SIMPLE $n$-ASSOCIATIVE RINGS

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This paper is concerned with certain classes of nonassociative rings. These rings are defined by first extending the associator \((a, b, c) = (ab)c - a(bc)\). The \(n\)-associator \((a_1, \cdots, a_n)\) is defined by

\[
\begin{align*}
(a_1, a_2) &= a_1a_2, \\
(a_1, \cdots, a_n) &= \sum_{k=0}^{n-2} (-1)^k (a_1, \cdots, a_k, a_{k+1}a_{k+2}, \cdots, a_n).
\end{align*}
\]

A ring is defined to be \(n\)-associative if the \(n\)-associator vanishes in the ring. It is shown that simple 4-associative and simple 5-associative rings are associative; simple \(2k\)-associative rings are \((2k-1)\) associative or have zero center; and simple, commutative \(n\)-associative rings, \(6 \leq n \leq 9\), are associative. The concept of rings which are associative of degree \(2k + 1\) is defined, and it is shown that simple, commutative rings which are associative of degree \(2k + 1\) are associative. The characteristic of the ring is slightly restricted in all but one of these results.

The concepts of the \(n\)-associator and \(n\)-associative rings were defined by A. H. Boers [1; Ch. 3 and Ch. 4]. Our results extend Boers’ main result that an \(n\)-associative division ring is associative with minor restriction on the characteristic [1; Th. 6]. We do not consider 2-associative rings.

To obtain our results, it is necessary to extend the concept of the \(n\)-associator. In a ring \(R\), define \(S(2j + 1, 2k + 1)\), \(1 \leq j \leq k\), by defining \(S(2j + 1, 2j + 1)\) to be the set of all finite sums of \((2j + 1)\)-associators with entries in \(R\), and then by defining \(S(2j + 1, 2k + 1)\), \(k > j\), to be the set of all finite sums of \((2j + 1)\)-associators \((a_1, \cdots, a_{2j+1})\) such that \((a_1, \cdots, a_{2j+1}) \in S(2j + 1, 2k - 1)\) and such that at least one of the \(2k - 1\) entries of \((a_1, \cdots, a_{2j+1})\) is in \(S(3, 3)\). For example, \(((a_1, a_2, a_3), a_4, a_5, (a_7, a_8, a_9)) \in S(3, 9)\).

Clearly, a ring \(R\) is \((2n + 1)\)-associative if and only if \(S(2n + 1, 2n + 1) = 0\) in \(R\). This leads us to call a ring \(R(2n + 1)\)-associative of degree \(2k + 1\) if \(S(2n + 1, 2k + 1) = 0\) in \(R\). No mention of degree will be made in case \(k = n\).

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Of particular interest are rings which are associative (3-associative) of degree $2k + 1$. In the first place, this in itself is an interesting extension of the concept of associativity. Consider the 4-dimensional algebra $A$ over an arbitrary field with basis $a_1, a_2, a_3, a_4$ such that $a_i^2 = a_2$, $a_ia_2 = a_3 - a_4$, $a_2a_1 = a_3$, and all other products zero. It can be verified that $S(3, 5) = 0$ in $A$ but that $A$ is not associative. Also, it turns out that a ring which is associative of degree $2k + 1$ is $(2k + 1)$-associative, but not conversely.

2. Preliminaries. We will need the following three identities derived by Boers.

(2.1) \[ (a_1, \ldots, a_n) = \sum_{k=1}^{n-3} (a_1, \ldots, a_k, (a_{k+1}, a_{k+2}, a_{k+3}), a_{k+4}, \ldots, a_n), \]

(2.2) \[ (a_1, \ldots, a_n) = \sum_{k=0}^{\lfloor (1/2)n - 1 \rfloor} \binom{k}{n-2k-1} (a_{n-2k}, \ldots, a_n) \]

for even $n$ where $\binom{r}{s}$ denotes the binomial coefficient [1; Ch. 3]. In a commutative ring, we have

(2.3) \[ (a_1, \ldots, a_n) = (-1)^{\left\lfloor \frac{1}{2}n - \frac{1}{2} \right\rfloor} (a_n, \ldots, a_1) \]

where $[x]$ denotes the greatest integer $\leq x$ [2; Th. A].

