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SIMPLE n -ASSOCIATIVE RINGS

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This paper is concerned with certain classes of nonassociative rings. These rings are defined by first extending the associator $(a, b, c) = (ab)c - a(bc)$. The n -associator (a_1, \dots, a_n) is defined by

$$(1.1) \quad \begin{aligned} (a_1, a_2) &= a_1 a_2, \\ (a_1, \dots, a_n) &= \sum_{k=0}^{n-2} (-1)^k (a_1, \dots, a_k, a_{k+1} a_{k+2}, \dots, a_n). \end{aligned}$$

A ring is defined to be n -associative if the n -associator vanishes in the ring. It is shown that simple 4-associative and simple 5-associative rings are associative; simple $2k$ -associative rings are $(2k - 1)$ associative or have zero center; and simple, commutative n -associative rings, $6 \leq n \leq 9$, are associative. The concept of rings which are associative of degree $2k + 1$ is defined, and it is shown that simple, commutative rings which are associative of degree $2k + 1$ are associative. The characteristic of the ring is slightly restricted in all but one of these results.

The concepts of the n -associator and n -associative rings were defined by A. H. Boers [1; Ch. 3 and Ch. 4]. Our results extend Boers' main result that an n -associative division ring is associative with minor restriction on the characteristic [1; Th. 6]. We do not consider 2-associative rings.

To obtain our results, it is necessary to extend the concept of the n -associator. In a ring R , define $S(2j + 1, 2k + 1)$, $1 \leq j \leq k$, by defining $S(2j + 1, 2j + 1)$ to be the set of all finite sums of $(2j + 1)$ -associators with entries in R , and then by defining $S(2j + 1, 2k + 1)$, $k > j$, to be the set of all finite sums of $(2j + 1)$ -associators (a_1, \dots, a_{2j+1}) such that $(a_1, \dots, a_{2j+1}) \in S(2j + 1, 2k - 1)$ and such that at least one of the $2k - 1$ entries of (a_1, \dots, a_{2j+1}) is in $S(3, 3)$. For example, $((a_1, a_2, a_3), a_4, a_6), a_6, (a_7, a_8, a_9) \in S(3, 9)$.

Clearly, a ring R is $(2n + 1)$ -associative if and only if $S(2n + 1, 2n + 1) = 0$ in R . This leads us to call a ring R $(2n + 1)$ -associative of degree $2k + 1$ if $S(2n + 1, 2k + 1) = 0$ in R . No mention of degree will be made in case $k = n$.

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Of particular interest are rings which are associative (3-associative) of degree $2k + 1$. In the first place, this in itself is an interesting extension of the concept of associativity. Consider the 4-dimensional algebra A over an arbitrary field with basis a_1, a_2, a_3, a_4 such that $a_1^2 = a_2$, $a_1 a_2 = a_3 - a_4$, $a_2 a_1 = a_3$, and all other products zero. It can be verified that $S(3, 5) = 0$ in A but that A is not associative. Also, it turns out that a ring which is associative of degree $2k + 1$ is $(2k + 1)$ -associative, but not conversely.

2. Preliminaries. We will need the following three identities derived by Boers.

$$(2.1) \quad (a_1, \dots, a_n) = \sum_{k=1}^{n-3} (a_1, \dots, a_k, (a_{k+1}, a_{k+2}, a_{k+3}), a_{k+4}, \dots, a_n),$$

$$(2.2) \quad (a_1, \dots, a_n) = \sum_{k=0}^{(1/2)n-1} \binom{(1/2)n-1}{k} (a_1, \dots, a_{n-2k-1}) (a_{n-2k}, \dots, a_n)$$

for even n where $\binom{r}{s}$ denotes the binomial coefficient [1; Ch. 3]. In a commutative ring, we have

$$(2.3) \quad (a_1, \dots, a_n) = (-1)^{[(1/2)n-(1/2)]} (a_n, \dots, a_1)$$

where $[x]$ denotes the greatest integer $\leq x$ [2; Th. A].

Next, we will need

LEMMA 2.1. $S(2j + 1, 2k + 1) \subset S(2m + 1, 2n + 1)$, $1 \leq m \leq j$, $m \leq n \leq k$.

