

Pacific Journal of Mathematics

SOME NEW RESULTS ON SIMPLE ALGEBRAS

DAVID JOSEPH RODABAUGH

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D. J. RODABAUGH

This paper deals with the problem of proving that a simple algebra (finite dimensional) has an identity element. The main result is contained in the following theorem. Let A be a simple algebra (char. $\neq 2$) in which $(x, x, x) = 0$ and $x^3 \cdot x = x^2 \cdot x^2$. If M is a subset of A such that $(A, M, A) = 0$ and $(M, A, A) \cup (M, A) \cup (A, A, M) \subseteq M$, then $M = 0$ or there is an identity element in A . This result is then used to prove the three following corollaries (char. $\neq 2$): (1) A simple power associative algebra with all commutators in the nucleus has an identity; (2) A simple power associative algebra with all associators in the middle center has an identity; (3) A simple antiflexible algebra in which $(x, x, x) = 0$ and A^+ is not nil has an identity.

For convenience in terminology, we define an algebra as a finite dimensional vector space on which a multiplication is defined that satisfies both distributive laws. An algebra is nilpotent if there is an integer k such that any product of k elements, no matter how associated, is zero. An element x in an algebra is nilpotent if the subalgebra generated by x is nilpotent. An algebra is nil if it consists entirely of nilpotent elements. A simple algebra is an algebra without proper ideals that is not nil. For char. $\neq 2$, define $x \cdot y = 1/2(xy + yx)$. The algebra A^+ is defined to be the vector space A with multiplication $x \cdot y$. In addition, define the commutator $(x, y) = xy - yx$ and the associator $(x, y, z) = (xy)z - x(yz)$. Using techniques similar to [3] we will prove the following theorem.

THEOREM 1. *If B is a subspace of an algebra A , there cannot be a nil subset M proper in B with:*

- (a) $B = M + MB$
- (b) M a subalgebra
- (c) $(M, M, B) = 0$.

Define $x^1 = x$ and for $k > 1$ define $x^k = x^{k-1}x$. Using this theorem, the following theorem is proved.

THEOREM 2. *Let A be a simple algebra (char. $\neq 2$) in which*

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$(x, x, x) = 0$ and $x^3 \cdot x = x^2 \cdot x^2$. If M is a subset of A such that

- (a) $(M, A, A) \subseteq M$
- (b) $(M, A) \subseteq M$
- (c) $(A, A, M) \subseteq M$
- (d) $(A, M, A) = 0$

then $M = 0$ or there is an identity element in A .

It should be remarked that, if (d) could be replaced by $(A, M, A) \subseteq M$, then $M = A$ would satisfy (a), (b), (c) and (d) in any algebra.

Many simple algebras are known to have identities. If the center (the set of x such that $(x, y) = (x, y, z) = (y, z, x) = 0$ for all y and z in A) of a simple algebra is not zero, the algebra has an identity element [4]. In addition, there are many identities which, if satisfied by a simple algebra, force that algebra to have an identity element.

In § 4, Theorem 2 will be used to prove the existence of an identity element in three classes of algebras.

The author would like to thank the referee for suggesting conditions (a), (b), (c) and (d) in a more general form than was first submitted.

2. Proof of the first theorem. Let us assume that M is a subset proper in B that satisfies (a), (b) and (c) of Theorem 1. We will show that M is not nil. From (c), it is clear that M is an associative subalgebra even though A is not even assumed to be power associative. The following fact is easily obtained by induction. In an associative algebra such as M , $\prod_{i=1}^n a_i = a_1 \cdots a_n$.