Next, we will need

**Lemma 2.1.** $S(2j + 1, 2k + 1) \subseteq S(2m + 1, 2n + 1), 1 \leq m \leq j, m \leq n \leq k$.

**Proof.** It is immediate from the definition of $S(2j + 1, 2k + 1)$ that $S(2j + 1, 2k + 1) \subseteq S(2j + 1, 2n + 1), j \leq n \leq k$. Hence we need only show that $S(2j + 1, 2n + 1) \subseteq S(2m + 1, 2n + 1), 1 \leq m \leq j$. The result is obvious if $j = m$. Assume that $S(2j - 1, 2n + 1) \subseteq S(2m + 1, 2n + 1), j > m$. Then by (2.1), $S(2j + 1, 2n + 1) \subseteq S(2j - 1, 2n + 1) \subseteq S(2m + 1, 2n + 1)$, and we are finished.

Let $A$ and $B$ be subsets of a ring. Define $AB$ to be the set of all finite sums of elements of the form $ab$ such that $a \in A$, $b \in B$.

Let $I(3, 2k + 1) = S(3, 2k + 1) + S(3, 2k + 1)R$. The next lemma is a generalization of the fact that $I(3, 3)$ is an ideal of an arbitrary ring $R$ [4; p. 985].

**Lemma 2.2.** $I(3, 2k + 1)$ is a right ideal of an arbitrary ring $R$ for $k = 1, 2, \ldots$.

**Proof.** Let $I = I(3, 2k + 1), S = S(3, 2k + 1)$. We have
The key to our results is

**Lemma 2.3.** If \( R \) is a simple, commutative ring, and if \( A = \{ a \in R \mid a S(3, 2k + 1) = 0 \} \), then \( A = 0 \) or \( S(3, 2k + 1) = 0 \).

**Proof.** We can assume that \( R = I(3, 2k + 1) \), for otherwise we are finished by Lemma 2.2. If \( K \) is the ideal generated by \( AR \), then \( K \subseteq S(3, 3) \). Indeed, \( AR = AI(3, 2k + 1) \subseteq S(3, 3) \). Define \( R_x : y \rightarrow yx \).

Assume that we have shown that \( aR_{z_1} \cdots R_{z_n} \in S(3, 3) \) for \( a \in A \) and for every choice of \( n \) elements \( x_1, \ldots, x_n \in R, n > 1 \). Then

\[
aR_{z_1} \cdots R_{z_n} R_{z_{n+1}} = ((aR_{z_1} \cdots R_{z_{n-1}})x_n)x_{n+1}
\]

\[
= (aR_{z_1} \cdots R_{z_{n-1}}, x_n, x_{n+1}) + aR_{z_1} \cdots R_{z_{n-1}}R_{x_{n}x_{n+1}} \in S(3, 3).
\]

If \( K = 0 \), then \( AR = 0 \) which implies \( A = 0 \) and we are finished. Hence assume \( K = R \). Thus \( R = S(3, 3) \) which implies \( R = S(3, 2k + 1) \) by induction since each element can now be replaced by a sum of associators. Therefore \( AR = 0 \), and hence \( A = 0 \).

To ease the computations of the proofs which follow, we define

\[
T(n, i, j) = \sum_{m=i}^{j} (a_1, \ldots, a_m, (a_{m+1}, a_{m+2}, a_{m+3}), a_{m+4}, \ldots, a_n)
\]

where \( 0 \leq i \leq j \leq n - 3 \). Note that (2.1) becomes \( (a_1, \ldots, a_n) = T(n, 0, n - 3) \). The next lemma, whose proof follows easily from the definition of \( T(n, i, j) \) (with the use of (2.3) in the case of part (d)), contains all the additional facts about \( T(n, i, j) \) that we will need.

**Lemma 2.4.**

(a) \( T(n, i, j) - T(n, i, k) = T(n, k + 1, j), k < j \);

(b) \( T(n, i, j) - T(n, k, j) = T(n, i, k - 1), i < k \);

(c) \( S(2k + 1, 2k + 1) \) consists of all finite sums of elements of the form \( T(2k + 3, i, i), i = 0, 1, \ldots, 2k \); and

(d) if \( 0 = T(n, i, j)a \) is an identity in a commutative ring, then so is \( 0 = T(n, n - 3 - j, n - 3 - i)a \).

We will not cite Lemma 2.4 when we use it.

Finally, the nucleus \( N \) of a ring \( R \) is defined by \( N = \{ u \in R \mid (u, x, y) = (x, u, y) = (x, y, u) = 0 \quad \text{for all} \ x, y \in R \} \). The center \( C \) of \( R \) is defined by \( C = \{ c \in N \mid cx = xc \quad \text{for all} \ x \in R \} \).
3. \textit{n}-Associative rings. In what follows, we will use the fact that (1.1) implies that if \( R \) is an \( n \)-associative ring, then \( R \) is \( k \)-associative for all \( k \geq n \).

