SEMI-ALGEBRAS THAT ARE LOWER SEMI-LATTICES

Edward Joseph Barbeau
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This paper is concerned with uniformly closed sets of continuous real-valued functions defined on a compact Hausdorff space that are at the same time semi-algebras (wedges closed under multiplication) and lower semi-lattices. The principal result is that any such set can be represented as an intersection of lower semi-lattice semi-algebras of three elementary types. This is an adaptation of a similar theorem of Choquet and Deny for lower semi-lattice wedges. A modified form of the theorem is also given for the case that the lower semi-lattice semi-algebra is in fact a lattice.

Throughout, $E$ denotes a compact Hausdorff space and $C(E)$ the family of all continuous real-valued functions defined on $E$. For two functions $f$ and $g$ in $C(E)$, the functions $f \cap g$ and $f \cup g$, defined respectively for any point $\eta$ of $E$ by

$$(f \cap g)(\eta) = \min\{f(\eta), g(\eta)\}; \quad (f \cup g)(\eta) = \max\{f(\eta), g(\eta)\},$$

are also in $C(E)$. A subset $P$ of $C(E)$ is

(a) a lower semi-lattice if and only if $f, g \in P \Rightarrow f \cap g \in P$,
(b) an upper semi-lattice if and only if $f, g \in P \Rightarrow f \cup g \in P$,
(c) a lattice if and only if $P$ is both an upper and a lower semi-lattice,
(d) a wedge if and only if $f, g \in P \Rightarrow \alpha f + \beta g \in P$, for any non-negative real numbers $\alpha, \beta$,
(e) a semi-algebra if and only if $P$ is a wedge and $f, g \in P \Rightarrow fg \in P$,
(f) closed under squaring if and only if $f \in P \Rightarrow f^2 \in P$.

Choquet and Deny [4] determined those uniformly closed wedges contained in $C(E)$ which are semi-lattices in terms of certain classes of Radon measures which generate the dual wedge. The theorem for the lower semi-lattice case can be formulated as follows. Let $\sigma$ be a positive Radon measure and $\xi$ be a point of $E$. Define the sets

$$L_{\sigma, \xi} = \{f: f \in C(E), \sigma(f) \leq f(\xi)\},$$
$$L_{\sigma} = \{f: f \in C(E), \sigma(f) \leq 0\}.$$

Each of these is a uniformly closed wedge which is a lower semi-lattice. We use $W'$ to denote the dual wedge of all those Radon measures which take nonnegative values on the wedge $W$, and $\delta_{\xi}$ to
denote the Radon measure with unit mass all concentrated at the point $\xi$ of $E$.

**Theorem 1** (Choquet-Deny). Let $W$ be a uniformly closed wedge which is a lower semi-lattice contained in $C(E)$. Suppose that $\mathcal{L}_1$ is the family of all pairs $(\sigma, \xi)$, with $\xi$ a point of $E$ and $\sigma$ a positive Radon measure satisfying $\sigma(\{\xi\}) = 0$, such that $\delta_\xi - \sigma \in W'$; suppose that $\mathcal{L}_2$ is the family of all positive Radon measures $\sigma$ such that $-\sigma \in W'$. Then

$$W = \left[ \cap \{L_{\sigma,\xi}: (\sigma, \xi) \in \mathcal{L}_1\} \right] \cap \left[ \cap \{L_{\sigma}: \sigma \in \mathcal{L}_2\} \right].$$

For the proof, see [4]; the result is valid even if $\mathcal{L}_1$ is void or $\mathcal{L}_2$ consists of the zero measure alone. (The convention that a void intersection is the whole of the space is adopted.) An analogous theorem holds for upper semi-lattices. These results were originally given in a more general setting with the underlying space not necessarily compact, but with the function space given the topology of uniform convergence on compacta.

F. F. Bonsall, [1], [2], considered the relationship between lattice and algebraic properties of a function wedge. He showed that any uniformly closed semi-algebra $A$ containing the function 1 and contained in $C^+(E)$ (the set of all nonnegative functions in $C(E)$) is a lattice if and only if it has the “type 1 property”, i.e.,

$$f \in A \implies f/(1 + f) \in A.$$ 

In addition, he gave an interesting characterization of such semi-algebras as sets of functions monotone with respect to certain quasi-orderings on $E$. In [2], Bonsall gave intersection theorems for certain closed wedges and semi-algebras contained in $C^+(E)$ which were upper semi-lattices and permitted reduction by constants. (A subset $K$ of $C(E)$ permits reduction by constants if and only if $f \in K$, $\lambda \geq 0 \implies (f - \lambda) \cup 0 \in K$.)

The main purpose of the present paper is to show that any uniformly closed lower semi-lattice semi-algebra contained in $C(E)$ is an intersection of ones of certain elementary types. The result obtained does not require the full force of the multiplication property of a semi-algebra, but only closure under squaring; its proof depends heavily on Theorem 1. In the final section, a similar intersection theorem for lattices is deduced. Unlike earlier results for semi-algebras, the theorems here are not restricted to nonnegative functions.

