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Let H be a Hilbert space with orthonormal basis $\{f_j\}_{j=1}^{\infty}$. If the operator T is defined on H by $Tf_i = a_i f_{i+1}$ for $i = 1, 2, \cdots$, where $|a_i| \leqq |a_{i+1}| \leqq M$ for $i = 1, 2, \cdots$, then T will be called a monotone shift. The first section of the paper examines some of the elementary properties of such operators.

Every monotone shift is hyponormal. The central portion of the paper aims at discovering which monotone shifts are subnormal. Necessary and sufficient conditions are given in terms of the $\{a_i\}$. These conditions make it easy to show that even the first four coefficients $(a_1 < a_2 < a_3 < a_4)$ may "prevent" a shift from being subnormal. However, for any $a_1 < a_2 < a_3$ there does exist a monotone shift with these as its initial terms. In fact, the unique minimal one is constructed.

A complete description is given of subnormal monotone shifts for which $|a_{j_0}| = |a_{j_0+1}|$ for some j_0. The paper concludes with counter-examples constructed from the machinery developed.

We are tacitly assuming that $\lim_{j\to\infty} |a_j|$ exists, i.e., T is a bounded operator. If $|a_j| = |a_{j+1}|$ for $j = 1, 2, \cdots$, then T is (up to unitary equivalence) simply a multiple of the justly famous unilateral shift.

We recall that an operator T on a Hilbert space H is subnormal if it is the restriction of a normal operator to an invariant subspace. The terms "point", "continuous" and "residual spectrum" have their usual meaning and are designated by $\sigma_P(\cdot)$, $\sigma_C(\cdot)$ and $\sigma_R(\cdot)$ respectively.

THEOREM 1. *Let T be a monotone shift on H where $A = \lim_{j\to\infty} |a_j|$, then*

(i) $\|T\| = A$

(ii) $\sigma_R(T) = \{z: |z| < A\}$

(iii) $\sigma_P(T^*) = \{z: |z| < A\}$

(iv) $\sigma_C(T) = \sigma_C(T^*) = \{z: |z| = A\}$.

Proof. Surely (i) is clear.

For $|z_0| < A$, consider the vector $g = \sum_{n=1}^{\infty} z_0^n b_n f_n$ where $b_k = 1$ if $a_{k-1} = 0$, $a_k \neq 0$ and $b_{n+1} = z_0 b_n/\bar{a}_n$ for $n > k$. Since $|z_0/\bar{a}_n| \leqq r < 1$ for n sufficiently large, $g \in H$. But $(T^* - zI)g = 0$ so g is the desired eigenvector, proving (iii). The relation $b_{n+1} = z_0 b_n/\bar{a}_n$ is necessary which implies the eigenvalue z_0 is of multiplicity one.

For any z, it is clear that $(T - zI)h \neq 0$ for $h \in H$, and $h \neq 0$.

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Now for an arbitrary operator, $z_0 \in \sigma_P(T^*)$ implies $\bar{z}_0 \in \sigma_P(T) \cup \sigma_R(T)$ and (ii) follows.

It is not hard to see that

$$\sigma_P(T) \cap \{z: |z| = A\} = \phi = \sigma_P(T^*) \cap \{z: |z| = A\}$$

and hence (iv) is proved.

REMARK. If one considered a two-sided monotone shift $Tf_j = a_j f_{j+1}$ for $j = 0, \pm 1, \cdots$, then the above theorem is easily modified, the spectrum of T now being the annulus $B \leqq |z| \leqq A$ where $A = \lim_{j\to\infty} |a_j|$ and $B = \lim_{j\to-\infty} |a_j|$.

THEOREM 2. *If T is a monotone shift then T is hyponormal; that is $||Tg|| \geqq ||T^*g||$ for $g \in H$.*

Proof. Clear.

THEOREM 3. *If T is a monotone shift then T is not a spectral operator, (for $T \neq 0$).*

Proof. Assume T is a spectral operator; then T^* is also a spectral operator which can be written as $S + N$ where S is a scalar operator and N a commuting quasi-nilpotent. By Theorem 1 we know that every point in the disc $|z| < ||T||$ is an eigenvalue of T^*. Invoking Corollary 5 to Theorem 1 of [2] every point in the disc $|z| < ||T||$ is an eigenvalue of S. But S is similar to a normal operator which must have an uncountable number of distinct eigenvalues. Thus our original assumption was ill founded.

Next we would like explicitly exhibit a normal extension B of T assuming T to be subnormal. The normal extension we construct will be minimal, (see [1], [3] for a discussion of subnormal operators). From now on we will assume $a_i \neq 0$ for $i = 1, 2, \cdots$. There is no real loss in generality if we do so. For if $a_1 = a_2 = \cdots = a_n = 0$, then the subspace M spanned by $\{\varphi_1, \cdots, \varphi_n\}$ is a reducing subspace of T and moreover T is normal on M. Hence we are really only concerned with what takes place in $M^\perp$. Let us designate the original basis for $H \equiv H^{(1)}$ by $\{\varphi_j^{(1)}\}_{j=1}^{\infty}$. We shall use the terms a_j and $a_j^{(1)}$ interchangeably as a notational convenience. It will be to our advantage to assume that $a_j^{(1)} > 0$, for $j = 1, 2, \cdots$, and we may do this by simply rotating the $\varphi_j^{(1)}$'s of the original basis by $e^{i\theta_j}$.