\textbf{Theorem 3.1.} \textit{If} \( R \) \textit{is a simple} \( 4 \)-associative or \( 5 \)-associative ring \textit{of characteristic not 2}, \textit{then} \( R \) \textit{is associative}.

\textit{Proof.} By (2.2) with \( n = 6 \), we have

\[(3.1) \quad 0 = S(3, 3)^2.\]

Since \( I(3, 3) \) is an ideal of \( R \), we can assume that \( R = I(3, 3) \), for otherwise \( S(3, 3) = 0 \). Hence (3.1) yields \( S(3, 3)R \subseteq S(3, 3) \). Therefore \( R = S(3, 3) \), but then we have \( S(3, 3)R = 0 = RS(3, 3) \) by (3.1). Thus \( S(3, 3) = 0 \).

Theorem 3.1 extends a result of Boers [3; p. 126] who has also shown it to be false for characteristic 2.

\textbf{Theorem 3.2.} \textit{Let} \( R \) \textit{be a simple} \( 2k \)-associative ring \textit{for} \( k \geq 2 \). \textit{Then} \( C = 0 \) \textit{or} \( R \) \textit{is} \( (2k - 1) \)-associative where \( C \) \textit{is the center of} \( R \).

\textit{Proof.} We first show that if \( N \) is the nucleus of \( R \), then any \( (2j + 1) \)-associator with an entry \( u \in N \) vanishes. Indeed, if \( j = 1 \), the result follows by the definition of \( N \). Assume that we have established the result for \( j = i \). By (2.1), \( (a_1, \ldots, a_{2i+3}) = T(2i+3, 0, 2i) \). Each \( n \)-associator in \( T(2i+3, 0, 2i) \) is a \( (2i + 1) \)-associator. Hence if \( u \in N \) is an entry of \( (a_1, \ldots, a_{2i+3}) \), then \( (a_1, \ldots, a_{2i+3}) = 0 \).

Now, use (2.2) with \( n = 2k \). In the resulting identity, let \( a_{2k} \in C \). Then, since \( C \subseteq N \), we have \( S(2k - 1, 2k - 1)C = 0 \). Therefore \( C = 0 \) or \( S(2k - 1, 2k - 1) = 0 \) since \( R \) is simple and the annihilators of \( C \) may be shown to form an ideal of \( R \).

Theorems 3.1 and 3.2 imply

\textbf{Corollary 3.1.} \textit{If} \( R \) \textit{is a simple} \( 6 \)-associative ring \textit{of characteristic not 2}, \textit{then} \( C = 0 \) \textit{or} \( R \) \textit{is associative}.

We now turn our attention to commutative rings.

\textbf{Theorem 3.3.} \textit{If} \( R \) \textit{is a simple, commutative} \( 6 \)-associative or \( 7 \)-associative ring \textit{of characteristic not 2 or 3}, \textit{then} \( R \) \textit{is associative}.

\textit{Proof.} By Theorem 3.1, it is sufficient to show that \( S(5, 5) = 0 \).
Let \( n = 8 \) and \( 10 \) in (2.2) to obtain

\[
(3.2) \quad 0 = (a_1, \cdots, a_5) (a_6, a_7, a_8) + (a_1, a_2, a_3) (a_4, \cdots, a_8)
\]

and

\[
(3.3) \quad 0 = S(5, 5)^2.
\]

Next, use (2.1) and (3.3) to obtain \( T(5, 0, 2) S(5, 5) = 0 \) which yields \( T(7, 0, 2) S(3, 3) = 0 \) upon application of (3.2). Hence we may assume that \( T(7, 0, 2) = 0 \), for otherwise \( S(3, 3) = 0 \) by Lemma 2.3. Using (2.1), we compute

\[
0 = T(7, 0, 4) - T(7, 0, 2) = T(7, 3, 4);
\]

hence \( T(7, 0, 1) = 0 \) by (2.3). Thus \( T(7, 2, 2) = 0 \) since \( T(7, 2, 2) = T(7, 0, 2) - T(2, 0, 1) \). Hence we have \( T(5, 2, 2) S(5, 5) = 0 \) using (3.2). Thus \( S(3,5)S(5,5) = 0 \) upon using (2.3), (2.1) with \( n = 5 \), and (3.3), in that order. Application of Lemma 2.3 and then Lemma 2.1 completes the proof.