Proof. It is immediate from the definition of $S(2j + 1, 2k + 1)$ that $S(2j + 1, 2k + 1) \subset S(2j + 1, 2n + 1)$, $j \leq n \leq k$. Hence we need only show that $S(2j + 1, 2n + 1) \subset S(2m + 1, 2n + 1)$, $1 \leq m \leq j$. The result is obvious if $j = m$. Assume that $S(2j - 1, 2n + 1) \subset S(2m + 1, 2n + 1)$, $j > m$. Then by (2.1), $S(2j + 1, 2n + 1) \subset S(2j - 1, 2n + 1) \subset S(2m + 1, 2n + 1)$, and we are finished.

Let A and B be subsets of a ring. Define AB to be the set of all finite sums of elements of the form ab such that $a \in A$, $b \in B$.

Let $I(3, 2k + 1) = S(3, 2k + 1) + S(3, 2k + 1)R$. The next lemma is a generalization of the fact that $I(3, 3)$ is an ideal of an arbitrary ring R [4; p. 985].

LEMMA 2.2. $I(3, 2k + 1)$ is a right ideal of an arbitrary ring R for $k = 1, 2, \dots$.

Proof. Let $I = I(3, 2k + 1)$, $S = S(3, 2k + 1)$. We have

$$IR \subset SR + SR \cdot R \subset SR + (S, R, R) .$$

However, by Lemma 2.1, $(S, R, R) \subset S$. Hence $IR \subset I$.

The key to our results is

LEMMA 2.3. *If R is a simple, commutative ring, and if $A = \{a \in R \mid aS(3, 2k + 1) = 0\}$, then $A = 0$ or $S(3, 2k + 1) = 0$.*

Proof. We can assume that $R = I(3, 2k + 1)$, for otherwise we are finished by Lemma 2.2. If K is the ideal generated by AR , then $K \subset S(3, 3)$. Indeed, $AR = AI(3, 2k + 1) \subset S(3, 3)$. Define $R_x: y \rightarrow yx$. Assume that we have shown that $aR_{x_1} \cdots R_{x_n} \in S(3, 3)$ for $a \in A$ and for every choice of n elements $x_1, \dots, x_n \in R, n > 1$. Then

$$\begin{aligned} aR_{x_1} \cdots R_{x_n} R_{x_{n+1}} &= ((aR_{x_1} \cdots R_{x_{n-1}})x_n)x_{n+1} \\ &= (aR_{x_1} \cdots R_{x_{n-1}}, x_n, x_{n+1}) + aR_{x_1} \cdots R_{x_{n-1}}R_{x_n x_{n+1}} \in S(3, 3) . \end{aligned}$$

If $K = 0$, then $AR = 0$ which implies $A = 0$ and we are finished. Hence assume $K = R$. Thus $R = S(3, 3)$ which implies $R = S(3, 2k + 1)$ by induction since each element can now be replaced by a sum of associators. Therefore $AR = 0$, and hence $A = 0$.

To ease the computations of the proofs which follow, we define

$$T(n, i, j) = \sum_{m=i}^j (a_1, \dots, a_m, (a_{m+1}, a_{m+2}, a_{m+3}), a_{m+4}, \dots, a_n)$$

where $0 \leq i \leq j \leq n - 3$. Note that (2.1) becomes $(a_1, \dots, a_n) = T(n, 0, n - 3)$. The next lemma, whose proof follows easily from the definition of $T(n, i, j)$ (with the use of (2.3) in the case of part (d)), contains all the additional facts about $T(n, i, j)$ that we will need.

LEMMA 2.4.

- (a) $T(n, i, j) - T(n, i, k) = T(n, k + 1, j), k < j;$
- (b) $T(n, i, j) - T(n, k, j) = T(n, i, k - 1), i < k;$
- (c) $S(2k + 1, 2k + 3)$ consists of all finite sums of elements of the form $T(2k + 3, i, i), i = 0, 1, \dots, 2k;$ and
- (d) if $0 = T(n, i, j)a$ is an identity in a commutative ring, then so is $0 = T(n, n - 3 - j, n - 3 - i)a$.

We will not cite Lemma 2.4 when we use it.

Finally, the nucleus N of a ring R is defined by $N = \{u \in R \mid (u, x, y) = (x, u, y) = (x, y, u) = 0 \text{ for all } x, y \in R\}$. The center C of R is defined by $C = \{c \in N \mid cx = xc \text{ for all } x \in R\}$.