Now set $B^*\varphi_j^{(1)} = a_{j-1}^{(1)}\varphi_{j-1}^{(1)} + b_j^{(2)}\varphi_j^{(2)}$ where $||\varphi_j^{(2)}|| = 1$ and $\varphi_j^{(2)}$ is orthogonal to $H^{(1)}$ for $j = 1, 2, \cdots$. Since $||B^*\varphi_j^{(1)}||^2 = (a_{j-1}^{(1)})^2 + |b_j^{(2)}|^2 = ||B\varphi_j^{(1)}||^2 = (a_j^{(1)})^2$ we may conclude that $|b_j^{(2)}|^2 = (a_j^{(1)})^2 - (a_{j-1}^{(1)})^2$, $(a_0^{(1)} = 0)$. We claim that the $\varphi_j^{(2)}$'s are orthogonal, for

$$b_i^{(2)}b_j^{(2)}(\varphi_i^{(2)}, \varphi_j^{(2)}) = (B^*\varphi_i^{(1)}, B^*\varphi_j^{(1)}) = (B\varphi_i^{(1)}, B\varphi_j^{(1)}) = 0\,,$$

when $i \neq j$. We are assuming for the moment that $b_j^{(2)} \neq 0$ for $j = 1, 2, \cdots$. Since the $\varphi_j^{(2)}$'s are determined up to a scalar of modulus one we may take $b_j^{(2)} = [(a_j^{(1)})^2 - (a_{j-1}^{(1)})^2]^{1/2}$. Now let us consider the possibility that $b_{j_0}^{(2)} = 0$. This would mean $a_{j_0}^{(1)} = a_{j_0-1}^{(1)}$ or that $\| T^*\varphi_{j_0}^{(1)} \| = \| T\varphi_{j_0}^{(1)} \|$. But by Theorem 4 of [5] the set of vectors $\{f: \| T^*f \| = \| Tf \|\}$ forms a closed invariant subspace of T. Thus $\| T\varphi_k^{(1)} \| = a_k^{(1)} = a_{j_0}^{(1)}$ for $k \geqq j_0$ and thus $b_k^{(2)} = 0$ for $k \geqq j_0$.

We will designate by $H^{(2)}$ the space spanned by $\{\varphi_j^{(2)}\}$ for $b_j^{(2)} \neq 0$. So far we know that $(B\varphi_j^{(2)}, \varphi_j^{(1)}) = b_j^{(2)} = [(a_j^{(1)})^2 - (a_{j-1}^{(1)})^2]^{1/2}$. Now

$$\begin{aligned}(B\varphi_{j+1}^{(2)}, B\varphi_j^{(1)}) &= (B\varphi_{j+1}^{(2)}, T\varphi_j^{(1)}) = b_{j+1}^{(2)}a_j^{(1)} = (B^*\varphi_{j+1}^{(2)}, B^*\varphi_j^{(1)}) \\ &= (B^*\varphi_{j+1}^{(2)}, a_{j-1}^{(1)}\varphi_{j-1}^{(1)} + b_j^{(2)}\varphi_j^{(2)}) \\ &= (B^*\varphi_{j+1}^{(2)}, b_j^{(2)}\varphi_j^{(2)}) = b_j^{(2)}(\varphi_{j+1}^{(2)}, B\varphi_j^{(2)})\end{aligned}$$

thus $(B\varphi_j^{(2)}, \varphi_{j+1}^{(2)}) = a_j^{(1)}b_{j+1}^{(2)}/b_j^{(2)}$. We set $a_j^{(2)} \equiv a_j^{(1)}b_{j+1}^{(2)}/b_j^{(2)}$. Since $(\varphi_j^{(2)}, B^*B\varphi_k^{(1)}) = 0$ for $k \neq j - 1$, it follows that

$$\begin{aligned}0 &= (B^*\varphi_j^{(2)}, B^*\varphi_k^{(1)}) = (B^*\varphi_j^{(2)}, a_{k-1}^{(1)}\varphi_{k-1}^{(1)} + b_k^{(2)}\varphi_k^{(2)}) \\ &= (B^*\varphi_j^{(2)}, b_k^{(2)}\varphi_k^{(2)}) = b_k^{(2)}(\varphi_j^{(2)}, B\varphi_k^{(2)})\end{aligned}$$

or $(B\varphi_k^{(2)}, \varphi_j^{(2)}) = 0$ for $k \neq j - 1$. Thus $B\varphi_j^{(2)} = a_j^{(2)}\varphi_{j+1}^{(2)} + b_j^{(2)}\varphi_j^{(1)} + f_j$ where f_j is orthogonal to $H^{(1)} \oplus H^{(2)}$. We now wish to show that $f_j = 0$ for $j = 1, 2, \cdots$. We observe that $(Bf_j, B\varphi_k^{(1)}) = a_k^{(1)}(f_j, B^*\varphi_{k+1}^{(1)}) = 0$ for all j, k since $B^*\varphi_{k+1}^{(1)} \in H^{(1)} \oplus H^{(2)}$. But then

$$\begin{aligned}0 &= (B^*f_j, B^*\varphi_j^{(1)}) = (f_j, B(b_j^{(2)}\varphi_j^{(2)} + a_{j-1}^{(1)}\varphi_{j-1}^{(1)})) \\ &= (f_j, Bb_j^{(2)}\varphi_j^{(2)}) = b_j^{(2)}(f_j, a_j^{(2)}\varphi_{j+1}^{(2)} + b_j^{(2)}\varphi_j^{(1)} + f_j) \\ &= b_j^{(2)} \| f_j \|\end{aligned}$$

from which we may conclude $f_j = 0$. Thus we have $B\varphi_j^{(2)} = a_j^{(2)}\varphi_{j+1}^{(2)} + b_j^{(2)}\varphi_j^{(1)}$ and since $\{\varphi_j^{(2)}\}$ is a basis for $H^{(2)}$ and B is bounded by assumption, B is now defined on $H^{(1)} \oplus H^{(2)}$.

