MINIMAL GERSCHGORIN SETS. II

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The Gerschgorin Circle Theorem, which yields $n$ disks whose union contains all the eigenvalues of a given $n \times n$ matrix $A = (a_{i,j})$, applies equally well to any matrix $B = (b_{i,j})$ of the set $\Omega_A$ of $n \times n$ matrices with $b_{i,i} = a_{i,i}$ and $|b_{i,j}| = |a_{i,j}|$, $1 \leq i, j \leq n$. This union of $n$ disks thus bounds the entire spectrum $S(\Omega_A)$ of the matrices in $\Omega_A$. The main result of this paper is a precise characterization of $S(\Omega_A)$, which can be determined by extensions of the Gerschgorin Circle Theorem based only on the use of positive diagonal similarity transformations, permutation matrices, and their intersections.

Given any $n \times n$ complex matrix $A = (a_{i,j})$, it is well known that the simplest of Gerschgorin arguments, which depends upon row sums of the moduli of off-diagonal entries of the matrix $X^{-1}AX$, $X$ a positive diagonal matrix, yields the union of $n$ disks which contains all the eigenvalues of $A$. It is clear that this union of $n$ disks necessarily contains all the eigenvalues of any $n \times n$ matrix in the set $\Omega_A$ defined as follows: $B = (b_{i,j}) \in \Omega_A$ if $b_{i,i} = a_{i,i}$, $1 \leq i \leq n$, and $|b_{i,j}| = |a_{i,j}|$ for all $1 \leq i, j \leq n$, $i \neq j$. Hence, this union of $n$ Gerschgorin disks can be viewed as giving bounds for the entire spectrum $S(\Omega_A) = \{z | \det(zI - B) = 0 \text{ for some } B \in \Omega_A\}$ of the set $\Omega_A$.

It is logical to ask to what extent the spectrum $S(\Omega_A)$ can be more precisely determined by extensions of Gerschgorin's original argument [3]. In the previous paper [6], it was shown that

\[(1.1) \quad \partial G(\Omega_A) \subset S(\Omega_A) \subset G(\Omega_A),\]

where $G(\Omega_A)$ is the minimal Gerschgorin set deduced from $A$ and $\partial G(\Omega_A)$ is its boundary. The first inclusion of (1.1) states that every point of the boundary $\partial G(\Omega_A)$ of the minimal Gerschgorin set is then an eigenvalue of some $B \in \Omega_A$. We now extend the results of [6] by making use of results of Schneider [4], and Camion and Hoffman [1]. In so doing, we shall \textit{precisely} determine $S(\Omega_A)$.

To begin, let $P_\phi = (\delta_{i,\phi(i)})$ be an $n \times n$ permutation matrix, where $\phi$ is a permutation of the integers $1 \leq i \leq n$ and $\delta_{i,j}$ is the Kronecker delta function, and let $X = \text{diag}(x_1, x_2, \ldots, x_n)$, where $x > 0$. Given $B \in \Omega_A$, we define the $n \times n$ matrix $M^\phi(x)$ by

\[(1.2) \quad M^\phi(x) = (X^{-1}BX - \lambda I)P_\phi = (m_{i,j}),\]

so that

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Following Schneider [4], if \( \lambda \) is an eigenvalue of \( B \), then \( M^\phi(x) \) is surely singular and thus not strictly diagonally dominant. Hence,

\[
|m_{i,i}| \leq \sum_{j \neq i} |m_{i,j}|
\]

must be true for at least one \( i, 1 \leq i \leq n \). Defining first

\[
A_i(x) \equiv \left( \sum_{j \neq i} |a_{i,j}| x_j \right) / x_i , \quad 1 \leq i \leq n ,
\]

then (1.4) implies that either

\[
|\lambda - a_{i,i}| \leq A_i(x) \quad \text{if} \quad \phi(i) = i ,
\]

or

\[
2 |a_{i,\phi(i)}| / x_i \leq |\lambda - a_{i,i}| + A_i(x) \quad \text{if} \quad \phi(i) \neq i .
\]

For any complex number \( \sigma \), we consequently define

\[
r^\phi_\lambda(\sigma; x) \equiv A_i(x) - |\sigma - a_{i,i}| \quad \text{if} \quad \phi(i) = i ,
\]

and let

\[
r^\phi_\lambda(\sigma; x) \equiv |\sigma - a_{i,i}| + A_i(x) - 2 |a_{i,\phi(i)}| x_{\phi(i)}/x_i \quad \text{if} \quad \phi(i) \neq i .
\]

With this, we next define the set \( G^\phi_i(x) \) as

\[
G^\phi_i(x) = \{ \sigma \mid r^\phi_\lambda(\sigma; x) \geq 0 \} , \quad 1 \leq i \leq n .
\]

If \( \phi(i) = i \), then \( G^\phi_i(x) \) reduces to the familiar Gerschgorin disk

\[ |z - a_{i,i}| \leq A_i(x) . \]

If \( \phi(i) \neq i \), we observe from (1.7') that \( G^\phi_i(x) \) is the closed exterior of a disk, and is thus an \textit{unbounded} set.