LEMMA 2.1. *If a_1, \dots, a_n are in M and if x is in B then $(\prod_{i=1}^n a_i)x = a_1(a_2 \cdots (a_n x) \cdots)$.*

LEMMA 2.2. *There exists an $x \neq 0$ in B and an a in M with $x = ax$.*

Proof. For any x in $B \cap M'$ (set theoretic complement), there is a set of elements a_1, \dots, a_n, b in M and a set y_1, \dots, y_n in $B \cap M'$ with $x = b + \sum_{i=1}^n a_i y_i$. Throughout this proof we will use a, b with subscripts to indicate members of M and y with subscripts to indicate members of $B \cap M'$. Since M is proper in B , we have $B \cap M' \neq \emptyset$. Let y_0 be in $B \cap M'$.

Clearly, $y = \sum_{i=1}^n a_i y_{i1} + b_1$. Since y_0 is not in M , some j exists for which $a_{1j} y_{1j}$ is not in M . Define $y_1 = y_{1j}$ and $a_1 = a_{1j}$. Suppose

$a_1, \dots, a_k, y_1, \dots, y_k$ have been defined for all $k < m$ with $a_1(a_2 \dots (a_k y_k) \dots)$ in $B \cap M'$. Since y_{m-1} is not in M , $y_{m-1} = \sum_{i=1}^n a_{mi} y_{mi} + b_m$. There must be some j for which $a_1(a_2 \dots (a_{m-1}(a_{mj} y_{mj})) \dots)$ is in $B \cap M'$ because $a_1(a_2 \dots (a_{m-1} y_{m-1}) \dots)$ is in $B \cap M'$ and M is a subalgebra. Defining $y_m = y_{mj}$ and $a_m = a_{mj}$, we have inductively defined for all n the set $y_1, \dots, y_n, a_1, \dots, a_n$ with $(\prod_{i=1}^n a_i) y_n = a_1(a_2 \dots (a_n y_n) \dots)$ in $B \cap M'$. For $j < k$, let $m_{jk} = \prod_{i=j}^k a_i$. Also, let $r_j = \prod_{i=1}^j a_i$ so that when $1 < j < k$, $r_{j-1} m_{jk} = r_k$. From finite dimensionality, there is an s with $m_{1s} y_s, \dots, m_{ss} y_s$ a linearly dependent set. If $m_{js} y_s$ were in M for $j > 1$, then $r_{j-1}(m_{js} y_s) = r_s y_s$ would be in M . This is not true by the construction of y_s so $m_{js} y_s$ is not in M and hence is not zero for all j . Because the elements $m_{1s} y_s, \dots, m_{ss} y_s$ are dependent, there exist field elements $\alpha_1, \dots, \alpha_s$ not all zero with $\sum_{i=1}^s \alpha_i m_{is} y_s = 0$. Letting $t = \max \{i: \alpha_i \neq 0\}$ will give

$$m_{ts} y_s = -\sum_{i=1}^{t-1} (\alpha_i/\alpha_t) m_{is} y_s = -\left[\sum_{i=1}^{t-1} (\alpha_i/\alpha_t) m_{i,t-1} \right] m_{ts} y_s .$$

Letting $x = m_{ts} y_s$ and $a = -\sum_{i=1}^{t-1} (\alpha_i/\alpha_t) m_{i,t-1}$, will give $x = ax$ with $x \neq 0$ in B and a in M .

We are now able to prove that M is not nil. By induction $x = ax = a(ax) = a^2x = \dots = a^kx$ for all k . Since x is not zero, a is not nilpotent and M is not nil. Since M is an associative algebra, we have the following result.

COROLLARY. *If B is a subspace of an algebra A and if M is a proper subset of B with:*

- (a) $M + MB = B$
- (b) M a subalgebra
- (c) $(M, M, B) = 0$

then there is an idempotent in M .

3. Proof of the second theorem. Throughout this section we will assume that A is a simple algebra in which $(x, x, x) = 0$ and $x^3 \cdot x = x^2 \cdot x^2$. In addition, assume $\text{char.} \neq 2$. Given a subset of A that satisfies (a), (b), (c) and (d) of Theorem, it is necessary to find a subalgebra of A satisfying the same conditions.