It is perhaps worth while to write it down in matrix form on $H^{(1)} \oplus H^{(2)}$:

$$\left[\begin{array}{ccc|ccc} 0 & & & b_1^{(2)} & & \\ a_1^{(1)} & 0 & & & b_2^{(2)} & \\ & a_2^{(1)} & 0 & & & b_3^{(2)} \\ & & a_3^{(1)} \cdot & & & \cdot \\ & & \cdot \ \cdot & & & \cdot \\ & & \cdot \ \cdot & & & \cdot \\ \hline & & \cdot & 0 & & \\ & & & a_1^{(2)} & 0 & \\ & 0 & & & a_2^{(2)} & 0 \\ & & & & a_3^{(2)} \cdot & \\ & & & & \cdot \ \cdot & \\ & & & & \cdot \ \cdot & \\ & & & & & \cdot \end{array}\right]$$

where $b_j^{(2)} = ((a_j^{(1)})^2 - (a_{j-1}^{(1)})^2)^{1/2}$ and $b_j^{(2)}a_j^{(2)} = b_{j+1}^{(2)}a_j^{(1)}$. We have assumed that all the $a_j^{(1)}$'s $\neq 0$ and have shown that if $b_{j_0}^{(2)} = 0$ then $b_k^{(2)} = 0$ for $k \geqq j_0$. In this case, $H^{(2)}$ would be finite dimensional.

We will now assume B has been defined on $H^{(1)}, \cdots, H^{(n)}$ where $\{\varphi_j^{(1)}\}, \cdots, \{\varphi_j^{(n)}\}$ are the respective bases and $B\varphi_j^{(k)} = a_j^{(k)}\varphi_{j+1}^{(k)} + b_j^{(k)}\varphi_j^{(k-1)}$ for $k = 2, \cdots, n$. We also note that all $b_j^{(k)} \neq 0$. That is, if $b_1^{(k)}, \cdots, b_{j_k}^{(k)}$ are nonzero and $b_{j_k+1}^{(k)} = 0$ then $b_j^{(k)} = 0$ for $j \geqq j_k + 1$ and $H^{(k)}$ is spanned by $\{\varphi_1^{(k)}, \cdots, \varphi_{j_k}^{(k)}\}$. Further $a_j^{(k)} \neq 0$ for $j = 1, \cdots, j_k - 1$. The above can be considered to be the induction hypothesis.

We set $B^*\varphi_j^{(n)} = a_{j-1}^{(n)}\varphi_{j-1}^{(n)} + b_j^{(n+1)}\varphi_j^{(n+1)}$ for $j = 1, 2, \cdots$, where $\| \varphi_j^{(n+1)} \| = 1$, and $\varphi_j^{(n+1)}$ is orthogonal to $H^{(n)} \bigoplus \cdots \bigoplus H^{(n)}$. Since $\| B^*\varphi_j^{(n)} \| = \| B\varphi_j^{(n)} \|$ we have $| b_j^{(n+1)} |^2 = (a_j^{(n)})^2 + (b_j^{(n)})^2 - (a_{j-1}^{(n)})^2 (a_0^{(n)} = 0$ by definition). Now

$$\begin{aligned}(B^*\varphi_i^{(n)}, B^*\varphi_j^{(n)}) &= b_i^{(n+1)}b_j^{(n+1)}(\varphi_i^{(n+1)}, \varphi_j^{(n+1)}) \\ &= (B\varphi_i^{(n)}, B\varphi_j^{(n)}) = 0 \qquad \text{for} \quad i \neq j.\end{aligned}$$

Thus $\varphi_i^{(n+1)}$ and $\varphi_j^{(n+1)}$ are orthogonal for we may assume $b_i^{(n+1)}$, $b_j^{(n+1)}$ are nonzero else $\varphi_{i \text{ or } j}^{(n+1)}$ would not be defined. Thus $b_j^{(n+1)}$ may be taken positive. Consider

$$\begin{aligned}(\varphi_{j+1}^{(n+1)}, B^*B\varphi_j^{(n)}) &= (\varphi_{j+1}^{(n+1)}, B^*(a_j^{(n)}\varphi_{j+1}^{(n)} + b_j^{(n)}\varphi_j^{(n-1)})) \\ &= (\varphi_{j+1}^{(n+1)}, a_j^{(n)}b_{j+1}^{(n+1)}\varphi_{j+1}^{(n+1)}) = a_j^{(n)}b_{j+1}^{(n+1)} \\ &= (\varphi_{j+1}^{(n+1)}, BB^*\varphi_j^{(n)}) = (\varphi_{j+1}^{(n+1)}, B(a_{j-1}^{(n)}\varphi_{j-1}^{(n)} + b_j^{(n+1)}\varphi_j^{(n+1)})) \\ &= (\varphi_{j+1}^{(n+1)}, b_j^{(n+1)}B\varphi_j^{(n+1)}) .\end{aligned}$$

This implies $B\varphi_j^{(n+1)} = b_j^{(n+1)}\varphi_j^{(n)} + a_j^{(n+1)}\varphi_{j+1}^{(n+1)} + f_j$ where $(f_j, \varphi_{j+1}^{(n+1)}) = 0$ and $a_j^{(n+1)}$ satisfies the relation $a_j^{(n+1)}b_j^{(n+1)} = a_j^{(n)}b_{j+1}^{(n+1)}$, for $j = 1, 2, \cdots$.

Now let us clear up the zero, nonzero relation between the a's and the b's. By the induction hypothesis either all $a_j^{(n)}$'s $\neq 0$ for $j = 1, 2, \cdots$, or there are a finite number and all but the last is nonzero. In either case assume $b_k^{(n+1)} = 0$ and $b_{k+1}^{(n+1)} \neq 0$; then $a_k^{(n+1)}b_k^{(n+1)} = 0 = a_k^{(n)}b_{k+1}^{(n+1)} \neq 0$ (for there are at least k nonzero $a_j^{(n)}$'s since the dim $H^{(n+1)} \leqq$ dim $H^{(n)}$). This is impossible. Now assume $b_j^{(n+1)} \neq 0$ for $j = 1, 2, \cdots, k$, and $b_j^{(n+1)} = 0$ for $j > k$, i.e., $H^{(n+1)}$ is k-dimensional. Then $a_j^{(n+1)} = a_j^{(n)}b_{j+1}^{(n+1)}/b_j^{(n+1)}$ is well defined for $j \leqq k$ and since $H^{(n)}$ was at least k-dimensional, $a_j^{(n)} \neq 0$ for $j = 1, \cdots, k - 1$; hence $a_j^{(n+1)} \neq 0$ for $j = 1, \cdots, k - 1$ as we wished to show.