3. *n*-Associative rings. In what follows, we will use the fact that (1.1) implies that if R is an n -associative ring, then R is k -associative for all $k \geq n$.

THEOREM 3.1. *If R is a simple 4-associative or 5-associative ring of characteristic not 2, then R is associative.*

Proof. By (2.2) with $n = 6$, we have

$$(3.1) \quad 0 = S(3, 3)^2.$$

Since $I(3, 3)$ is an ideal of R , we can assume that $R = I(3, 3)$, for otherwise $S(3, 3) = 0$. Hence (3.1) yields $S(3, 3)R \subset S(3, 3)$. Therefore $R = S(3, 3)$, but then we have $S(3, 3)R = 0 = RS(3, 3)$ by (3.1). Thus $S(3, 3) = 0$.

Theorem 3.1 extends a result of Boers [3; p. 126] who has also shown it to be false for characteristic 2.

THEOREM 3.2. *Let R be a simple $2k$ -associative ring for $k \geq 2$. Then $C = 0$ or R is $(2k - 1)$ -associative where C is the center of R .*

Proof. We first show that if N is the nucleus of R , then any $(2j + 1)$ -associator with an entry $u \in N$ vanishes. Indeed, if $j = 1$, the result follows by the definition of N . Assume that we have established the result for $j = i$. By (2.1), $(a_1, \dots, a_{2i+3}) = T(2i+3, 0, 2i)$. Each n -associator in $T(2i+3, 0, 2i)$ is a $(2i+1)$ -associator. Hence if $u \in N$ is an entry of (a_1, \dots, a_{2i+3}) , then $(a_1, \dots, a_{2i+3}) = 0$.

Now, use (2.2) with $n = 2k$. In the resulting identity, let $a_{2k} \in C$. Then, since $C \subset N$, we have $S(2k - 1, 2k - 1)C = 0$. Therefore $C = 0$ or $S(2k - 1, 2k - 1) = 0$ since R is simple and the annihilators of C may be shown to form an ideal of R .

Theorems 3.1 and 3.2 imply

COROLLARY 3.1. *If R is a simple 6-associative ring of characteristic not 2, then $C = 0$ or R is associative.*

We now turn our attention to commutative rings.

THEOREM 3.3. *If R is a simple, commutative 6-associative or 7-associative ring of characteristic not 2 or 3, then R is associative.*

Proof. By Theorem 3.1, it is sufficient to show that $S(5, 5) = 0$.

Let $n = 8$ and 10 in (2.2) to obtain

$$(3.2) \quad 0 = (a_1, \dots, a_5) (a_6, a_7, a_8) + (a_1, a_2, a_3) (a_4, \dots, a_8)$$

and

$$(3.3) \quad 0 = S(5, 5)^2.$$

Next, use (2.1) and (3.3) to obtain $T(5, 0, 2)S(5, 5) = 0$ which yields $T(7, 0, 2)S(3, 3) = 0$ upon application of (3.2). Hence we may assume that $T(7, 0, 2) = 0$, for otherwise $S(3, 3) = 0$ by Lemma 2.3. Using (2.1), we compute $0 = T(7, 0, 4) - T(7, 0, 2) = T(7, 3, 4)$; hence $T(7, 0, 1) = 0$ by (2.3). Thus $T(7, 2, 2) = 0$ since $T(7, 2, 2) = T(7, 0, 2) - T(2, 0, 1)$. Hence we have $T(5, 2, 2)S(5, 5) = 0$ using (3.2). Thus $S(3, 5)S(5, 5) = 0$ upon using (2.3), (2.1) with $n = 5$, and (3.3), in that order. Application of Lemma 2.3 and then Lemma 2.1 completes the proof.

THEOREM 3.4. *If R is a simple, commutative 8-associative or 9-associative ring of characteristic not 2, 3, or 5, then R is associative.*

Proof. By Theorem 3.3, it is sufficient to show that $S(7, 7) = 0$.

