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ON A PROBLEM OF J. F. RITT

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In the Ritt algebra $R\{u, v\} = R(u_0, v_0, u_1, v_1, u_2, v_2, \dots)$ where the derivation is such that $y'_i = y_{i+1}$ for $y = u$ or v , consider the differential ideal $\Omega = [uv] = ((uv), (uv)_1, (uv)_2, \dots)$. Let $P = u_{i_1} \cdots u_{i_m} v_{j_1} \cdots v_{j_n}$ be a power product in u, v and their derivatives. For sufficiently large q , it is known that $P^q \equiv 0 [uv]$. Power products of the form $u_i v_j$ are of particular interest; one of J. F. Ritt's unsolved problems is to find the smallest q such that $(u_i v_j)^q \equiv 0 [uv]$. The purpose of this paper is to solve this problem in the special case $i = 1$. The main theorem is: The smallest q such that $(u_1 v_j)^q \equiv 0 [uv]$ is $2 + j$. Part of the solution involves generalizing some results of D. G. Mead and part is an application of the well-known reduction process of H. Levi.

Basic Notions. H. Levi's reduction process [1] starts with replacing a factor $u_a v_b$ of $P = u_{i_1} \cdots u_{i_m} v_{j_1} \cdots v_{j_n} = u_a v_b Q$ by the other terms in $(uv)_{a+b}$. Thus we obtain the congruence

$$P \equiv - \sum_{\substack{i=0 \\ i \neq a}}^{a+b} \binom{a+b}{i} u_i v_{a+b-i} Q [uv].$$

Repeated substitutions of this kind eventually permit P to be congruent to a linear combination of terms which are not in $[uv]$, unless all of the coefficients are zero. The unwieldy coefficients involved in this reduction process are simplified by M -congruences which may be introduced in the following way: Let S be the set

$$(a_0, a_1, a_2, \dots; b_0, b_1, b_2, \dots; c_0, c_1, c_2, \dots)$$

and A the free algebra on S over R . Define an (algebra) homomorphism $h: A \rightarrow R\{u, v\}$ by $h(r) = r$ if $r \in R$, $h(a_i) = u_i/i!$, $h(b_i) = v_i/i!$, and $h(c_i) = (uv)_i/i!$. There exists an isomorphism of A/K into $R\{u, v\}$, where K is the kernel of h and this isomorphism becomes a differential isomorphism by defining $(y_i)_1 = (i+1)y_{i+1}$ for $y_i = a_i, b_i$ or c_i and leaving the derivative in R unchanged. Now consider the algebra obtained by replacing $a_i, b_i,$ and c_i by u_i, v_i and $(uv)_i$ respectively. The relation

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$$(uv)_n = \sum_{i=0}^n \binom{n}{i} u_i v_{n-i}$$

has become

$$(uv)_n = \sum_{i=0}^n u_i v_{n-i} .$$

An M -congruence in the original algebra $R\{u, v\}$ is an ordinary congruence in this isomorphic image of $R\{u, v\}$.

Throughout this paper, the notation and terminology of [1] and [2] are used. All congruences are M -congruences and are taken modulo $[uv]$.

2. **Minimum weight sequences.** Let $R\{u, v\}$ be a Ritt algebra in the indeterminates u and v , $\Omega = [uv]$ the differential ideal generated by the form $X = uv$, and $P = u_{i_1} \cdots u_{i_m} v_{j_1} \cdots v_{j_n}$ a power product of weight $w = \sum_{i=1}^m i_p + \sum_{i=1}^n j_q$, and signature (m, n) . For each (m_i, n_i) , with $1 \leq m_i \leq m$ and $1 \leq n_i \leq n$, take the minimum weight of all factors of P of signature (m_i, n_i) . Subtract $m_i n_i$ from that minimum. The result is one number for each pair (m_i, n_i) . The set of these excess weights is called the weight matrix of P . A theorem of H. Levi [1] says that if there is a negative entry in the weight matrix of P , then P is in $[uv]$. To reduce the number of excess weights that need to be computed, the idea of an ordered power product and the concept of a minimum weight sequence are introduced.

