\right) + (p-1)! \left(\frac{q+1}{p-1}\right)\left(\frac{r}{p-1}\right)\right\} .
\]

It remains to show that every triple \((j, k, l)\) with \( k \geq j - 1 \) can be constructed from triples where the formula is known to hold. Now first consider the cases \((\alpha) j = k\) and \((\beta) j = k + 1\). By rearrangement of the power product \( v_j u_i^{k-1} v_i^1 u_{i+1-j} ,\)

\[
(6) \quad m(j - 1, j - 1, l - 1) = m(j, j - 1, l) ,
\]

so that if the formula is true for any case \((\alpha)\), then it is true for the next \((\beta)\) case, for all \( l \).

Start with the case \((0, 0, l)\), \( v_0^{j-1} u_{j+1} .\) The formula is obviously true for this case (and indeed for \((0, k, l)\)) with the convention that

\[
(j - 1)! \left(\frac{k}{j - 1}\right)\left(\frac{l}{j - 1}\right) = 0 .
\]

Then by (6) the formula must hold also for \((1, 0, l)\). By (5)

\[
m(1, 1, l) = -m(1, 0, l) - lm(0, 0, l - 1)
\]

and

\[
m(1, 2, l) = -m(1, 1, l) - lm(0, 1, l - 1) .
\]

In general, if the formula holds for \((j, j, l)\) (some particular \( j \) and all \( l \)), then it holds for \((j + 1, j, l - 1)\), all \( l - 1 \). Using (5), the formula is then true for \((j + 1, j + 1, l)\). Therefore, the formula holds for all cases \( \alpha \) and \( \beta \); that is, if \( k = j \) or \( k = j - 1 \).
Consider next the case $k = j + 1$. For each $l$ by (5),

$$m(1, 3, l) = - m(1, 2, l) - lm(0, 2, l - 1).$$

The formula holds for $(0, 2, l - 1)$ since it holds for all $(0, k, l)$ and for $(1, 2, l)$ as shown above; and, hence, the formula holds for $(1, 3, l)$. In general, if the formula holds for $(j, j + p, l)$ and for $(j - 1, j + p, l)$ for all $l$, then by (5)

$$m(j, j + p + 1, l) = - m(j, j + p, l) - lm(j - 1, j + p, l - 1)$$

and the formula holds for $k = j + p + 1$. Thus the formula holds for all $k \geq j$, and the proof of the theorem is complete.

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