**Theorem 3.4.** If \( R \) is a simple, commutative 8-associative or 9-associative ring of characteristic not 2, 3, or 5, then \( R \) is associative.

**Proof.** By Theorem 3.3, it is sufficient to show that \( S(7, 7) = 0 \).

If we let \( n = 10, 12, \) and \( 14 \) in (2.2), we get

\[
(3.4) \quad 0 = 2(a_1, \cdots, a_7) (a_8, a_9, a_{10}) + 3(a_1, \cdots, a_8) (a_9, \cdots a_{10})
\]

\[
+ 2(a_1, a_2, a_3) (a_4, \cdots, a_{10}),
\]

\[
(3.5) \quad 0 = (a_1, \cdots, a_7) (a_8, \cdots, a_{12}) + (a_1, \cdots, a_8) (a_9, \cdots, a_{12}),
\]

and

\[
(3.6) \quad 0 = S(7, 7)^2.
\]

Our first goal is to establish

\[
(3.7) \quad S(7, 9)S(5, 5) = 0 = S(7, 7)S(5, 7).
\]

Applying (2.1) to (3.6) yields \( T(7, 0, 4) S(7, 7) = 0 \), to which we apply (3.5) to obtain \( T(9, 0, 4) S(5, 5) = 0 \). Using (2.1), we compute

\[
0 = (T(9, 0, 6) - T(9, 0, 4)) S(5, 5) = T(9, 5, 6) S(5, 5)
\]

which implies \( T(9, 0, 1) S(5, 5) = 0 \). Then

\[
0 = (T(9, 0, 4) - T(9, 0, 1)) S(5, 5) = T(9, 2, 4) S(5, 5),
\]

to which we apply (3.5) to obtain \( T(7, 2, 4) S(7, 7) = 0 \). Hence we have \( T(7, 0, 2) S(7, 7) = 0 \). Using (3.5) again yields \( T(9, 0, 2) S(5, 5) = 0 \). Thus we can compute

\[
0 = (T(9, 0, 2) - T(9, 0, 1)) S(5, 5)
\]

to obtain

\[
(3.8) \quad T(9, 2, 2) S(5, 5) = 0 = T(9, 4, 4) S(5, 5).
\]

Computing \( 0 = (T(9, 2, 4) - T(9, 2, 2) - T(9, 4, 4)) S(5, 5) \) yields
Applying (3.5) to (3.9), we have $T(7, 3, 3) S(7, 7) = 0$, from which we obtain $T(7, 1, 1) S(7, 7) = 0$, and hence (3.5) implies

(3.10) \[ 0 = T(9, 1, 1) S(5, 5) = 0. \]

Computing $0 = (T(9, 0, 1) - T(9, 1, 1)) S(5, 5)$ yields $T(9, 0, 0) S(5, 5) = 0$, which, along with (3.8), (3.9), and (3.10), implies (3.7) after using (3.5).

Our next goal is to establish

(3.11) \[ 0 = S(5, 7). \]

Let $a_i = (x_1, x_2, x_3), a_i = x_{i+2}, \ i > 1$ in (3.4); then let $a_i = x_1, a_i = (x_2, x_3, x_4), a_i = x_{i+2}, \ i > 2$ in (3.4); and then let $a_i = x_1, a_i = x_2, a_i = (x_3, x_4, x_5), a_i = x_{i+2}, \ i > 3$ in (3.4). Add the resulting identities and apply (2.1) to obtain

(3.12) \[ 0 = 2T(9, 0, 2) (x_{10}, x_{11}, x_{12}) + 3T(7, 0, 2) (x_8, \ldots, x_{12}) \\
+ 2(x_1, \ldots, x_5) (x_6, \ldots, x_8) \]

where the $T$'s are now written in terms of the $x_i's$. Substitute $(x_{10}, x_{11}, x_{12})$ for $x_{10}, x_{11}, x_{12}$ in (3.12); then substitute $(x_{11}, x_{12}, x_{13})$ for $x_{11}$ and $x_{14}$ for $x_{12}$ in (3.12); and then substitute $(x_{12}, x_{13}, x_{14})$ for $x_{12}$ in (3.12). Add the resulting identities, and use (2.1) and (3.7) to obtain

(3.13) \[ 0 = T(7, 0, 2) \sum_{i=9}^{11} (x_8, \ldots, x_{i+1}, x_{i+2}, x_{i+3}, \ldots, x_{14}) \]

Applying (2.1) and (3.7) to (3.13) and then using (2.3), we get, after subtracting the resulting identity from (3.13) with subscripts relabeled,

(3.14) \[ 0 = T(7, 0, 2) (x_8, x_9, (x_{10}, x_{11}, x_{12}), x_{13}, x_{14}) \]

If we apply (3.4) to (3.14), then (2.1) with $n = 5$ and (3.7), we obtain

(3.15) \[ 0 = T(9, 0, 2) S(3, 5) \]

upon using (2.3) and then (2.1) and (3.7).