An ordered power product Q is of the form: $Q = f_1 f_2 \cdots f_n$, where for each $i, i = 1, 2, \dots, n, f_i = u_{d(i)}$ or $f_i = v_{d(i)}$. For each $m \leq n, Q_m = f_1 \cdots f_m$ is an initial factor of Q , of, say, signature (k, l) and weight w . The excess weight of Q_m is $e_m = w - kl$. Thus Q has a sequence of excess weights of its initial factors, $\{e_1, \dots, e_n\}$. The sequence $\{e_1, \dots, e_n\}$ is called the initial factor weight sequence of Q .

Let P be any power product of signature (m, n) . Let Q_d be any factor of P of signature $(k, l), k + l = d$ and smallest excess weight, w_d .

DEFINITION 2.1. The sequence $\{w_1, \dots, w_{m+n}\}$ is the minimum weight sequence of P .

A simple method of constructing $\{w_1, \dots, w_{m+n}\}$ is shown next.

LEMMA 2.2. $Q_d \mid Q_{d+1}$.

Proof. The u (and v) subscripts of Q_i , $t = 1, \dots, m + n$, form a nondecreasing sequence of natural numbers; call them $r(i)$ and $s(i)$ respectively. Among all pairs (x, y) , $x + y = d$, by hypothesis,

$$\sum_1^x r(i) + \sum_1^y s(i) - xy$$

is a minimum for $x = k$ and $y = l$. We claim that the corresponding minimum for $d + 1$ is attained by either $x = k + 1$, $y = l$ or $x = k$, $y = l + 1$. Consider the two expressions:

$$(1) \quad \sum_1^{k+j+1} r(i) + \sum_1^{l-j} s(i) - (k+j+1)(l-j)$$

$$(2) \quad \sum_1^{k+j} r(i) + \sum_1^{l-j+1} s(i) - (k+j)(l-j+1).$$

Subtract (2) from (1). The difference

$$\Delta = r(k+j+1) + k + 2j - s(l-j+1) - l$$

should be positive for (1) to be larger for $j > 0$ than for $j = 0$.

Since w_d is a minimum,

$$r(k+1) + (k+1) \geq s(l) + l.$$

Therefore, if $j > 0$, $2j > 1$, $r(k+j+1) \geq r(k+1)$ and $s(l-j) \leq s(l)$. Thus $\Delta > 0$ and Q_{d+1} must be $Q_d u_r$ or $Q_d v_s$.

LEMMA 2.3. Q_{d+1} is $Q_d u_r$ or $Q_d v_s$ according as

$$s - r + l - k > 0$$

or

$$s - r + l - k < 0.$$

(For $s - r + l - k = 0$, either u_r or v_s yields a minimum.)

Proof. Let $P = Q_d u_r$ and the excess weight of P ,

$$e_{d+1} = w_d + r - (k+1)l.$$

Replace u_r by v_s and call the new excess weight e_{d+1}^* ,

$$e_{d+1}^* = w_d + s - (k)(l+1).$$

The change gives a smaller excess weight if and only if

$$e_{d+1}^* - e_{d+1} < 0,$$

or

$$s - r + l - k < 0.$$

The following theorems are simple consequences of Lemmas 2.2 and 2.3, and Levi's theorem.

THEOREM 2.4. *For any power product P , there is an ordered power product P^* whose initial factor weight sequence is the minimum weight sequence of P .*

THEOREM 2.5. *If an entry in the minimum weight sequence of P is negative, then $P \equiv 0$ [uv].*

3. Generalized α -terms. If $P = u_{i_1} \cdots u_{i_m} v_{j_1} \cdots v_{j_n}$ and $j_q \geq m$ for $q = 1, 2, \dots, n$, then P is called an α -term. For example, there is exactly one α -term of signature (m, n) and weight mn , namely $u_0^m v_m^n$. The initial factor weight sequence of $u_0^m v_m^n$ is its minimum weight sequence and consists entirely of zeros. In this section, a study is made of all power products with minimum weight sequences consisting entirely of zeros.