Defining \( G^\phi(x) \) to be the union of the sets \( G^\phi_i(x) \):

\[
G^\phi(x) \equiv \bigcup_{i=1}^n G^\phi_i(x) ,
\]

the inequalities of (1.6) and (1.6') show that if \( \lambda \in S(\Omega_A) \), then \( \lambda \in G^\phi_i(x) \) for some \( i \), and hence \( \lambda \in G^\phi(x) \). Thus, \( S(\Omega_A) \subset G^\phi(x) \) for \textit{every} \( x > 0 \), and we then have that

\[
G^\phi(\Omega_A) \equiv \bigcap_{x > 0} G^\phi(x) ,
\]

called the \textit{minimal Gerschgorin set relative to the permutation \( \phi \)}, is such that

\[
S(\Omega_A) \subset G^\phi(\Omega_A)
\]

for \textit{every} permutation \( \phi \). It is clear that \( G^\phi(\Omega_A) \) is a \textit{closed} set for
any permutation φ. Since $G_φ(x)$ is a bounded set only when $φ(i) = i$, it follows that $G_φ(Ω_4)$ is a bounded set only when φ is the identity permutation. We remark that the results of [6] are for the special case when φ is the identity permutation.

Since (1.11) is valid for any permutation φ, it then follows that

\[(1.12)\]

\[S(Ω_4) ⊂ H(Ω_4),\]

where

\[(1.13)\]

\[H(Ω_4) = \bigcap_φ G_φ(Ω_4).\]

In § 2, we first characterize (Theorem 1) the minimal Gerschgorin sets $G_φ(Ω_4)$, and then show (Theorem 2) that their boundaries $∂G_φ(Ω_4)$ are subsets of $S(Ω_4)$. Finally, using a result of Camion and Hoffman [1], we prove (Theorem 3) in § 3 our main result that

\[(1.14)\]

\[S(Ω_4) = H(Ω_4).\]

Summarizing, the now elementary Gerschgorin Circle Theorem [3], applied to a particular matrix $A$, actually gives eigenvalue bounds for a set $Ω_4$ of related matrices. Our main result is that the exact spectrum $S(Ω_4)$ of $Ω_4$ can be determined from extensions of the Gerschgorin Circle Theorem based only on positive diagonal similarity transformations, permutation matrices, and intersections.

In § 4, we include an extension of a result of [6] concerning the number of eigenvalues of any $B ∈ Ω_4$ in a bounded component of $G_φ(Ω_4)$. Finally, in § 5 we include several examples to show how $S(Ω_4)$ can be determined.

2. The Function $ν_φ(σ)$. In order to determine $G_φ(Ω_4)$, let $σ$ be any complex number, and consider the real $n \times n$ matrix $Q_φ(σ) = (q_{i,j})$ whose entries are defined by

\[(2.1)\]

\[q_{i,j} = (-1)^{δ_{i,j}} |a_{i,φ(j)} - σδ_{i,φ(j)}|, \quad 1 ≤ i, j ≤ n.\]

Since the off-diagonal entries of $Q_φ(σ)$ are nonnegative, then $Q_φ(σ)$ is essentially nonnegative [2; 5, p. 260], and hence we can associate with the matrix $Q_φ(σ)$ the real number $ν_φ(σ)$, where $ν_φ(σ)$ is the (possibly multiple) eigenvalue of $Q_φ(σ)$ with largest real part. From the Perron-Frobenius theory of nonnegative matrices [5, pp. 46-47], $ν_φ(σ)$ corresponds to a nonnegative eigenvector $y ≥ 0$, i.e., $Q_φ(σ)y = ν_φ(σ)y$, and it is further known that

\[(2.2)\]

\[ν_φ(σ) = \inf_{u > 0} \max_{1 ≤ i ≤ n} \left\{ \frac{(Q_φ(σ)u)_i}{u_i} \right\}.\]
We remark that $\nu_\phi(\sigma)$ is a continuous function of $\sigma$.

**THEOREM 1.** Let $A = (a_{i,j})$ be an $n \times n$ complex matrix, let $\phi$ be any permutation, and let $\sigma$ be a complex number. Then, $\sigma \in G^\phi(\Omega_A)$ if and only if $\nu_\phi(\sigma) \geq 0$.