If we let $n = 10, 12$, and 14 in (2.2), we get

$$(3.4) \quad 0 = 2(a_1, \dots, a_7) (a_8, a_9, a_{10}) + 3(a_1, \dots, a_5) (a_6, \dots, a_{10}) \\ + 2(a_1, a_2, a_3) (a_4, \dots, a_{10}),$$

$$(3.5) \quad 0 = (a_1, \dots, a_7) (a_8, \dots, a_{12}) + (a_1, \dots, a_5) (a_6, \dots, a_{12}),$$

and

$$(3.6) \quad 0 = S(7, 7)^2.$$

Our first goal is to establish

$$(3.7) \quad S(7, 9)S(5, 5) = 0 = S(7, 7)S(5, 7).$$

Applying (2.1) to (3.6) yields $T(7, 0, 4)S(7, 7) = 0$, to which we apply (3.5) to obtain $T(9, 0, 4)S(5, 5) = 0$. Using (2.1), we compute $0 = (T(9, 0, 6) - T(9, 0, 4))S(5, 5) = T(9, 5, 6)S(5, 5)$ which implies $T(9, 0, 1)S(5, 5) = 0$. Then $0 = (T(9, 0, 4) - T(9, 0, 1))S(5, 5) = T(9, 2, 4)S(5, 5)$, to which we apply (3.5) to obtain $T(7, 2, 4)S(7, 7) = 0$. Hence we have $T(7, 0, 2)S(7, 7) = 0$. Using (3.5) again yields $T(9, 0, 2)S(5, 5) = 0$. Thus we can compute $0 = (T(9, 0, 2) - T(9, 0, 1))S(5, 5)$ to obtain

$$(3.8) \quad T(9, 2, 2)S(5, 5) = 0 = T(9, 4, 4)S(5, 5).$$

Computing $0 = (T(9, 2, 4) - T(9, 2, 2) - T(9, 4, 4))S(5, 5)$ yields

$$(3.9) \quad 0 = T(9, 3, 3) S(5, 5) .$$

Applying (3.5) to (3.9), we have $T(7, 3, 3) S(7, 7) = 0$, from which we obtain $T(7, 1, 1) S(7, 7) = 0$, and hence (3.5) implies

$$(3.10) \quad 0 = T(9, 1, 1) S(5, 5) = 0 .$$

Computing $0 = (T(9, 0, 1) - T(9, 1, 1)) S(5, 5)$ yields $T(9, 0, 0) S(5, 5) = 0$ which, along with (3.8), (3.9), and (3.10), implies (3.7) after using (3.5).

Our next goal is to establish

$$(3.11) \quad 0 = S(5, 7)^2 .$$

Let $a_1 = (x_1, x_2, x_3)$, $a_i = x_{i+2}$, $i > 1$ in (3.4); then let $a_1 = x_1$, $a_2 = (x_2, x_3, x_4)$, $a_i = x_{i+2}$, $i > 2$ in (3.4); and then let $a_1 = x_1$, $a_2 = x_2$, $a_3 = (x_3, x_4, x_5)$, $a_i = x_{i+2}$, $i > 3$ in (3.4). Add the resulting identities and apply (2.1) to obtain

$$(3.12) \quad 0 = 2T(9, 0, 2) (x_{10}, x_{11}, x_{12}) + 3T(7, 0, 2) (x_8, \dots, x_{12}) \\ + 2(x_1, \dots, x_5) (x_6, \dots, x_{12})$$

where the T 's are now written in terms of the x 's. Substitute (x_{10}, x_{11}, x_{12}) for x_{10} , x_{13} for x_{11} , and x_{14} for x_{12} in (3.12); then substitute (x_{11}, x_{12}, x_{13}) for x_{11} and x_{14} for x_{12} in (3.12); and then substitute (x_{12}, x_{13}, x_{14}) for x_{12} in (3.12). Add the resulting identities, and use (2.1) and (3.7) to obtain

$$(3.13) \quad 0 = T(7, 0, 2) \sum_{i=9}^{11} (x_8, \dots, (x_{i+1}, x_{i+2}, x_{i+3}), \dots, x_{14}) .$$

Applying (2.1) and (3.7) to (3.13) and then using (2.3), we get, after subtracting the resulting identity from (3.13) with subscripts relabeled,

$$(3.14) \quad 0 = T(7, 0, 2) (x_8, x_9, (x_{10}, x_{11}, x_{12}), x_{13}, x_{14}) .$$

If we apply (3.4) to (3.14), then (2.1) with $n = 5$ and (3.7), we obtain $0 = T(9, 0, 2) ((x_{10}, x_{11}, x_{12}), x_{13}, x_{14})$ which yields

$$(3.15) \quad 0 = T(9, 0, 2) S(3, 5)$$

upon using (2.3) and then (2.1) and (3.7).