DEFINITION 3.1. A minimum weight sequence consisting entirely of zeros is called a *zero-sequence*.

DEFINITION 3.2. A power product whose minimum weight sequence is a zero-sequence is called a *generalized α -term*.

The power product $P = u_{i_1} \cdots u_{i_m} v_{j_1} \cdots v_{j_n}$ may be symbolically written $P = UV$. The *degree* of P , $d(P)$, is the sum of the degrees of U and V , $d(P) = m + n$. The *weight* of P , $w(P)$, is the sum of the weights of U and V , $w(P) = w(U) + w(V)$. $w(U)$ is called the *u -weight* of P .

LEMMA 3.3. *Let $P = UV$ be a generalized α -term of degree $d = m + n$. Then P has either u_n or v_m as a factor, but not both u_n and v_m as factors.*

Proof. By Theorem 2.4, let Q be an initial factor of P such that $d(Q) = d(P) - 1$ and the excess weight of Q is zero. By Lemma 2.3, either

$$(1) \quad Qu_r = P$$

or

$$(2) \quad Qv_s = P.$$

If (1) holds, then $w(Q) = mn - r$. Also, $w(Q) = (m - 1)n$, hence

$r = n$. If (2) holds, by similar reasoning, $m = s$.

Suppose that u_n and v_m are both factors of P . Then let $T = P/u_n v_m$. The excess weight of T is -1 and P can not be a generalized α -term.

THEOREM 3.4. *If P and P^* are generalized α -terms of degree d such that $P = UV$ and $P^* = UV^*$, then $V = V^*$.*

Proof. The proof is by induction on d . For $d = 1$, $V = V^* = 1$ if $U = u_0$, and if $U = 1$, $V = V^* = v_0$. Assume that the theorem holds for power products whose degree is less than d . By Lemma 3.3, two cases may be distinguished:

Case 1. u_n is a factor of U ,

Case 2. u_n is not a factor of U .

In Case 1, consider $Q = P/u_n$ and $Q^* = P^*/u_n$. The degree of Q and Q^* is less than d and the induction hypothesis may be applied to Q and Q^* yielding the conclusion that $V = V^*$. In Case 2, v_m must be a factor of V and V^* . The induction hypothesis may then be applied to $R = P/v_m$ and $R^* = P^*/v_m$, yielding the conclusion that $V/v_m = V^*/v_m$; and hence, V and V^* are the same.

THEOREM 3.5. *If A is a generalized α -term of signature (m, n) , then*

$$A \equiv (-1)^J u_0^m v_0^n,$$

where J is the u -weight of A .

Proof. The proof is by induction on J . If $J = 0$, A is an α -term and the theorem is true. By the theorem of [3], assume that A involves u_0 and let

$$A = u_0^a v_a^b u_b^c \dots$$

By Lemma 1 of [3], with the roles of u and v interchanged, $i = a$, $j = b$, and $P = u_0^a v_a^{b-1}$,

$$A \equiv -u_0^a v_a^{b-1} u_{b-1} v_{a+1} u_b^{c-1} \dots$$

The term on the right is a generalized α -term and has u -weight one less than the u -weight of A . Using the induction hypothesis, the proof may be completed.

Since $u_0^m v_m^n$ is not in $[uv]$ by [1], the following remark summarizes this section.

THEOREM 3.6. *No generalized α -term is in $[uv]$.*

4. A reduction theorem. In ([2], p. 429), D. G. Mead has several theorems for $[y^2]$, which, for practical purposes, reduce the degree of the power product under consideration. Similar results may be developed for $[uv]$, with minor complications due to the asymmetrical definition of an α -term. With the help of the symmetry theorems of [4], or [3], proper reduction theorems may be obtained.

For the rest of the paper, adopt the notation of [2]. Let P be any power product of excess weight zero and A the unique α -term of the same weight and signature as P . The M -congruence $P \equiv cA$ is also written $m(P) = c$.