Let us go back to the defining relation

$$B\varphi_j^{(n+1)} = a_j^{(n+1)}\varphi_{j+1}^{(n+1)} + b_j^{(n+1)}\varphi_j^{(n)} + f_j .$$

Since

$$(\varphi_i^{(n+1)}, B^*B\varphi_j^{(n)}) = 0 \qquad \text{for} \quad i \neq j + 1$$

we have

$$(\varphi_i^{(n+1)}, BB^*\varphi_j^{(n)}) = (\varphi_i^{(n+1)}, b_j^{(n+1)}(a_j^{(n+1)}\varphi_{j+1}^{(n+1)} + b_j^{(n+1)}\varphi_j^{(n)} + f_j))$$
$$+ (\varphi_i^{(n+1)}, Ba_{j-1}^{(n)}\varphi_{j-1}^{(n)}) = (\varphi_i^{(n+1)}, f_j) = 0 \qquad \text{for} \quad i \neq j+1.$$

Now $(\varphi_{j+1}^{(n+1)}, f_j) = 0$ by definition and hence f_j is orthogonal to $H^{(1)} \oplus \cdots \oplus H^{(n+1)}$. But

$$\begin{aligned}(f_j, B^*B\varphi_j^{(n)}) &= (f_j, B^*(a_j^{(n)}\varphi_{j+1}^{(n)} + b_j^{(n)}\varphi_j^{(n-1)})) \\ &= 0 = (f_j, BB^*\varphi_j^{(n)}) \\ &= (f_j, B(a_{j-1}^{(n)}\varphi_{j-1}^{(n)} + b_j^{(n+1)}\varphi_j^{(n+1)})) \\ &= (f_j, b_j^{(n+1)}B\varphi_j^{(n+1)}) \\ &= (f_j, b_j^{(n+1)}(a_j^{(n+1)}\varphi_{j+1}^{(n+1)}b_j^{(h+1)}\varphi_j^{(h)} + f_j)) \\ &= b_j^{(n+1)} \,||\, f_j \,|| \qquad \text{and since} \quad b_j^{(n+1)} \neq 0\end{aligned}$$

we conclude $f_j = 0$. Thus $B\varphi_j^{(n+1)} = a_j^{(n+1)}\varphi_{j+1}^{(n+1)} + b_j^{(n+1)}\varphi_j^{(n)}$ where $b_j^{(n+1)} \neq 0$ for $j = 1, \cdots, (\dim H^{(n+1)}$ and $a_j^{(n+1)} \neq 0$ for $j = 1, \cdots, (\dim H^{(n+1)} - 1)$. It should help to present B as an infinite matrix with respect to the basis $\{\varphi_j^{(n)}\}$ where $j = 1, 2, \cdots, \dim H^{(n)}$ and $n = 1, 2, \cdots$.

$$B = \begin{vmatrix} F^{(1)} & G^{(2)} & 0 & & & \\ 0 & F^{(2)} & G^{(3)} & & 0 & \\ 0 & 0 & F^{(3)} & \cdot & & \\ & & & \cdot & \cdot & \\ & 0 & & & \cdot & \cdot \\ & & & & & \cdot \end{vmatrix}$$

where

$$F^{(n)} = \begin{vmatrix} 0 & & & & & \\ a_1^{(n)} & 0 & & & & \\ & a_2^{(n)} & 0 & & & \\ & & a_3^{(n)} & \cdot & & \\ & & & \cdot & \cdot & \\ & & & & \cdot & \cdot \\ & & & & & \cdot \end{vmatrix}$$

and

$$G^{(n)} = \begin{vmatrix} b_1^{(n)} & & & & & \\ & b_2^{(n)} & & & & \\ & & b_3^{(n)} & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ & & & & & \cdot \end{vmatrix}$$

Now that we have a normal extension of T in concrete form it is not difficult to read off some necessary conditions for the subnormality of T. In fact, it is clear that:

(I) $(a_j^{(n)})^2 + (b_j^{(n)})^2 \geqq (a_{j-1}^{(n)})^2$ for $n = 2, 3, \cdots$, and $j = 1, 2, \cdots$, (otherwise it would not be possible to define $b_j^{(n+1)}$ compatibly with the normality of B).

(II) If $b_j^{(n)} = 0$ then $b_{j+1}^{(n)} = 0$ for $j = 1, 2, \cdots$, and $n = 2, 3, \cdots$, (else the equation $a_j^{(n)} b_j^{(n)} = a_j^{(n-1)} b_{j+1}^{(n)}$ could not be satisfied).