**Proof.** From the definitions of $Q^\phi(\sigma)$ in (2.1) and $r^\phi_i(\sigma; x)$ in (1.7)-(1.7'), it follows that

$$r^\phi_i(\sigma; x) = \left(\frac{x_{\phi(i)}}{x_i}\right) \left(\frac{(Q^\phi(\sigma)z)_i}{z_i}\right), \text{ where } z_i \equiv x_{\phi(i)}.$$ 

Now, if $\sigma \in G^\phi(\Omega_A)$, then $\sigma \in G^\phi(x)$ for every $x > 0$. But for every $x > 0$, there is an $i$ such that $\sigma \in G^\phi_i(x)$, so that $r^\phi_i(\sigma; x) \geq 0$. Since $x > 0$, then $(x_{\phi(i)}/x_i)$ is positive for all $1 \leq i \leq n$, and it therefore follows from (2.2) that

$$\max_{1 \leq i \leq n} [(Q^\phi(\sigma)z)_i/z_i] \geq 0 \text{ for every } x > 0.$$ 

Clearly, as $x > 0$ runs over all positive vectors, so does the corresponding vector $z > 0$. Hence, $\nu_\phi(\sigma) \geq 0$ from (2.2). Conversely, assume that $\nu_\phi(\sigma) \geq 0$. From (2.2) and (2.3), it follows that $r^\phi_i(\sigma; x) \geq 0$ for some $i$ for every $x > 0$. Hence, $\sigma \in G^\phi(x)$ for every $x > 0$, and thus $\sigma \in G^\phi(\Omega_A)$, which completes the proof.

Our interest turns now to the boundary $\partial G^\phi(\Omega_A)$ of the minimal Gershgorin set $G^\phi(\Omega_A)$. As usual, it is defined by

$$\partial G^\phi(\Omega_A) = \overline{G^\phi(\Omega_A)'} \cap \overline{G^\phi(\Omega_A)},$$

where $\overline{G^\phi(\Omega_A)'}$ is the closure of the complement $G^\phi(\Omega_A)'$ of $G^\phi(\Omega_A)$. It follows from Theorem 1 that $G^\phi(\Omega_A)'$ is the set of all $\sigma$ which satisfy $\nu_\phi(\sigma) < 0$. Similarly, the boundary $\partial G^\phi(\Omega_A)$ of the minimal Gershgorin set is the set of all $\sigma$ for which $\nu_\phi(\sigma) = 0$, and to which there exists a sequence of complex numbers $\{z_j\}_{j=1}^\infty$ with $\lim_{j \to \infty} z_j = \sigma$ such that $\nu_\phi(z_j) < 0$.

As in [6], we now show that every point of the boundary $\partial G^\phi(\Omega_A)$ is an eigenvalue of some matrix $B \in \Omega_A$.

**THEOREM 2.** Let $A = (a_{i,j})$ be an $n \times n$ complex matrix, and let $\phi$ be any permutation. If $\nu_\phi(\sigma) = 0$, then $\sigma$ is an eigenvalue of some matrix $B \in \Omega_A$, and thus $\sigma \in S(\Omega_A)$.

**Proof.** If $\nu_\phi(\sigma) = 0$, then there exists a vector $y \geq 0$ with $y \neq 0$ such that $Q^\phi(\sigma)y = 0$. Writing $(\sigma - a_{k,k}) = |\sigma - a_{k,k}| \exp(i\psi_k)$, $1 \leq k \leq n$, let the $n \times n$ matrix $B = (b_{i,j})$ be defined by
(2.5) \[ b_{k,j} = a_{k,k} - b_{k,j} = |a_{k,j}| \exp i \left\{ \psi_k + \pi \left[ 1 + \delta_{k,\phi(k)} + \delta_{j,\phi(k)} \right] \right\}, \ k \neq j. \]

It is evident that \( B \in \Omega_A \), and if \( y_j = z_{\phi(j)} \), it can be verified (upon considering separately the cases when \( \phi(i) = i \) and \( \phi(i) \neq i \)) that \( Q^\phi(\sigma)y = 0 \) is equivalent to

\[ \sum_{j=1}^{n} b_{k,j}z_j = \sigma z_k, \quad 1 \leq k \leq n. \]

Since \( y \neq 0 \), then \( z \neq 0 \), and we conclude from (2.6) that \( \sigma \) is an eigenvalue of \( B \), which completes the proof.