Because of the asymmetry introduced into the situation by the multiplication, one cannot trivially obtain a corresponding result for upper semi-lattice semi-algebras. It seems that the class of these
semi-algebras is much more extensive and varied than the class of lower semi-lattices, so that a complete determination is still in the future.

By abuse of notation, we use, for any Radon measure $\sigma$, the symbol $\sigma$ to refer both to the continuous linear functional defined on $C(E)$ and to the corresponding regular measure defined on the Borel subsets of $E$, but no confusion will result from this. The support of $\sigma$ in $E$ is denoted by $S(\sigma)$.

2. The Principal Result. Let $\sigma$ be a positive Radon measure with support $S(\sigma)$, $\xi$ and $\zeta$ be two points of $E$ and $N$ a closed subset of $E$. Then it is clear that each of the sets

$$A_{\sigma, \xi} = \{f: f \in C(E), \sigma(f) \leq f(\xi), 0 \leq f(\eta) \leq f(\xi) \ (\forall \eta \in S(\sigma))\}$$

$$B_{\xi, \zeta} = \{f: f \in C(E), f(\xi) = f(\zeta)\}$$

$$C_N = \{f: f \in C(E), f(\eta) = 0 \ (\forall \eta \in N)\}$$

is a uniformly closed semi-algebra which is a lower semi-lattice, and that any intersection of sets of these forms is such a semi-algebra. It will be shown that every uniformly closed lower semi-lattice semi-algebra is an intersection of sets of the forms $A_{\sigma, \xi}$, $B_{\xi, \zeta}$ and $C_N$.

**Lemma 1.** Let $A$ be a closed subwedge of $C(E)$ closed under squaring, and suppose that $\delta_1 - \sigma \in A$ where $\sigma$ is a positive Radon measure on $E$ and $\xi$ is a point of $E$. Then, for $f \in A$,

$$|f(\eta)| \leq |f(\xi)|$$

whenever $\eta \in S(\sigma)$.

**Proof.** If $f \in A$, then $f^2 \in A$. Suppose $f(\xi) = 0$. Then $(-\sigma)(f^2) = 0$, so that $f^2$ vanishes almost everywhere ($\sigma$). Hence $f(\eta) = f^2(\eta) = 0$ whenever $\eta \in S(\sigma)$.

On the other hand, if $f \in A$ and $f(\xi) = \lambda \neq 0$, then $g = \lambda^{-\xi}f^2 \in A \cap C^+(E)$ and $g(\xi) = 1$. Define

$$G = \{\eta: \eta \in E, g(\eta) > 1\} = \{\eta: \eta \in E, |f(\eta)| > |f(\xi)|\}.$$ 

If $G$ is void, then $|f(\eta)| \leq |f(\xi)|$ for $\eta$ belonging to $E$, and, a fortiori, to $S(\sigma)$. If $G$ is nonvoid, then $G$, being open, is $\sigma$-integrable. Let $K$ be any compact subset of $G$, and let $\lambda_K = \inf \{g(\eta): \eta \in K\}$. Since $g$ attains its minimum on $K$, $\lambda_K > 1$. For $m$ any power of 2, $g^m$ belongs to $A$, and $\phi_K$, the characteristic function of $K$, satisfies $\phi_K \leq \lambda_K^{-m}g^m$, so that

$$\sigma(K) \leq \lambda_K^{-m}\sigma(g^m) \leq \lambda_K^{-m}g^m(\xi) = \lambda_K^{-m}.$$
Hence \( \sigma(K) = 0 \). Since \( \sigma(G) = \sup \{ \sigma(K): K \text{ compact}, K \subseteq G \} = 0 \), then \( G \cap S(\sigma) = \emptyset \) and the result follows.

**Lemma 2.** Let \( \mathcal{L}_1 \) and \( \mathcal{L}_2 \) be the families defined as in Theorem 1 with respect to the uniformly closed wedge \( A \), and suppose that \( A \) is closed under squaring. Let

\[ N = \{ \eta: \eta \in E, f(\eta) = 0 \ (\forall f \in A) \}. \]

Then:

(a) \( \sigma \in \mathcal{L}_2 \) if and only if \( S(\sigma) \subseteq N \);

(b) if \( (\sigma, \xi) \in \mathcal{L}_1 \) and every function in \( A \) takes a nonnegative value at the point \( \xi \), then \( A \subseteq A_{\sigma, \xi} \);

(c) if \( (\sigma, \xi) \in \mathcal{L}_1 \) and some function in \( A \) takes a negative value at the point \( \xi \), then there exists closed disjoint subsets \( M_0 \) and \( M_1 \) of \( S(\sigma) \) (either possibly void) such that

\[
\begin{align*}
& (i) \quad M_0 \cup M_1 = S(\sigma), \\
& (ii) \quad M_0 \subseteq N, \\
& (iii) \quad \eta \in M_1 = A \subseteq B_{\xi, \eta}, \\
& (iv) \quad \sigma(M_1) = 1.
\end{align*}
\]

**Proof.** (a) If \( A \subseteq L_\sigma \) and \( f \in A \), then \(( - \sigma ) f^2 = 0\), from which \( f^2 \), and hence \( f \), vanishes on \( S(\sigma) \). This part is now clear.