(III) There exists a constant M such that $|a_j^{(n)}| \leqq M$ and $|b_j^{(n)}| \leqq M$ for $n = 2, 3, \cdots$, and $j = 1, 2, \cdots$, (else the normal extension would not be bounded). These conditions are also sufficient. More precisely:

THEOREM 4. *Let $\{a_j^{(1)}\}$ be a sequence such that $0 < a_1^{(1)} \leqq a_2^{(1)} \leqq \cdots$ and define $T\varphi_j^{(1)} = a_j^{(1)} \varphi_{j+1}^{(1)}$ for $j = 1, 2, \cdots$, where $\{\varphi_j^{(1)}\}_{j=1}^{\infty}$ is an orthonormal basis for $H^{(1)}$. The operator T is subnormal if and only if*

(I) $(a_j^{(n)})^2 + (b_j^{(n)})^2 - (a_{j-1}^{(n)})^2 \geqq 0$

(II) $b_j^{(n)} = 0 \Rightarrow b_{j+1}^{(n)} = 0$

(III) *there exists a constant M such that $|a_j^{(n)}| \leqq M$ and $|b_j^{(n)}| \leqq M$ for $n = 2, 3, \cdots$, and $j = 1, 2, \cdots$, where*

$$(b_j^{(n+1)} = + [(a_j^{(n)})^2 + (b_j^{(n)})^2 - (a_{j-1}^{(n)})^2]^{1/2}$$

and $a_j^{(n+1)} = a_j^{(n)} b_{j+1}^{(n+1)} / b_j^{(n+1)}$ (if $b_{j_0}^{(n)} = 0$ then $a_{j_0}^{(n)}$ is taken to be zero).

Proof. The necessity of (I), (II) and (III) has already been proved. In the other direction, since conditions (I) and (II) are satisfied the recursive definitions of the $a_j^{(n+1)}$ and $b_j^{(n+1)}$ make sense. We now set $B\varphi_j^{(n)} = a_j^{(n)} \varphi_{j+1}^{(n)} + b_j^{(n)} \varphi_j^{(n-1)}$ where $\{\varphi_j^{(n)}\}$ is an orthonormal basis for $H^{(n)}$. If $b_{j_0}^{(n)} = 0$ then $\varphi_{j_0}^{(n)}$ is omitted from the basis. With this definition it should be clear from the former construction that

$$(B\varphi_j^{(n)}, B\varphi_k^{(m)}) = (B^*\varphi_j^{(n)}, B^*\varphi_k^{(m)}) \qquad \text{for all} \quad n, m, j, k.$$

(Here as elsewhere all should be interpreted as vacuously covering the case when $\varphi_j^{(n)}$ has been omitted from the basis.) Thus B is defined and satisfies $\|Bx\| = \|B^*x\|$ on finite linear combinations of the basis elements. From (III) we may infer that B is bounded: certainly $\|B\| \leqq 2M$. Hence B is defined and is normal on all of $\hat{H} = \sum \oplus H^{(n)}$.

We remark that the $a_j^{(1)}$'s were taken to be real only as a notational convenience. To modify the theorem for an arbitrary monotone shift one need only add absolute value signs in the appropriate places.

Let $a_1, \cdots, a_n$ be given. If there exist elements $a_{n+1}, a_{n+2}, \cdots$, such that $T\varphi_j = a_j \varphi_{j+1}$ for $j = 1, 2, \cdots$, is a subnormal operator, we will call T a subnormal completion of $a_1, \cdots, a_n$.

In Theorem 5 we will show that for any a_1, a_2, a_3 where $|a_1| < |a_2| < |a_3|$, there exists a subnormal completion. However, one can specify a_1, a_2, a_3, a_4, where $a_1 < a_2 < a_3 < a_4$, in such a manner that they can not be completed to form a subnormal monotone shift. This can be easily done by making use of necessary condition (I).

Consider $(a_3^{(2)})^2 + (b_3^{(2)})^2 \geqq (a_2^{(2)})^2$ which is equivalent to:

$$(a_3^2 - a_2^2) + a_3^2 \frac{a_4^2 - a_3^2}{a_3^2 - a_2^2} \geqq a_2^2 \frac{a_3^2 - a_2^2}{a_2^2 - a_1^2}$$

or

$$a_4^2 \geqq a_3^2 + \frac{a_1^2}{a_3^2} \frac{(a_3^2 - a_2^2)^2}{a_2^2 - a_1^2} \,. \tag{1}$$

[1]

If a_4 is chosen so as not to satisfy (1) then clearly no choice of the remaining a_j's will make the resulting shift subnormal. For example, if $a_1 = 1$, $a_2 = 2^{1/2}$, $a_3 = 3^{1/2}$, then for $3^{1/2} \leqq a_4 < (10/3)^{1/2}$ there exists no subnormal completion.

THEOREM 5. *Given a_1, a_2, a_3 where $0 < |a_1| < |a_2| < |a_3|$ then there exists a subnormal completion of a_1, a_2, a_3. Moreover, if S is any subnormal completion of a_1, a_2, a_3, then*

$$\| S \|^2 \geqq 1/2 \Bigg[|a_2|^2 \frac{|a_3|^2 - |a_1|^2}{|a_2|^2 - |a_1|^2} + \left(\left[|a_2|^2 \frac{|a_3|^2 - |a_1|^2}{|a_2|^2 - |a_1|^2} \right]^2 - 4 |a_1|^2 |a_2|^2 \frac{|a_3|^2 - |a_2|^2}{|a_2|^2 - |a_1|^2} \right)^{1/2} \Bigg] .$$

Further there is, up to unitary equivalence, exactly one subnormal completion for which equality holds.

Proof. We may take a_1, a_2, a_3 to be positive. The a_n's for the minimal completion are given by the following rule:

$$a_n^2 \frac{a_{n+1}^2 - a_n^2}{a_n^2 - a_{n-1}^2} = a_{n-1}^2 \frac{a_n^2 - a_{n-1}^2}{a_{n-1}^2 - a_{n-1}^2} - (a_n^2 - a_{n-1}^2) \,. \tag{2}$$

Since $a_{n-1}^2/(a_{n-2}^2 - a_{n-2}^2) > 1$ the right side is strictly positive and $a_{n+1} > a_n$. The defining relation (2) is simply the necessary condition $(a_j^{(2)})^2 + (b_j^{(2)})^2 - (a_{j-1}^{(2)})^2 \geqq 0$ with equality holding.