DEFINITION 3.1. The set $\mathcal{N} = \{\text{subsets of } A \text{ that satisfy (a), (b), (c) and (d) of Theorem 2}\}$.

LEMMA 3.1. *If M is in \mathcal{N} then either $M = 0$ or $M + MA = A$.*

This lemma is obvious for $M + MA$ is an ideal in A . From now

on, assume that \mathcal{N} contains a nonzero member. Each member of \mathcal{N} can easily be extended to a subspace of A , \mathcal{N} is closed under set union, and A is finite dimensional. Consequently, there is a largest member of \mathcal{N} which we will call N .

LEMMA 3.2. *The set N is an associative subalgebra with $N + NA = A$.*

Proof. Clearly N is a nonzero subspace so $N + NA = A$. In any ring, the following identities are satisfied:

$$(1) \quad (xy, z) + (yz, x) + (zx, y) = (x, y, z) + (y, z, x) + (z, x, y)$$

$$(2) \quad (wx, y, z) - (w, xy, z) + (w, x, yz) = w(x, y, z) + (w, x, y)z.$$

If x and y are in N , then from (1), we have

$$(3) \quad (xy, z) = (y, z, x) - (yz, x) - (zx, y).$$

Hence, (xy, z) is in N . Equation (2) implies the fact that $(w, xy, z) = 0$. If w and x are in N , then $(wx, y, z) = (w, xy, z) + w(x, y, z)$ which is in $N + N^2$. Also, (x, y, xw) is in $N + N^2$. Consequently, $N + N^2$ is in \mathcal{N} so $N + N^2 \subseteq N$. The fact that N is associative is obvious.

The proof of Lemma 3.2 can be used to construct a subalgebra P satisfying (a), (b), (c) and (d) of Theorem 2 in a quite different way. First, extend M to a subspace of A and call this subspace M . Then for $k \geq 1$, define $M_k = M_{k-1}^2 + M_{k-1}$. Induction and the proof of Lemma 3.2 imply M_k satisfies (a), (b), (c) and (d) of Theorem 2 for all k . By finite dimensionality, for some k , $M_k = M_{k+1}$. The set $P = M_k$ is a subalgebra satisfying (a), (b), (c) and (d). It is clear that $P + PA = A$.

We know that an associative simple algebra has an identity element. Therefore, if $A = N$, then A has an identity element. From now on, assume $A \neq N$. Theorem 1 implies N is not nil so there is an idempotent in N . An idempotent e is principal if there is no idempotent g with $eg = ge = 0$. Because N is associative, there is an idempotent that is principal in N . We will call this idempotent f . It cannot be assumed that f is principal in A . We will need the following definition.

DEFINITION 3.2. For any subset B of A , $B_{ij}(e) = \{x \text{ in } B \text{ such that } ex = ix \text{ and } xe = jx\}$. Define B_{ij} by $B_{ij} = B_{ij}(f)$.

Because N is an associative algebra with f in N , $N = N_{11} + N_{10} + N_{01} + N_{00}$. In addition, $N_{ij}N_{jk} \subseteq N_{ik}$ and, for $j \neq k$, $N_{ij}N_{km} = 0$. Up to now, we have not used the fact that $(x, x, x) = 0$. Linearizing this

yields this multilinear identity:

$$(4) \quad (x, y, z) + (y, z, x) + (z, x, y) + (z, y, x) \\ + (y, x, z) + (x, z, y) = 0.$$

If we now let $x = z = f$ and divide by 2, we obtain $(f, y, f) = 0$. Therefore, we have $(f, f, y) = (f, y, f) = (y, f, f) = 0$ which is all that is needed to prove this lemma:

LEMMA 3.3. *Relative to f , $A = A_{11} + A_{10} + A_{01} + A_{00}$.*

LEMMA 3.4. *The subspaces N_{ij} and A_{ij} obey the following laws:*

$$(5) \quad A_{ij}A_{km} = 0 \text{ if } j \neq k$$

$$(6) \quad N_{ij}A_{jk} \cup A_{ij}N_{jk} \subseteq N_{ik} \text{ if } i \neq k$$

$$(7) \quad N_{ij}A_{jk} \subseteq A_{i1} + A_{i0}$$

$$(8) \quad A_{ij}N_{jk} \subseteq A_{1k} + A_{0k}$$

$$(9) \quad N_{10}A_{01} \subseteq N_{11} + N_{10}$$

$$(10) \quad N_{01}A_{10} \subseteq N_{00} + N_{01}$$

$$(11) \quad N_{ii}A_{ii} \subseteq A_{ii} + N.$$

Proof. To prove (5), let x be in A_{ij} and let y be in A_{km} in $(x, f, y) = 0$. The relations (7) and (8) result from $(f, N_{ij}, A_{jk}) = 0$ and $(A_{ij}, N_{jk}, f) = 0$. In proving (6), we first let $y = f, z = n_{ij}$ (in N_{ij}) and $x = x_{jk}$ (in A_{jk}) in (4) to obtain $(f, x_{jk}, n_{ij}) + (n_{ij}, x_{jk}, f) = 0$. Using (5), we find that $(f, x_{jk}, n_{ij}) = jx_{jk}n_{ij} - f(x_{jk}n_{ij}) = 0$. Hence, $(n_{ij}x_{jk})f = kn_{ij}x_{jk}$. This together with (7) implies $N_{ij}A_{jk} \subseteq A_{ik}$. In a similar way, (4), (5) and (8) imply $A_{ij}N_{jk} \subseteq A_{ik}$. We will now prove $N_{ij}A_{jk} \subseteq N_{ik}$. The proof of $A_{ij}N_{jk} \subseteq N_{jk}$ is similar. Using z and x as defined above, $xz = 0$ and (z, x) is in N . Therefore $zx = (z, x)$ is in $N \cap A_{ik}$ and we have proved (6). We now take n in N_{10} and a in A_{01} . From (7) and (8) we have $na = b_{11} + b_{10}$ and $an = c_{10} + c_{00}$ with b_{ij}, c_{ij} , in A_{ij} . We also see that b_{11}, c_{11} and $b_{10} - c_{10}$ are in N because (n, a) is in N . Equation (4) implies $(n, a, f) + (f, a, n) = 0$. Upon substitution we get $(b_{11} + b_{10})f - (b_{11} + b_{10}) - f(c_{10} + c_{00}) = 0$ or $-b_{10} - c_{10} = 0$. Therefore $b_{10} = 1/2(b_{10} - c_{10})$ is in N_{10} . This proves (9) and a similar argument proves (10). In [1], a linearization of $(x^2 \cdot x) \cdot x = x^2 \cdot x^2$ was all that was needed to prove that $A_{11} \cdot A_{11} \subseteq A_{11}$ and $A_{00} \cdot A_{00} \subseteq A_{00}$. The proof of Lemma 3.4 is completed by observing that this implies $N_{ii}A_{ii} \subseteq N_{ii} \cdot A_{ii} + (N_{ii}, A_{ii}) \subseteq A_{ii} + N$.

LEMMA 3.5. *The algebra $A = N + N_{11}A_{11} + N_{00}A_{00}$.*

Proof. As a result of (5) and Lemma 3.2, $A = N + \sum N_{ij}A_{jk}$. From (6), $N_{11}A_{10} + N_{10}A_{00} \subseteq N_{10}$ and $N_{00}A_{01} + N_{01}A_{11} \subseteq N_{01}$. The relations in (9) and (10) complete the proof.