In order to prove a somewhat stronger result, let \( \sigma \in \partial G^\phi(\Omega_A) \). Then, \( \nu(\sigma) = 0 \) and \( \sigma \in S(\Omega_A) \). But as \( S(\Omega_A) \subseteq G^\phi(\Omega_A) \) from (1.11), we have the

**COROLLARY 1.** Let \( A \) be an \( n \times n \) complex matrix. Then, for any permutation \( \phi \),

\[ dG^\phi(\Omega_A) \subseteq dS(\Omega_A). \]

In [6], an interesting geometrical property of the boundary \( \partial G^\phi(\Omega_A) \) was given when \( \phi \) was the identity permutation, and \( A \) was assumed to be irreducible. In that case, each boundary point of \( G^\phi(\Omega_A) \) was shown to be the intersection of \( n \) Gerschgorin circles. An analogous result is true for an arbitrary permutation \( \phi \), under slightly stronger hypotheses.

**COROLLARY 2.** Let \( A \) be an \( n \times n \) complex matrix, let \( \phi \) be any permutation, and let \( \sigma \in \partial G^\phi(\Omega_A) \). If \( Q^\phi(\sigma) \) is irreducible, then there exists a vector \( x > 0 \) such that \( \sigma \in dG^\phi_i(x) \) for all \( 1 \leq i \leq n \).

**Proof.** If \( Q^\phi(\sigma) \) is irreducible, then \( Q^\phi(\sigma) \) is essentially positive [5, p. 257]. Thus, there exists a vector \( z > 0 \) such that \( Q^\phi(\sigma)z = \nu(\sigma)z \). But, if \( \sigma \in dG^\phi(\Omega_A) \), then \( \nu(\sigma) = 0 \), and \( Q^\phi(\sigma)z = 0 \). Letting \( x > 0 \) be defined component-wise by \( z_i = x_{\phi(i)} \), it then follows from (2.3) that \( r^\phi_i(\sigma; x) = 0 \) for all \( 1 \leq i \leq n \). Now, \( r^\phi_i(\sigma; x) \) is obviously a continuous function of \( \sigma \) from (1.7)–(1.7'), and from (1.8) we deduce that \( \partial G^\phi_i(x) = \{ \mu | r^\phi_i(\mu; x) = 0 \} \). Hence, \( \sigma \in \partial G^\phi_i(x) \) for all \( 1 \leq i \leq n \), which completes the proof.

We remark that if \( \phi \) is the identity permutation, then \( Q^\phi(\sigma) \) is irreducible for any \( \sigma \) if and only if \( A \) is irreducible. For general \( \phi \), it is not difficult to show that \( A \) irreducible implies that \( Q^\phi(\sigma) \) is irreducible when \( \sigma \neq a_{i,i} \) for any \( i \).

3. **Main Result.** We shall now show that \( S(\Omega_A) = H(\Omega_A) = \bigcap_\phi G^\phi(\Omega_A) \). Since \( S(\Omega_A) \subseteq H(\Omega_A) \) by (1.12), it suffices to prove that
$S(\Omega_A)' \subset H(\Omega_A)'$, where $S(\Omega_A)'$ denotes the complement of $S(\Omega_A)$. This last inclusion will follow quite easily from the following theorem of Camion and Hoffman [1]:

Given an arbitrary $n \times n$ complex matrix $B = (b_{i,j})$, let $\hat{\Omega}_B$ be the set of all matrices $C = (c_{i,j})$ with $|c_{i,j}| = |b_{i,j}|$ for all $1 \leq i, j \leq n$. Then, if all matrices $C \in \hat{\Omega}_B$ are nonsingular, there exists a positive diagonal matrix $X = \text{diag}(x_1, \ldots, x_n)$, $x_i > 0$, and a permutation matrix $P_\phi = (\delta_{i,\phi(j)})$ such that the matrix $M = BXP_\phi = (m_{i,j})$ is strictly diagonally dominant, i.e.,

\begin{equation}
|m_{i,i}| > \sum_{j \neq i} |m_{i,j}| \quad \text{for all} \quad 1 \leq i \leq n .
\end{equation}

We first prove

**Lemma 1.** $\sigma \in S(\Omega_A)'$ if and only if each $R \in \hat{\Omega}_{A-\sigma I}$ is nonsingular.

**Proof.** It is clear that each $R \in \hat{\Omega}_{A-\sigma I}$ can be uniquely expressed as $R = D(B - \sigma I)$, where $D = \text{diag} \left( e^{i\psi_1}, \ldots, e^{i\psi_n} \right)$, $\psi_j$ is real, and $B \in \Omega_A$. Then, $\sigma \in S(\Omega_A)'$ implies that $\det (B - \sigma I) \neq 0$ for any $B \in \Omega_A$. But as $|\det D| = 1$, then $\det R = \det D \cdot \det (B - \sigma I) \neq 0$ for any $R \in \hat{\Omega}_A$. The converse follows similarly.