Next we will find a more convenient expression for a_n. From (2) we have:

$$\frac{a_{n+1}^2 - a_n^2}{a_n^2 - a_{n-1}^2} = \frac{a_{n-2}^2}{a_n^2} \cdot \frac{a_n^2 - a_{n-1}^2}{a_{n-1}^2 - a_{n-2}^2} \tag{3}$$

and so by induction,

$$\frac{a_{n+1}^2 - a_n^2}{a_n^2 - a_{n-1}^2} = \frac{a_2^2 \cdot a_1^2}{a_n^2 \cdot a_{n-1}^2} \cdot \frac{a_3^2 - a_2^2}{a_2^2 - a_1^2} = \frac{C}{a_n^2 a_{n-1}^2} \tag{4}$$

[1] Note that a_j^2 means $(a_j)^2$ not $a_j^{(2)}$

by definition of C. Thus

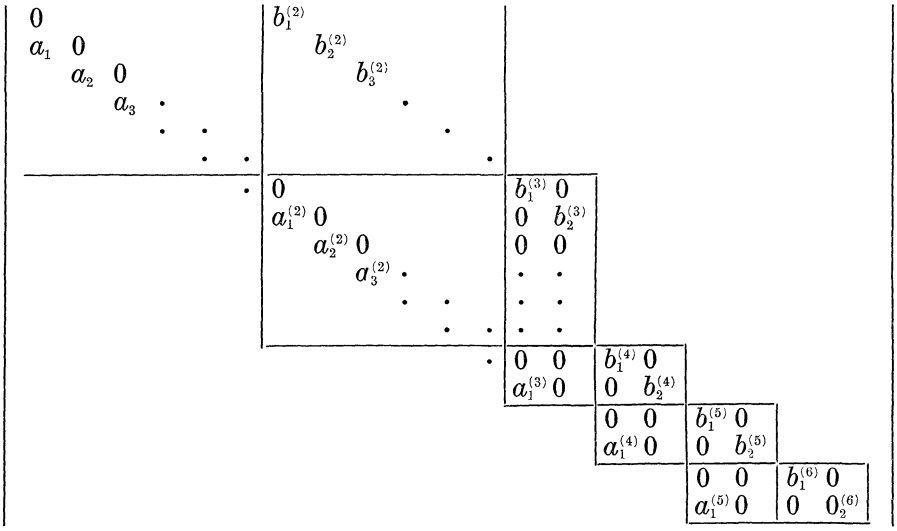
$$a_{n+1}^2 = a_n^2 + C\left[\frac{1}{a_{n-1}^2} - \frac{1}{a_n^2}\right] \tag{5}$$

and again by telescoping:

$$a_{n+1}^2 = a_2^2 + C\left[\frac{1}{a_1^2} - \frac{1}{a_n^2}\right]. \tag{6}$$

From (6) we can see that $\{a_n\}$ converges and we will shortly compute the limit.

Is the operator T defined from the a_n's actually subnormal? Let us write down its normal extension.



$$a_n^{(2)} = a_n\left[\frac{a_{n+1}^2 - a_n^2}{a_n^2 - a_{n-1}^2}\right]^{1/2}, \qquad b_n^{(2)} = [a_{n+1}^2 - a_n^2]^{1/2}.$$

We see from (2) that

$$a_n^2 \frac{a_{n+1}^2 - a_n^2}{a_n^2 - a_{n-1}^2} \leqq a_{n-1}^2 \frac{a_n^2 - a_{n-1}^2}{a_{n-1}^2 - a_{n-2}^2}$$

and hence the $a_n^{(2)}$'s are bounded and obviously the $b_n^{(2)}$'s are. Now if it can be shown that $a_1^{(k)}$, $b_1^{(k)}$, $b_2^{(k)}$, for $k = 3, 4, \cdots$, can be defined to satisfy the normality relations and moreover remain bounded then we are finished. The defining relations are:

(i) $a_1^{(n)} b_1^{(n)} = a_1^{(n-1)} b_2^{(n)}$

(ii) $(b_1^{(n+1)})^2 = (a_1^{(n)})^2 + (b_1^{(n)})^2$

and

(iii) $(b_2^{(n+1)})^2 = (b_2^{(n)})^2 - (a_1^{(n)})^2$.

There is no difficulty in defining $a_1^{(n)}$ from (i) and $b_1^{(n+1)}$ from (ii). From (ii) we see that $b_1^{(n+1)}$'s are increasing and $b_1^{(n+1)} > a_1^{(n)}$. Hence from (i) we have $1 < b_1^{(n)}/a_1^{(n-1)} = b_2^{(n)}/a_1^{(n)}$ or $(b_2^{(n)})^2 - (a_2^{(n)})^2 - (a_1^{(n)})^2 > 0$ so there is no obstacle to defining $b_2^{(n+1)}$ by (iii). Now to show boundedness. From (iii) we see $\{b_2^{(n)}\}$ is decreasing and thus converges which also implies that $\{a_1^{(n)}\}$ converges to zero. Combining (ii) and (iii) we have

$$(b_1^{(n+1)})^2 - (b_1^{(n)})^2 = (b_2^{(n)})^2 - (b_2^{(n+1)})^2 .$$

Thus

$$(b_1^{(n+1)})^2 - (b_2^{(3)})^2 = (b_2^{(3)})^2 - (b_2^{(n+1)})^2$$

which implies $\{b_1^{(n)}\}$ converges since $\{b_2^{(n)}\}$ does. Now B the normal extension of T must be a bounded operator.

We will compute the norm of this completion from (6). Let $A = \lim_{j\to\infty} |\, a_j \,|^2 = ||\, T \,||^2$. Then (6) yields

$$A = a_2^2 + C\left[\frac{1}{a_1^2} - \frac{1}{A}\right] \tag{7}$$

or

$$A^2 - (a_2^2 + C/a_1^2)A - C = 0 .$$

Recalling that $C = a_1^2 a_2^2[(a_3^2 - a_2^2)/(a_2^2 - a_1^2)]$ and selecting the proper root of the quadratic equation yields

$$A = \frac{1}{2}\left[a_2^2\frac{a_3^2 - a_1^2}{a_2^2 - a_1^2} + \left\{\left[a_2^2\frac{a_3^2 - a_1^2}{a_2^2 - a_1^2}\right]^2 - 4a_1^2a_2^2\frac{a_3^2 - a_2^2}{a_2^2 - a_1^2}\right\}^{1/2}\right] \tag{8}$$

(the expression inside the radical is positive).