LEMMA 4.1. *Let $P = u_{i_1} \cdots u_{i_m} v_{j_1} \cdots v_{j_n}$. Then*

$$(1) \quad m(P) = m(u_0 u_{i_1} \cdots u_{i_m} v_{j_1+1} v_{j_2+1} \cdots v_{j_n+1})$$

$$(2) \quad m(P) = (-1)^m m(v_0 u_{i_1+1} u_{i_2+1} \cdots u_{i_m+1} v_{j_1} \cdots v_{j_n}).$$

Proof. The proof of part 1 is the same as that of Lemma 1, p. 429, [2]. For part 2, the proof of part 1 with the u_i and v_j interchanged may be used, with the symmetry theorem of [4] or [3].

5. The main theorem. By Levi's theorem $(u_i v_j)^t \equiv 0$ for $t = i + j + 1$. For $i = 0$, $u_0^j v_j^j$ is an α -term and not congruent to zero, so that $1 + j$ is the smallest power of $u_0 v_j$ in $[uv]$. We conjecture that, in general, the smallest power of $u_i v_j$ in $[uv]$ is $i + j + 1$. Theorem 5.1 implies that the conjecture is true for $i = 1$.

THEOREM 5.1.

$$(u_i v_j)^{1+j} \equiv (-1)^{1+j} (1+j)! (u_0 v_{1+j})^{1+j},$$

for $j \geq 0$.

Proof. Equivalently we may show that

$$m(u_i v_j)^{1+j} = (-1)^{1+j} (1+j)!, \quad \text{for } j \geq 0.$$

The proof is by induction on j ; since $u_i v_0 \equiv -u_0 v_1$, $m(u_i v_0) = -1$ and the theorem is true for $j = 0$. Assume that

$$m(u_i v_t)^{1+t} = (-1)^{1+t} (1+t)! \quad \text{for } t < j.$$

By Levi's theorem, $Q = u_0 u_1^j v_j^{j+1} \equiv 0$, and hence, $Q' \equiv 0$. Thus

$$(u_1 v_j)^{1+j} + 2j(u_0 u_1^{j-1} u_2 v_j^{j+1}) + (j+1)^2 u_0 u_1^j v_j^j v_{j+1} \equiv 0.$$

Applying Lemma 4.1 results in the equation,

$$(1) \quad m(u_1 v_j)^{1+j} = -2jm(u_1^{j-1} u_2 v_{j-1}^{j+1}) - (j+1)^2 m(u_1^j v_{j-1}^j v_j).$$

Since $(u_1^j v_{j-1}^{j-1})' \equiv 0$, we also have

$$(2) \quad m(u_1^{j-1} u_2 v_{j-1}^{j-1}) = -\frac{j(j+1)}{2j} m(u_1^j v_{j-1}^j v_j).$$

Substituting (2) into (1) yields,

$$m(u_1 v_j)^{1+j} = -(1+j) m(u_1^j v_{j-1}^j v_j).$$

Apply the induction hypothesis to the term on the right, to complete the proof.

6. More general results. Using the same methods, more general results may be obtained.

THEOREM 6.1. For $j \geq 0$, $k > j - 1 \geq 0$, $l \geq j - 1 \geq 0$

$$m(v_j u_1^k v_k^l u_{l-1-j}) = (-1)^{k+l-j+1} \left\{ j! \binom{k}{j} \binom{l}{j} + (j-1)! \binom{k}{j-1} \binom{l}{j-1} \right\}.$$