Application of (3.4) to (3.15); then use of (2.1) and (3.7) followed by (2.3) yields $T(7, 0, 2) S(5, 7) = 0$. Hence using (2.1) and (3.7) we compute $0 = (T(7, 0, 4) - T(7, 0, 2)) S(5, 7) = T(7, 3, 4) S(5, 7)$ which yields

$$(3.16) \quad 0 = T(7, 0, 1) S(5, 7) .$$

Computing $0 = (T(7, 0, 2) - T(7, 0, 1)) S(5, 7)$, we obtain

$$(3.17) \quad 0 = T(7, 2, 2) S(5, 7) .$$

Returning to (3.15), we can assume that $T(9, 0, 2) = 0$, for otherwise $S(7, 7) = 0$ by Lemma 2.3 and Lemma 2.1. Hence we have $T(9, 4, 6) = 0$. Computing $0 = T(9, 0, 6) - T(9, 0, 2) - T(9, 4, 6)$, we obtain $T(9, 3, 3) = 0$ which we apply to (3.4) to get

$$(3.18) \quad 0 = 2(x_1, x_2, x_3) c + 3T(7, 3, 3) (s_1, \dots, s_5)$$

where $c = ((x_4, x_5, x_6), x_7, s_1, \dots, s_5)$ and where at least one of s_1, s_2, s_3, s_4 , or $s_5 \in S(3, 3)$. Let $x_3 = z \in S(3, 3)$ in (3.18) and use (3.17) to obtain $0 = (x_1, x_2, z)c$, to which we apply (2.3) and then (2.1) and (3.7) to get $S(3, 5) c = 0$. Since $c \in S(3, 5)$ by Lemma 2.1, Lemma 2.3 implies that $c = 0$. Hence (3.18) yields $T(7, 3, 3) S(5, 7) = 0$, and therefore $T(7, 1, 1) S(5, 7) = 0$. Now, recalling (3.16), we compute $0 = (T(7, 0, 1) - T(7, 1, 1)) S(5, 7)$ to obtain $T(7, 0, 0) S(5, 7) = 0$, but then $T(7, 4, 4) S(5, 7) = 0$, and we have established (3.11).

Equations (3.4) and (3.11) yield

$$0 = ((x_4, x_5, x_6), x_7, x_8) (x_9, x_{10}, s_1, s_2, x_1, x_2, x_3)$$

where s_1 or $s_2 \in S(3, 3)$ since $c = 0$, to which we apply (2.3) and then (2.1) and (3.7) to obtain $0 = (x_9, x_{10}, s_1, s_2, x_1, x_2, x_3) S(3, 5)$ which in turn, using (3.4) and (3.11), yields $0 = (x_9, x_{10}, s) T(9, i, i)$ for $i = 4, 5$, and 6 and where $s \in S(3, 3)$. Thus, by (2.3) and then (2.1) and (3.7), we have $S(3, 5) T(9, i, i) = 0$ for $i = 4, 5$, and 6. Therefore, we have $S(3, 5) S(7, 9) = 0$ since $T(9, 3, 3) = 0$. Lemmas 2.3 and 2.1 then imply that

$$(3.19) \quad 0 = S(7, 9) .$$

Equations (3.4) and (3.19) imply $0 = S(5, 5) T(7, i, i)$, $i = 0, 1$, which yields

$$(3.20) \quad 0 = S(5, 5) T(7, i, i), \quad i = 0, 1, 3, 4 .$$

Using (2.3), (3.4), and (3.19), we compute

$$3T(7, 2, 2) S(5, 5) \subset -2T(5, 2, 2) S(7, 7) \subset 2T(5, 0, 0) S(7, 7) = 0 ,$$

the equality by (3.4), (3.19), and (3.20). Hence

$$(3.21) \quad S(5, 7) S(5, 5) = 0 .$$

Finally (3.4), (3.19), and (3.21) imply that $S(3, 5) S(7, 7) = 0$. Lemma 2.3 and Lemma 2.1 then imply $S(7, 7) = 0$, and we are finished.