Application of (3.4) to (3.15); then use of (2.1) and (3.7) followed by (2.3) yields $T(7, 0, 2) S(5, 7) = 0$. Hence using (2.1) and (3.7) we compute $0 = (T(7, 0, 4) - T(7, 0, 2)) S(5, 7) \equiv T(7, 3, 4) S(5, 7) \ \text{which yields}$

(3.16) \[ 0 = T(7, 0, 1) S(5, 7) \].
Computing $0 = (T(7, 0, 2) - T(7, 0, 1)) S(5, 7)$, we obtain

\begin{equation}
0 = T(7, 2, 2) S(5, 7) .
\end{equation}

Returning to (3.15), we can assume that $T(9, 0, 2) = 0$, for otherwise $S(7, 7) = 0$ by Lemma 2.3 and Lemma 2.1. Hence we have $T(9, 4, 6) = 0$. Computing $0 = T(9, 0, 6) - T(9, 0, 2) - T(9, 4, 6)$, we obtain $T(9, 3, 3) = 0$ which we apply to (3.4) to get

\begin{equation}
0 = 2(x_1, x_2, x_3) c + 3T(7, 3, 3) (s_1, \ldots, s_5)
\end{equation}

where $c = ((x_4, x_5, x_6), x_7, s_1, \ldots, s_5)$ and where at least one of $s_1, s_2, s_3, s_4$, or $s_5 \in S(3, 3)$. Let $x_3 = z \in S(3, 3)$ in (3.18) and use (3.17) to obtain $0 = (x_1, x_2, z)c$, to which we apply (2.3) and then (2.1) and (3.7) to get $S(3, 5) c = 0$. Since $c \in S(3, 5)$ by Lemma 2.1, Lemma 2.3 implies that $c = 0$. Hence (3.18) yields $T(7, 3, 3) S(5, 7) = 0$, and therefore $T(7, 1, 1) S(5, 7) = 0$. Now, recalling (3.16), we compute $0 = (T(7, 0, 1) - T(7, 1, 1)) S(5, 7)$ to obtain $T(7, 0, 0) S(5, 7) = 0$, but then $T(7, 4, 4) S(5, 7) = 0$, and we have established (3.11).

Equations (3.4) and (3.11) yield

$$
0 = ((x_4, x_5, x_6), x_7, x_8) (x_9, x_{10}, s_1, s_2, x_1, x_2, x_3)
$$

where $s_1$, or $s_2 \in S(3, 3)$ since $c = 0$, to which we apply (2.3) and then (2.1) and (3.7) to obtain $0 = (x_9, x_{10}, s_1, s_2, x_1, x_2, x_3) S(3, 5)$ which in turn, using (3.4) and (3.11), yields $0 = (x_9, x_{10}, s) T(9, i, i)$ for $i = 4, 5,$ and $6$ and where $s \in S(3, 3)$. Thus, by (2.3) and then (2.1) and (3.7), we have $S(3, 5) T(9, i, i) = 0$ for $i = 4, 5,$ and $6$. Therefore, we have $S(3, 5) S(7, 9) = 0$ since $T(9, 3, 3) = 0$. Lemmas 2.3 and 2.1 then imply that

\begin{equation}
0 = S(7, 9) .
\end{equation}

Equations (3.4) and (3.19) imply $0 = S(5, 5) T(7, i, i)$, $i = 0, 1$, which yields

\begin{equation}
0 = S(5, 5) T(7, i, i), \quad i = 0, 1, 3, 4 .
\end{equation}

Using (2.3), (3.4), and (3.19), we compute

$$
3T(7, 2, 2) S(5, 5) \subset -2T(5, 2, 2) S(7, 7) \subset 2T(5, 0, 0) S(7, 7) = 0 ,
$$

the equality by (3.4), (3.19), and (3.20). Hence

\begin{equation}
S(5, 7) S(5, 5) = 0 .
\end{equation}

Finally (3.4), (3.19), and (3.21) imply that $S(3, 5) S(7, 7) = 0$. Lemma 2.3 and Lemma 2.1 then imply $S(7, 7) = 0$, and we are finished.