Now, suppose $\sigma \in S(\Omega_A)'$. From Lemma 1 and the result of Camion and Hoffman applied to $B = A - \sigma I$, there exists a positive diagonal matrix $X = \text{diag}(x_1, \ldots, x_n)$ and a permutation matrix $P_\phi = (\delta_{i,\phi(j)})$ such that the matrix $M = (A - \sigma I)XP_\phi = (m_{i,j})$ is strictly diagonally dominant, where

\begin{equation}
m_{i,j} = (a_{i,\phi(j)} - \sigma\delta_{i,\phi(j)})x_{\phi(j)} .
\end{equation}

Comparing (3.2) with the definition of $Q^\phi(\sigma)$ in (2.1) and setting $z_j = x_{\phi(j)}$, $1 \leq j \leq n$, (3.1) can be equivalently expressed as

\begin{equation}
0 > \sum_{j \neq i} |m_{i,j}| - |m_{i,i}| = (Q^\phi(\sigma)z)_i , \quad 1 \leq i \leq n .
\end{equation}

Since $z > 0$, it follows from (2.2) that $\nu_\phi(\sigma) < 0$, and hence from Theorem 1 we deduce that $\sigma \in G^\phi(\Omega_A)$. Consequently, $\sigma \in S(\Omega_A)$ implies that $\sigma \in G^\phi(\Omega_A)$, which in turn implies that $\sigma \in H(\Omega_A)$, or

\begin{equation}
S(\Omega_A)' \subset H(\Omega_A)' .
\end{equation}

This, coupled with the result that $S(\Omega_A) \subset H(\Omega_A)$, gives us

**Theorem 3.** Let $A = (a_{i,j})$ be any $n \times n$ complex matrix. Then
$S(\Omega_\Lambda) = H(\Omega_\Lambda)$.

4. Disconnected minimal gerschgorin sets. A familiar result of Gerschgorin [3] states that if $k$ disks of the Gerschgorin set $G^\prime(x)$ (where $I$ is the identity permutation) are disjoint from the remaining $n - k$ disks, then these $k$ disks contain exactly $k$ eigenvalues of any matrix $B \in \Omega_\Lambda$. In this section, we give a generalization of this result (cf. Theorem 5 of [6]). For a given $n \times n$ matrix $A = (a_{i,j})$ and an arbitrary permutation $\phi$, let $G^\phi_\Lambda(\Omega_\Lambda)$ denote the nonempty disjoint closed connected components of the minimal Gerschgorin set $G^\phi(\Omega_\Lambda)$:

\[ G^\phi(\Omega_\Lambda) = \bigcup_{j=1}^{m} G^\phi_j(\Omega_\Lambda), \quad 1 \leq m \leq n. \]

For each bounded component $G^\phi_j(\Omega_\Lambda)$, let the order $s^\phi_j$ be defined as the number of diagonal elements $a_{i,j}$ of $A$ contained in $G^\phi_j(\Omega_\Lambda)$ for which $\phi(i) = i$. We shall show that each matrix $B \in \Omega_\Lambda$ contains exactly $s^\phi_j$ eigenvalues in each bounded component $G^\phi_j(\Omega_\Lambda)$ of the minimal Gerschgorin set $G^\phi(\Omega_\Lambda)$.

To begin, we enlarge the set $\Omega_\Lambda$. An $n \times n$ matrix $B = (b_{i,j})$ is defined to be an element of the extended set $\Omega_\phi$ if

\[ \begin{align*}
&b_{i,i} = a_{i,i}, \; 1 \leq i \leq n; \; |b_{i,\phi(i)}| \geq |a_{i,\phi(i)}|, \; \phi(i) \neq i, \\
&|b_{i,j}| \leq |a_{i,j}|, \; 1 \leq i, j \leq n, \; \text{for which } j \neq i \; \text{and} \; j \neq \phi(i).
\end{align*} \]

Clearly, $\Omega_\Lambda \subset \Omega_\phi$.

**Lemma 2.** Given $B \in \Omega_\phi$, then $G^\phi(\Omega_B) \subset G^\phi(\Omega_\Lambda)$.

**Proof.** For any vector $u > 0$ and any complex number $\sigma$, consider the vector $Q^\phi_B(\sigma)u$, where we are using an obvious subscript notation. With $B \in \Omega_\phi$, one verifies from (4.2) and (2.1) that $Q^\phi_B(\sigma)u \leq Q^\phi(\sigma)u$ for any $u > 0$ and any $\sigma$, from which it follows that

\[ \max_{1 \leq i \leq n} \left\{ \frac{(Q^\phi_B(\sigma)u)_i}{u_i} \right\} \leq \max_{1 \leq i \leq n} \left\{ \frac{(Q^\phi(\sigma)u)_i}{u_i} \right\}. \]

Thus, from (2.2), $\nu_{\phi, B}(\sigma) \leq \nu_{\phi, \Lambda}(\sigma)$. Hence, by Theorem 1, $\sigma \in G^\phi(\Omega_B)$ implies that $\sigma \in G^\phi(\Omega_\Lambda)$, which completes the proof.