LEMMA 3.6. *The set $B = N_{10}N_{01} + N_{10} + N_{01} + N_{01}N_{10}$ is an associative ideal in A .*

Proof. The laws $N_{ij}N_{jk} \subseteq N_{ik}$ and $N_{ij}N_{mk} = 0$ for $j \neq m$ easily establish the fact that B is an associative ideal in N . Using Lemma 3.5, we need only prove that $B(N_{11}A_{11})$, $(N_{11}A_{11})B$, $(N_{00}A_{00})B$ and $B(N_{00}A_{00})$ are in B . We have $(N_{11}A_{11})(N_{10} + N_{01}) \subseteq (A_{11} + N)(N_{10} + N_{01}) \subseteq A_{11}N_{10} + B \subseteq B$. Also, $(N_{10} + N_{01})(N_{11}A_{11}) = [(N_{10} + N_{01})N_{11}]A_{11} \subseteq N_{01}A_{11} \subseteq N_{01} \subseteq B$. Interchanging 0 and 1 will prove that $(N_{00}A_{00})(N_{10} + N_{01})$ and $(N_{01} + N_{10})(N_{00}A_{00})$ are in B . Therefore $(N_{10} + N_{01})A$ and $A(N_{10} + N_{01})$ are in B . Now, $(N_{10}N_{01})A = N_{10}(N_{01}A) \subseteq N_{10}B \subseteq B$ and $A(N_{10}N_{01}) = (AN_{10})N_{01} \subseteq BN_{01} \subseteq B$. Similarly $(N_{01}N_{10})A$ and $A(N_{01}N_{10})$ are in B so B is an associative ideal in A .

We have assumed that $N \neq A$ which implies $B = 0$. Therefore $A = N_{11} + N_{11}A_{11} + N_{00} + N_{00}A_{00}$. The relations in (7) and (11) together with the fact that $N = N_{11} + N_{00}$ imply $N_{11}A_{11} \subseteq A_{11}$ and $N_{00}A_{00} \subseteq A_{00}$.

We are now able to complete the proof of Theorem 2. We have $A = A_{11} + A_{00}$ with $A_{11} = N_{11} + N_{11}A_{11}$ and $A_{00} = N_{00} + N_{00}A_{00}$. The result in Theorem 1 and the fact that N_{00} is nil imply $N_{00} = A_{00}$. Therefore, using (5) and $N_{00}^2 \subseteq N_{00}$, we see that N_{00} is an ideal of A . Consequently, f is the identity element of A .

4. Applications. We will mention several cases where Theorem 2 can be used to prove the existence of an identity element in a simple algebra. The following two corollaries are obvious since simple associative and simple commutative power associative algebras have identities. The nucleus of an algebra A is $\{x: xy = yx \text{ for all } y \text{ in } A\}$.

COROLLARY 4.1. *For char. $\neq 2$, a simple power associative algebra with all commutators in the nucleus has an identity element.*

COROLLARY 4.2. *For char. $\neq 2$ a simple power associative algebra with all associators in the middle center ($\{x: (x, y) = (y, x, z) = 0$ for all y and $z\}$) has an identity element.*

For our last application, let A be a simple algebra that satisfies the following:

$$(12) \quad (x, y, z) = (z, y, x) \quad (\text{the antiflexible law})$$

$$(13) \quad (x, x, x) = 0 .$$

In [2] it is proved that $x^2x^2 = x^3x = xx^3$ in A . In fact, A^+ is power associative [2]. Furthermore, if we let $Z = \{x: xy = yx \text{ for all } y \text{ in } A\}$, we can show by the arguments of [3] that $(A, Z, A) = 0$ and (Z, A, A) , (A, A, Z) are subsets of Z . Because A^+ is power associative, there is an idempotent e in A^+ if A^+ is not nil. Clearly, e is an idempotent in A . The arguments of [2] will show $A = A_{11}(e) + A_{00}(e)$. Therefore, e is in Z and $Z \neq 0$. We have proved this corollary.

COROLLARY 4.3. *If A is a simple anti flexible algebra (char. $\neq 2$) that satisfies $(x, x, x) = 0$ and if A^+ is not a nil algebra, then there is an identity element in A .*

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