Theorems 3.2, 3.3, and 3.4 imply...
COROLLARY 3.2. If $R$ is a simple, commutative 10-associative ring of characteristic not 2, 3, or 5, or if $R$ is a simple, commutative 8-associative ring of characteristic 5, then $R$ is associative or $C = 0$ where $C$ is the center of $R$.

4. Rings which are $n$-associative of degree $2k + 1$. An immediate corollary of Lemma 2.1 is

LEMMA 4.1. If $R$ is $(2m + 1)$-associative of degree $2n + 1$, then $R$ is $(2k + 1)$-associative for all $k$ such that $m \leq k$ and $n \leq k$.

The converse of Lemma 4.1 is false as can be seen by the following example. Let $A$ be the 13-dimensional commutative algebra with basis $u_1, u_2, \ldots, u_{13}$ satisfying $u_1^2 = u_4$, $u_1u_2 = u_3$, $u_1u_3 = u_8$, $u_1u_5 = u_7$, $u_1u_6 = u_9$, $u_9u_2 = u_14$, $u_9u_3 = u_9$, $u_9u_4 = u_6$, $u_9u_8 = - u_{10} + u_{11}$, $u_9u_{10} = u_{12}$, $u_9u_{11} = u_{12} - u_{13}$, the commutative law, and all other products zero. It can be verified that $A$ is 5-associative, but that $((u_1, u_2, u_2), u_2) = u_{13} \neq 0$, and hence $A$ is not associative of degree 5.

THEOREM 4.1. If $R$ is a simple, commutative ring of characteristic not a prime $\leq k$ which is associative of degree $2k + 1$, then $R$ is associative.

Proof. Assume that $k > 1$. We have $S(3, 2k + 1) = 0$. We will show that this implies that $S(3, 2k - 1) = 0$, from which the proof is completed by an obvious induction.

Applying Lemma 2.1 to $S(3, 2k + 1) = 0$, we have

(4.1) $0 = S(2j + 1, 2m + 1)$ for $j \geq 1$ and $m \geq k$.

Let $n = 2k + 2$ in (2.2). Then by (4.1) with $i = m = k$, we have

$k(a_1, a_2, a_3)(a_4, \ldots, a_{2k+2}) = - \sum_{i=1}^{k-2} \binom{k}{i} (a_1, \ldots, a_{2k-2i+1})(a_{2k-2i+2}, \ldots, a_{2k+2}),$

to which we apply (4.1) with $j = k - i$ and $m = 2k - i - 2$, $i = 1, \ldots, k - 2$, to obtain $0 = S(3, 2k - 1)S(2k - 1, 2k - 1)$. Therefore, by Lemma 2.3, $S(3, 2k - 1) = 0$ or $S(2k - 1, 2k - 1) = 0$. Assume that we have shown that $S(3, 2k - 1) = 0$ or $S(2k - 2j + 1, 2k - 2j + 1) = 0$. Assume that

(4.2) $0 = S(2k - 2j + 1, 2k - 2j + 1)$.

Let $n = 2k - 2j + 2$ in (2.2). Then, as above, we apply (4.2) and (4.1) with $m = 2k - j - i - 2$, $i = 1, \ldots, k - j - 2$, to obtain $0 =$
S(3, 2k - 1) S(2k - 2j - 1, 2k - 2j - 1). Therefore, as before, 
S(3, 2k - 1) = 0 or S(2k - 2j - 1, 2k - 2j - 1) = 0. Hence, S(3, 2k - 1) = 
0 or S(3, 3) = 0, and we are finished.

In view of Lemma 4.1, Theorem 4.1 is an extension of the results 
of §3 for a more restricted class of rings.

Finally, define \( S(3, 2k + 1)^n = S(3, 2k + 1)^{n-1} S(3, 2k + 1), \ n > 1. \) 
We have

**Corollary.** If \( R \) is a simple, commutative ring of characteristic 
not a prime \( \leq k \) in which \( S(3, 2k + 1)^n = 0 \) for some \( n \), then \( R \) is 
associative.

**Proof.** Because of Theorem 4.1, we need only show that 
\( S(3, 2k + 1) = 0 \) in \( R \). Assume \( n > 1 \). Then 
\( 0 = S(3, 2k + 1)^{n-1} S(3, 2k + 1) \). 
Lemma 2.3 and an easy induction yield 
\( S(3, 2k + 1) = 0. \)

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