For this extended set $\Omega_\phi$, we remark that it can be further shown that $S(\Omega_\phi) = G^\phi(\Omega_\Lambda)$ for any permutation $\phi$. This generalizes another result (Theorem 6) of [6].

In the spirit of Gerschgorin's original continuity argument [3], we prove
THEOREM 4. Let $A = (a_{i,j})$ be any $n \times n$ complex matrix, and let $\phi$ be any permutation. If $G^\phi(\Omega_A)$ has a bounded component $G^\phi(\Omega_A)$ of order $s^\phi$, then, for any matrix $B \in \Omega_A$, $B$ contains exactly $s^\phi$ eigenvalues in $G^\phi(\Omega_A)$.

Proof. For any $B = (b_{i,j}) \in \Omega_A$, consider the family of matrices $B_m(\alpha) = (b_{i,j}(\alpha))$ defined by

\[
\begin{cases}
    b_{i,i}(\alpha) = b_{i,i}, & 1 \leq i \leq n; \\
    b_{i,\phi(i)}(\alpha) = b_{i,\phi(i)}[m(1 - \alpha) + \alpha] \text{ when } \phi(i) \neq i; \\
    b_{i,j}(\alpha) = \alpha b_{i,j} \text{ for any } 1 \leq i, j \leq n \text{ for which } j \neq i \text{ and } j \neq \phi(i).
\end{cases}
\]

By definition, $B_m(\alpha) \in \Omega_A$ for all $0 \leq \alpha \leq 1$ and all $m \geq 1$, and $B_m(1) = B$. Moreover, $B_m(\alpha) \in \Omega_{B_m(\alpha)}$ for all $0 \leq \alpha \leq \alpha' \leq 1$. Thus, from Lemma 2, $G^\phi(\Omega_{B_m(\alpha)}) \subseteq G^\phi(\Omega_A)$ for all $0 \leq \alpha \leq 1$ and all $m \geq 1$, and it is clear that the set $G^\phi(\Omega_{B_m(\alpha)})$ increases monotonically with $\alpha$. We shall show that $B_m(0)$ has exactly $s^\phi$ eigenvalues in the bounded component $G^\phi(\Omega_A)$, and the theorem will follow by continuously increasing $\alpha$ from zero to unity.

From (4.4), the only possibly nonzero entries of the matrix $B_m(0)$ are $b_{i,i}(0)$ and $b_{i,\phi(i)}(0)$ where $\phi(i) \neq i$. Hence, by considering the disjoint cycles of the permutation $\phi$, we can find an $n \times n$ permutation matrix $P$ such that

\[
P B_m(0) P^T = \begin{bmatrix} B_{1,1} & 0 \\ B_{2,2} & \quad \quad 0 \\ \hline 0 & B_{N,N} \end{bmatrix}, \quad 1 \leq N < n.
\]

Here, $B_{1,1}$ is a diagonal matrix corresponding to all disjoint cycles with $\phi(i) = i$. The other matrices $B_{j,j}$ have the cyclic form

\[
B_{j,j} = \begin{bmatrix} b_{i,1}^{(j)} & b_{i,2}^{(j)} & 0 \\ \hline & b_{r_j-1,r_j}^{(j)} \\ 0 & b_{r_j,r_j}^{(j)} \end{bmatrix}, \quad 2 \leq j \leq N,
\]

where the off-diagonal entries of $B_{j,j}$ are, from (4.4), given by $mb_{i,\phi(i)}$, $\phi(i) \neq i$. Obviously, the eigenvalues of all the $B_{j,j}$ are the eigenvalues of $B_m(0)$.

The spectrum of matrices of the form (4.6) is discussed in Example 1 of the next section, and in §6 of [6]. We now assert that

\[
|b_{i,1}^{(j)} b_{i,2}^{(j)} \cdots b_{r_j,r_j}^{(j)}| \neq 0 \text{ for any } 2 \leq j \leq N.
\]