Before we show that the subnormal operator T which we constructed to complete a_1, a_2, a_3 is minimal in norm, we will prove that it is an efficient subnormal completion in another sense. Let $S\varphi_j = c_j\varphi_{j+1}$ be any subnormal operator with $c_i = a_i$ for $i = 1, 2, 3$. Rather than be inundated by absolute value signs we will assume that both the a_i's and the c_j's are positive.

Now the necessary condition (I)

$$(a_j^{(2)})^2 + (b_j^{(2)})^2 - (a_{j-1}^{(2)})^2 \geqq 0$$

applied to the operator S yields the following analogue to (2):

$$c_n^2\frac{c_{n+1}^2 - c_n^2}{c_n^2 - c_{n-1}^2} \geqq c_{n-1}^2\frac{c_n^2 - c_{n-1}^2}{c_{n-1}^2 - c_{n-2}^2} - (c_n^2 - c_{n-1}^2) \tag{9}$$

and if we argue as before we can reduce this to

$$c_{n+1}^2 \geqq a_2^2 + C\left[\frac{1}{a_1^2} - \frac{1}{c_n^2}\right]. \tag{10}$$

If the completion c_n is different from the one given above then for some first k we have $c_k > a_k$ (it can not be less by the necessary condition (I)). But then we have

$$c_{k+1}^2 \geqq a_2^2 + C\left[\frac{1}{a_1^2} - \frac{1}{c_k^2}\right] > a_2^2 + C\left[\frac{1}{a_1^2} - \frac{1}{a_k^2}\right] = a_{k+1}^2 .$$

Hence $c_j > a_j$ for $j \geqq k$.

Now we will prove the last statement of the theorem. Let S and c_j be as above. Let k be the smallest integer such that $a_{k+2} = c_{k+2}$ but $a_{k+3} < c_{k+3}$. Using the necessary condition (I) we arrive at

$$a_{n+1}^2 = a_n^2 + D\left[\frac{1}{a_{n-1}^2} - \frac{1}{a_n^2}\right] \tag{11a}$$

and

$$a_{n+1}^2 \geqq c_n^2 + \hat{D}\left[\frac{1}{c_{n-1}^2} - \frac{1}{c_n^2}\right] \tag{11b}$$

for $n \geqq k + 3$ where

$$D = a_{k+2}^2 a_{k+1}^2 \frac{a_{k+3}^2 - a_{k+2}^2}{a_{k+2}^2 - a_{k+1}^2} \qquad \text{and} \qquad \hat{D} = c_{k+2}^2 c_{k+1}^2 \frac{c_{k+3}^2 - c_{k+2}^2}{c_{k+2}^2 - c_{k+1}^2} .$$

These are obtained by a restricted telescoping of (3) and (9). Since $c_{k+1} = a_{k+1}$, $c_{k+2} = a_{k+2}$ and $c_{k+3} > a_{k+3}$, certainly $\hat{D} > D$. Now from (11a) and (11b) we obtain

$$a_{n+1}^2 = a_{k+2}^2 + D\left[\frac{1}{a_{k+1}^2} - \frac{1}{a_n^2}\right] \tag{12a}$$

and

$$c_{n+1}^2 \geqq c_n^2 + \hat{D}\left[\frac{1}{c_{k+1}^2} - \frac{1}{c_n^2}\right] \qquad \text{for} \quad n \geqq k + 3 . \tag{12b}$$

If we let $n \longrightarrow \infty$ in (12a) and (12b) we obtain

$$A = a_{k+2}^2 + D\left[\frac{1}{a_{k+1}^2} - \frac{1}{A}\right] \tag{13a}$$

and

$$\hat{A} \geqq c_{k+2}^2 + \hat{D}\left[\frac{1}{a_{k+1}^2} - \frac{1}{\hat{A}}\right] \tag{13b}$$

or

$$\hat{A} \geqq a_{k+2}^2 + \hat{D}\left[\frac{1}{a_{k+1}^2} - \frac{1}{\hat{A}}\right].$$

Now in (13a) the solution A is a strictly increasing function of D and since $\hat{D} > D$ we must have $\hat{A} > A$. But $\| T \|^2 = A < \hat{A} = \| S \|^2$ and thus we have shown there is exactly one subnormal completion of a_1, a_2, a_3 with the norm given by (8) (again we note that operators are specified up to unitary equivalence only).

REMARK. The above construction of a subnormal completion of a_1, a_2, a_3 can be used in slightly more general circumstances. It has been pointed out that a_1, a_2, a_3, a_4 may not have a subnormal completion. If however a_4 satisfies the necessary condition

$$a_4^2 \geqq a_3^2 + a_1^2 \frac{(a_3^2 - a_2^2)^2}{a_2^2 - a_1^2}$$

then one can obtain a subnormal completion by defining a_n for $n \geqq 5$ as in (2). When one has $a_1, \cdots, a_5$ this definition does not work for reasons sufficiently complicated to remain unmentioned here.