Theorems 3.2, 3.3, and 3.4 imply

COROLLARY 3.2. *If R is a simple, commutative 10-associative ring of characteristic not 2, 3, or 5, or if R is a simple, commutative 8-associative ring of characteristic 5, then R is associative or $C = 0$ where C is the center of R .*

4. Rings which are n -associative of degree $2k + 1$. An immediate corollary of Lemma 2.1 is

LEMMA 4.1. *If R is $(2m + 1)$ -associative of degree $2n + 1$, then R is $(2k + 1)$ -associative for all k such that $m \leq k$ and $n \leq k$.*

The converse of Lemma 4.1 is false as can be seen by the following example. Let A be the 13-dimensional commutative algebra with basis u_1, u_2, \dots, u_{13} satisfying $u_1^2 = u_4$, $u_1u_2 = u_3$, $u_1u_3 = u_8$, $u_1u_5 = u_7$, $u_1u_7 = u_{10}$, $u_1u_9 = u_{11}$, $u_2^2 = u_5$, $u_2u_3 = u_9$, $u_2u_4 = u_6$, $u_2u_8 = -u_{10} + u_{11}$, $u_2u_{10} = u_{12}$, $u_2u_{11} = u_{12} - u_{13}$, the commutative law, and all other products zero. It can be verified that A is 5-associative, but that $((u_1, u_1, u_2), u_2, u_2) = u_{13} \neq 0$, and hence A is not associative of degree 5.

THEOREM 4.1. *If R is a simple, commutative ring of characteristic not a prime $\leq k$ which is associative of degree $2k + 1$, then R is associative.*

Proof. Assume that $k > 1$. We have $S(3, 2k + 1) = 0$. We will show that this implies that $S(3, 2k - 1) = 0$, from which the proof is completed by an obvious induction.

Applying Lemma 2.1 to $S(3, 2k + 1) = 0$, we have

$$(4.1) \quad 0 = S(2j + 1, 2m + 1) \text{ for } j \geq 1 \text{ and } m \geq k.$$

Let $n = 2k + 2$ in (2.2). Then by (4.1) with $i = m = k$, we have

$$k(a_1, a_2, a_3)(a_4, \dots, a_{2k+2}) = - \sum_{i=1}^{k-2} \binom{k}{i} (a_1, \dots, a_{2k-2i+1})(a_{2k-2i+2}, \dots, a_{2k+2}),$$

to which we apply (4.1) with $j = k - i$ and $m = 2k - i - 2$, $i = 1, \dots, k - 2$, to obtain $0 = S(3, 2k - 1)S(2k - 1, 2k - 1)$. Therefore, by Lemma 2.3, $S(3, 2k - 1) = 0$ or $S(2k - 1, 2k - 1) = 0$. Assume that we have shown that $S(3, 2k - 1) = 0$ or $S(2k - 2j + 1, 2k - 2j + 1) = 0$. Assume that

$$(4.2) \quad 0 = S(2k - 2j + 1, 2k - 2j + 1).$$

Let $n = 2k - 2j + 2$ in (2.2). Then, as above, we apply (4.2) and (4.1) with $m = 2k - j - i - 2$, $i = 1, \dots, k - j - 2$, to obtain $0 =$

$S(3, 2k - 1) S(2k - 2j - 1, 2k - 2j - 1)$. Therefore, as before, $S(3, 2k - 1) = 0$ or $S(2k - 2j - 1, 2k - 2j - 1) = 0$. Hence, $S(3, 2k - 1) = 0$ or $S(3, 3) = 0$, and we are finished.

In view of Lemma 4.1, Theorem 4.1 is an extension of the results of §3 for a more restricted class of rings.

Finally, define $S(3, 2k + 1)^n = S(3, 2k + 1)^{n-1} S(3, 2k + 1)$, $n > 1$. We have

COROLLARY. *If R is a simple, commutative ring of characteristic not a prime $\leq k$ in which $S(3, 2k + 1)^n = 0$ for some n , then R is associative.*

Proof. Because of Theorem 4.1, we need only show that $S(3, 2k + 1) = 0$ in R . Assume $n > 1$. Then $0 = S(3, 2k + 1)^{n-1} S(3, 2k + 1)$. Lemma 2.3 and an easy induction yield $S(3, 2k + 1) = 0$.

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