Otherwise, $b_{k,\phi(k)} = 0$ for some integer $k$, where $\phi(k) \neq k$, and, as shown
in the next section, this implies that \( G^\psi(\Omega_A) \) is the entire complex plane. This contradicts the hypothesis that \( G^\psi(\Omega_A) \) has a bounded component. From (4.4), we can write the product in (4.7) as \( m^{r_j} \cdot K_j \), where \( K_j \) is independent of \( m \) and \( \alpha \). Then, it is readily verified that the eigenvalues \( \lambda \) of \( B_{j,i} \) satisfy

\[
(4.8) \quad \prod_{k=1}^{r_j} \left| b_{j,k} - \lambda \right| = m^{r_j} \cdot K_j, \quad 2 \leq j \leq N,
\]

for any \( B_m(0) \) derived from \( B \in \Omega_A \). Since \( B_m(0) \in \Omega_A^\psi \) for all \( m \geq 1 \), we may choose \( m \) to be arbitrarily large, and it is clear from (4.8) that the eigenvalues of \( B_{j,i} \) must lie in an unbounded component of \( G^\psi(\Omega_A) \) for any \( 2 \leq j \leq N \). Hence, the number of eigenvalues of \( B_m(0) \) which lie in the bounded component \( G^\psi_j(\Omega_A) \) is just the number of diagonal entries of \( B_{j,i} \) in \( G^\psi_j(\Omega_A) \), which by definition is precisely \( s^\psi_j \).

Now, increasing \( \alpha \) continuously from zero to unity, it follows that \( B \) has exactly \( s^\psi_j \) eigenvalues in \( G^\psi_j(\Omega_A) \), which completes the proof.

We remark that the order \( s^\psi_j \) of a bounded component \( G^\psi_j(\Omega_A) \) is a positive integer. For, if \( s^\psi_j \) were zero, no \( B \in \Omega_A \) would have an eigenvalue in \( G^\psi_j(\Omega_A) \), so that \( S(\Omega_A) \cap G^\psi_j(\Omega_A) \) would be empty, which is a contradiction.

5. Some examples. We now give three examples to illustrate our results concerning the sets \( S(\Omega_A), G^\psi(\Omega_A), \) and \( H(\Omega_A) \).

**Example 1.** It was previously shown [6] for the matrix

\[
(5.1) \quad A = \begin{bmatrix}
    a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\
    0 & a_{2,2} & a_{2,3} & \cdots & 0 \\
    & & \ddots & \ddots & \ddots \\
    & & & a_{n-1,n} & 0 \\
    & & & 0 & a_{n,n}
\end{bmatrix},
\]

where

\[
(5.2) \quad |a_{1,1}a_{2,2} \cdots a_{n,n}| = 1,
\]

that \( \partial G^I(\Omega_A) = S(\Omega_A) \), \( I \) being the identity permutation. Let \( \psi \) be the permutation\(^1\) \((1\ 2\ 3 \cdots n)\). If \( \phi \) is any permutation other than \( \psi \) or \( I \), there is a positive integer \( k \), \( 1 \leq k \leq n \), such that \( \phi(k) \neq k \), and \( \phi(k) \neq \psi(k) \), so that \( a_{k,\phi(k)} = 0 \). Thus, from (1.7'),

\[
(5.2) \quad r^\psi_k(\sigma; x) = |\sigma - a_{k,k}| + |a_{k,\psi(k)}| |x_{\psi(k)}/x_k| > 0
\]

for all \( x > 0 \), and for all complex numbers \( \sigma \). Hence, we deduce from

\(^1\) That is, in this section we are describing a permutation by its disjoint cycles.
(2.2), (2.3), and Theorem 1 that $G^\Phi(\Omega_A)$ is the entire complex plane. This argument shows more generally for an arbitrary matrix $A$ that any permutation $\phi$ which places a zero on the diagonal of $Q^\phi(\sigma)$ yields a minimal Gerschgorin set $G^\phi(\Omega_A)$ which is the entire complex plane.

For $\phi = I$, it was shown [6] for the matrix of (5.1) that

$$
(5.3) \quad G^I(\Omega_A) = \left\{ \sigma \mid \prod_{i=1}^{n} |\sigma - a_{i,i}| \leq 1 \right\}
$$

and in an identical fashion, we can show that

$$
(5.4) \quad G^\psi(\Omega_A) = \left\{ \sigma \mid \prod_{i=1}^{n} |\sigma - a_{i,i}| \geq 1 \right\}.
$$

Hence, it follows that

$$
(5.5) \quad S(\Omega_A) = H(\Omega_A) = G^I(\Omega_A) \cap G^\psi(\Omega_A) = \partial G^I(\Omega_A).
$$

**Example 2.** Consider the matrix

$$
(5.6) \quad A = \begin{bmatrix}
2 & 0 & 1 \\
0 & 1 & 1 \\
1 & 1 & 2
\end{bmatrix}.
$$

In this case, there are only three permutations, corresponding to $\phi = I$, $\phi = (13)$, and $\phi = (23)$, for which $G^\phi(\Omega_A)$ is not the entire complex plane, and it is readily verified that

$$
(5.7) \quad \begin{cases}
G^I(\Omega_A) = \{ \sigma \mid |2 - \sigma|^2 \cdot |1 - \sigma| \leq |1 - \sigma| + |2 - \sigma| \}, \\
G^{(13)}(\Omega_A) = \{ \sigma \mid |2 - \sigma|^2 \cdot |1 - \sigma| \geq |1 - \sigma| - |2 - \sigma| \}, \\
G^{(23)}(\Omega_A) = \{ \sigma \mid |2 - \sigma|^2 \cdot |1 - \sigma| \geq - |1 - \sigma| + |2 - \sigma| \}.
\end{cases}
$$