Proof. The proof is by induction on the triple (j, k, l) , and a useful equation is derived first. Differentiate each of the following congruences, for $j \geq 1$, $k \geq j$, $l \geq j - 1$:

$$(1) \quad v_{j-1} u_1^k v_k^l u_{l+1-j} \equiv 0$$

$$(2) \quad u_0 v_j u_1^k v_{k+1}^l u_{l+1-j} \equiv 0.$$

The resulting congruences yield, after applying Lemma 4.1, the equations:

$$(3) \quad \begin{aligned} m(v_{j-1} u_1^{k-1} u_2 v_k^l u_{l+1-j}) &= -\frac{j}{2k} m(v_j u_1^k v_k^l u_{l+1-j}) \\ &\quad - \frac{l(k+1)}{2k} m(v_{j-1} u_1^k v_{k+1}^{l-1} u_{l+1-j}) \\ &\quad - \frac{l+2-j}{2k} m(v_{j-1} u_1^k v_k^l u_{l+2-j}); \end{aligned}$$

$$(4) \quad m(v_{j-1} u_1^{k-1} u_2 v_k^l u_{l+1-j}) = -\frac{1}{2k} m(v_j u_1^{k+1} v_{k+1}^l u_{l+1-j})$$

$$\begin{aligned}
 & - \frac{(j+1)}{2k} m(v_j u_1^k v_k^l u_{l+1-j}) \\
 & - \frac{l(k+2)}{2k} m(v_{j-1} u_1^k v_{k+1}^{l-1} u_{l+1-j}) \\
 & - \frac{(l+2-j)}{2k} m(v_{j-1} u_1^k v_k^l u_{l+2-j}) .
 \end{aligned}$$

Combining (3) and (4), and representing $v_j u_1^k v_k^l u_{l+1-j}$ by (j, k, l) , the conclusion is:

$$(5) \quad m(j, k+1, l) = -m(j, k, l) - lm(j-1, k, l-1) .$$

If $m(p, q, r) = (-1)^{q+r-p+1} \left\{ p! \binom{q}{p} \binom{r}{p} + (p-1)! \binom{q}{p-1} \binom{r}{p-1} \right\}$, then (5) implies

$$\begin{aligned}
 m(p, q+1, r) &= (-1)^{(q+1)+r-p+1} \left\{ p! \binom{q+1}{p} \binom{r}{p} \right. \\
 &\quad \left. + (p-1)! \binom{q+1}{p-1} \binom{r}{p-1} \right\} .
 \end{aligned}$$

It remains to show that every triple (j, k, l) with $k \geq j-1$ can be constructed from triples where the formula is known to hold. Now first consider the cases (α) $j = k$ and (β) $j = k+1$. By rearrangement of the power product $v_j u_1^{j-1} v_{j-1}^l u_{l+1-j}$,

$$(6) \quad m(j-1, j-1, l-1) = m(j, j-1, l) ,$$

so that if the formula is true for any case (α) , then it is true for the next (β) case, for all l .

Start with the case $(0, 0, l)$, $v_0^{l+1} u_{l+1}$. The formula is obviously true for this case (and indeed for $(0, k, l)$) with the convention that

$$(j-1)! \binom{k}{j-1} \binom{l}{j-1} = 0 .$$

Then by (6) the formula must hold also for $(1, 0, l)$. By (5)

$$m(1, 1, l) = -m(1, 0, l) - lm(0, 0, l-1)$$

and

$$m(1, 2, l) = -m(1, 1, l) - lm(0, 1, l-1) .$$

In general, if the formula holds for (j, j, l) (some particular j and all l), then it holds for $(j+1, j, l-1)$, all $l-1$. Using (5), the formula is then true for $(j+1, j+1, l)$. Therefore, the formula holds for all cases α and β ; that is, if $k = j$ or $k = j-1$.

Consider next the case $k = j + 1$. For each l by (5),

$$m(1, 3, l) = -m(1, 2, l) - lm(0, 2, l - 1).$$

The formula holds for $(0, 2, l - 1)$ since it holds for all $(0, k, l)$ and for $(1, 2, l)$ as shown above; and, hence, the formula holds for $(1, 3, l)$. In general, if the formula holds for $(j, j + p, l)$ and for $(j - 1, j + p, l)$ for all l , then by (5)

$$m(j, j + p + 1, l) = -m(j, j + p, l) - lm(j - 1, j + p, l - 1)$$

and the formula holds for $k = j + p + 1$. Thus the formula holds for all $k \geq j$, and the proof of the theorem is complete.

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