THEOREM 6. *Let T be a monotone shift, $T\varphi_j = a_j\varphi_{j+1}$. Assume further that*

(i) $a_j \neq 0$ *for* $j = 1, 2, \cdots$

(ii) T *is subnormal*

(iii) $|a_k| = |a_{k+1}|$ *for some* k

then $|a_j| = |a_{j+1}|$ *for* $j = 2, 3, \cdots$, *and* a_1 *is arbitrary.*

Proof. We will assume that the a_j's are positive. Let k be the smallest integer such that $a_k = a_{k+1}$. If $k \leqq 2$ then by Theorem 4 of [5] we are through. For the case $k \geqq 3$ we now make use of Theorem 4. We observe that

$$a_k^{(2)} = a_k\left[\frac{a_{k+1}^2 - a_k^2}{a_k^2 - a_{k-1}^2}\right]^{1/2} = 0$$

and

$$a_{k-1}^{(2)} = a_{k-1}^2\left[\frac{a_k^2 - a_{k-1}^2}{a_{k-1}^2 - a_{k-1}^2}\right]^{1/2} \neq 0$$

and

$$b_k^{(2)} = [a_k^2 - a_{k-1}^2]^{1/2}.$$

Now a necessary condition for the subnormality of T is just that

$$0 \leqq (a_k^{(2)})^2 + (b_k^{(2)})^2 - (a_{k-1}^{(2)})^2$$
$$= (a_k^2 - a_{k-1}^2) - \frac{a_{k-1}^2}{a_{k-1}^2 - a_{k-2}^2}(a_k^2 - a_{k-1}^2) .$$

But that implies $(a_{k-1}^2)/a_{k-1}^2(a_{k-2}^2) \leqq 1$ or $a_{k-2} = 0$ contrary to hypothesis. This still leaves the existence of monotone shifts unsettled for $a_1 < a_2 = a_j$ for $j = 2, 3, \cdots$. We will show that these are all subnormal by writing down the subnormal extension. (The reader may do this by simply following the "directions" in Theorem 4.)

$$\left|\begin{array}{cccccc|cc|cc|cc|cc|c}
0 & & & & & & a_1 & 0 & & & & & & & \\
a_1 & 0 & & & & & 0 & p & & & & & & & \\
 & a_2 & 0 & & & & 0 & 0 & & & & & & & \\
 & & a_3 & \cdot & & & \cdot & \cdot & & & & & & & \\
 & & & \cdot & \cdot & & \cdot & \cdot & & & & & & & \\
 & & & & \cdot & \cdot & \cdot & \cdot & & & & & & & \\
\hline
 & & & & & \cdot & 0 & 0 & a_2 & & & & & & \\
 & & & & & & p & 0 & 0 & & & & & & \\
\hline
 & & & & & & & & 0 & a_2 & & & & & \\
\hline
 & & & & & & & & & 0 & a_2 & & & & \\
\hline
 & & & & & & & & & & 0 & \cdot & & & \\
 & & & & & & & & & & & \cdot & \cdot & & \\
 & & & & & & & & & & & & \cdot & \cdot & \\
 & & & & & & & & & & & & & \cdot &
\end{array}\right|$$

$$p = (a_2^2 - a_1^2)^{1/2}$$

COROLLARY. *The space $H^{(2)}$ is either infinite dimensional or* $\dim H^{(2)} \leqq 2$. *This is just another way of stating Theorem* (6).

REMARK. This corollary has an analogue for the spaces $H^{(n)}$. We state without proof that either $H^{(n)}$ is infinite dimensional or $\dim H^{(n)} \leqq n$. In the proof of Theorem 5 we constructed a subnormal operator where $\dim H^{(3)} = 2$. Given a_1, a_2, a_3, a_4, where a_4 satisfies the necessary condition (I), the construction yields an operator where $\dim H^{(3)} = 3$.

Using the last theorem we can construct several counter-examples. In [4] we encounter the following:

THEOREM. *If T is hyponormal and T^n is normal for some integer n then T is normal.*

The question arises whether the theorem remains valid with normal replaced by subnormal (in both hypothesis and conclusion). The answer is no. Define $T\varphi_1 = (1/4)\varphi_2$, $T\varphi_2 = (1/2)\varphi_3$, $T\varphi_i = \varphi_{i+1}$ for $i \geqq 3$. Clearly T is hyponormal, but not subnormal by the previous theorem, We claim T^n is subnormal for $n \geqq 2$. For $T^2\varphi_1 = (1/8)\varphi_3$, $T^2\varphi_2 = (1/2)\varphi_4$

and $T^2\varphi_i = \varphi_{i+2}$ for $i \geqq 3$. Thus if we let $g_j = \varphi_{2j-1}$ and $h_j = \varphi_{2j}$ for $j = 1, 2, \cdots$, then we may write T^2 as the direct sum of operators acting on $H_1 = \{\text{span of } g_j\}$ and $H_2 = \{\text{span of } h_j\}$. Now if $Fg_1 = (1/8)g_2$ and $F_1 g_j = g_{j+1}$ for $j \geqq 2$, then F_1 is subnormal on H_1. Similarly, if $F_2 h_1 = (1/2)h_2$ and $F_2 h_j = h_{j+1}$ for $j = 2$, then F_2 is subnormal on H_2. But T^2 on H equals $F_1 \oplus F_2$ on $H_1 \oplus H_2$ and hence T^2 is subnormal. For $n \geqq 3$, T^n can be shown to be subnormal by a similar argument.

We will now exhibit two subnormal operators which are similar but whose (minimal) normal extensions are not. Let $\{\varphi_j\}_{j=1}^{\infty}$ be an orthonormal basis for H. Define $T\varphi_j = \varphi_{j+1}$ for $j = 1, 2, 3, \cdots$, and $S\varphi_1 = (1/2)\varphi_2$, $S\varphi_j = \varphi_{j+1}$ for $j = 2, 3, \cdots$. Then S and T are subnormal by Theorem 6, they are similar and, in fact, the operator Q given by $Q\varphi_1 = (1/2)\varphi_1$ $Q\varphi_j = \varphi_j$ for $j \geqq 2$, effects the similarity. We claim that the minimal normal extensions are not similar. If they were, they would have to be unitarily equivalent since they are normal. The normal extension of T is unitary while the normal extension of S clearly is not which completes the argument.

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