The boundaries $\partial G^\phi(\Omega_A)$ are obviously determined by choosing the equality signs in (5.7). The spectrum $S(\Omega_A)$ in this case is a multiply connected region and is illustrated in Figure 1.
**Example 3.** Consider the matrix
\[
A = \begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
-1 & -5 & -1 & -1 \\
\end{bmatrix},
\]
which is the companion matrix of the polynomial
\[
p_A(z) = z^4 + z^3 + z^2 + 5z + 1.
\]
As previously shown, any permutation \(\phi\) which places a zero on the diagonal of \(Q^\phi(\sigma)\) yields a minimal Gerschgorin set \(G^\phi(\Omega_A)\) which is the entire complex plane. Consequently, we need consider only the permutations \(I, (1234), (234),\) and \((34)\). The associated minimal Gerschgorin sets are given by
\[
\begin{align*}
G^I(\Omega_A) &= \{\sigma \mid |\sigma|^3 \cdot 1 + \sigma \leq 1 + 5 |\sigma| + |\sigma|^3\}, \\
G^{(1234)}(\Omega_A) &= \{\sigma \mid |\sigma|^3 \cdot 1 + \sigma \geq 1 - 5 |\sigma| - |\sigma|^3\}, \\
G^{(234)}(\Omega_A) &= \{\sigma \mid |\sigma|^3 \cdot 1 + \sigma \geq -1 + 5 |\sigma| - |\sigma|^3\}, \\
G^{(34)}(\Omega_A) &= \{\sigma \mid |\sigma|^3 \cdot 1 + \sigma \geq -1 - 5 |\sigma| + |\sigma|^3\}.
\end{align*}
\]
The last minimal Gerschgorin set \(G^{(34)}(\Omega_A)\) is the entire complex plane, and thus yields no boundary components of \(S(\Omega_A)\). The set \(G^{(1234)}(\Omega_A)\) yields, however, two separate boundaries, and \(G^{(234)}(\Omega_A)\) has a bounded component. Applying Theorem 4, we can assert that each matrix of the set \(\Omega_A\) has exactly one eigenvalue in this component, and hence each matrix of \(\Omega_A\) has exactly one eigenvalue in the inner annular region of Figure 2.

These examples have interesting common features. In each ex-
ample, the minimum number of permutations necessary to define all the boundary components of $S(\Omega_\Lambda)$ does not exceed the order $n$ of the matrix $A$. Similarly, the total number of boundary components of $S(\Omega_\Lambda)$ does not exceed $2n$. We conjecture this to be true in general. We do point out that examples can be constructed where these upper bounds are attained.

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* Paul A. White, Acting Editor until J. Dugundji returns.
Henry A. Antosiewicz, *Boundary value problems for nonlinear ordinary differential equations* ................................................................. 191
Bernard Werner Levinger and Richard Steven Varga, *Minimal Gerschgorin sets. II* ................................................................. 199
Paul Camion and Alan Jerome Hoffman, *On the nonsingularity of complex matrices* ................................................................. 211
J. Chidambaramswamy, *Divisibility properties of certain factorials* ........ 215
J. Chidambaramswamy, *A problem complementary to a problem of Erdős* . . . 227
John Dauns, *Chains of modules with completely reducible quotients* .......... 235
Wallace E. Johnson, *Existence of half-trajectories in prescribed regions and asymptotic orbital stability* ................................................................. 243
Victor Klee, *Paths on polyhedra. II* ................................................................. 249
Edwin Haena Mookini, *Sufficient conditions for an optimal control problem in the calculus of variations* ................................................................. 263
Zane Clinton Motteler, *Existence theorems for certain quasi-linear elliptic equations* ................................................................. 279
David Lewis Outcalt, *Simple n-associative rings* ................................................................. 301
David Joseph Rodabaugh, *Some new results on simple algebras* ........ 311
Oscar S. Rothaus, *Asymptotic properties of groups generation* .......... 319
Ernest Edward Shult, *Nilpotence of the commutator subgroup in groups admitting fixed point free operator groups* ................................................................. 323
William Hall Sills, *On absolutely continuous functions and the well-bounded operator* ................................................................. 349
Joseph Gail Stampfli, *Which weighted shifts are subnormal* ........ 367
Donald Reginald Traylor, *Metrizability and completeness in normal Moore spaces* ................................................................. 381