(b) It must be shown that whenever \( \eta \in S(\sigma) \) and \( f \in A \), then \( 0 \leq f(\eta) \leq f(\xi) \). By Lemma 1, we know that \( |f(\eta)| \leq f(\xi) \). Suppose, if possible, that, for some point \( \zeta \) in \( S(\sigma) \), some positive \( \varepsilon \) and some \( f \in A \), \( f(\zeta) = -\varepsilon \). Choose a positive integer \( m \) such that \( f(\zeta) < me \), and let \( h = f \cap mf \). Then \( h \in A \) and \( |h(\xi)| = -h(\zeta) = me > f(\xi) = h(\zeta) \), so that Lemma 1 is contradicted.

(c) Let \( f \), \( g \in A \) and suppose that \( f(\xi) = -1 \), \( g(\xi) = +1 \). Then \( f + g \in A \) and \((f + g)(\xi) = 0 \), so that, by Lemma 1, \((f + g)(\eta) = 0 \) for every point \( \eta \) in \( S(\sigma) \). In particular, \((f + f^2)(\eta) = 0 \) for \( \eta \in S(\sigma) \), with the consequence that \( f \) takes only the values 0 and \( -1 \) on \( S(\sigma) \). Define \( M_0 = S(\sigma) \cap \{ \eta: f(\eta) = 0 \} \) and \( M_1 = S(\sigma) \cap \{ \eta: f(\eta) = -1 \} \). Evidently \( M_0 \) and \( M_1 \) are closed, disjoint sets satisfying (i).

Let \( h \in A \). If \( h(\xi) = 0 \), then, by Lemma 1, \( h \) vanishes everywhere on \( S(\sigma) \). If \( h(\xi) > 0 \), then the argument of the last paragraph with \( g \) replaced by \((h(\xi))^{-1}h \) yields \( f(\eta) + (h(\xi))^{-1}h(\eta) = 0 \) for each point \( \eta \) in \( S(\sigma) \). If \( h(\xi) < 0 \), then \(( - h(\xi))^{-1}h \in A \), and the argument of the last paragraph applied to \(( - h(\xi))^{-1}h \) and \( f^2 \) yields \(( - h(\xi))^{-1}h(\eta) + f^2(\eta) = 0 \) for each point \( \eta \) in \( S(\sigma) \). In any case

\[
h(\eta) = \begin{cases} 0 & (\eta \in M_0) \\ h(\xi) & (\eta \in M_1) \end{cases},
\]

so that (ii) and (iii) are true. Part (iv) may be seen by noting that, for \( h \in A \), \( h(\xi)\sigma(M_1) = \sigma(h) \leq h(\xi) \) and both positive and negative
values are possible for $h$ at $\xi$.

**Theorem 2.** Let $A$ be a uniformly closed subwedge of $C(E)$ such that (i) $A$ is a lower semi-lattice,

(ii) $A$ is closed under squaring.

Let $\mathcal{F}_1$ be the family of all pairs $(\sigma, \xi)$, with $\xi$ a point of $E$ and $\sigma$ a positive Radon measure satisfying $\sigma(\{\xi\}) = 0$, such that $A \subseteq A_{\sigma, \xi}$; let $\mathcal{F}_2$ be the family of all pairs $(\xi, \zeta)$ of distinct points of $E$ such that $A \subseteq B_{\xi, \zeta}$; let $N = \{\eta : \eta \in E, f(\eta) = 0 (\forall f \in A)\}$.

Then:

(I) \[ A = \left[ \bigcap \{ A_{\sigma, \xi} : (\sigma, \xi) \in \mathcal{F}_1 \} \right] \cap \left[ \bigcap \{ B_{\xi, \zeta} : (\xi, \zeta) \in \mathcal{F}_2 \} \right] \cap C_N. \]

**Proof.** By Theorem 1, with $\mathcal{L}_1$ and $\mathcal{L}_2$ defined with reference to $A$, we have that $A$ is the intersection of all sets of the form $L_{\sigma, \xi}$ with $(\sigma, \xi) \in \mathcal{L}_1$ and of the form $L_\sigma$ with $\sigma \in \mathcal{L}_2$. Denote by $F$ the set on the right hand side of (I). Clearly, $A \subseteq F$. On the other hand, if $f \in F$, then, by Lemma 2(a), $f \in L_\sigma$ for each $\sigma \in \mathcal{L}_2$. Let $(\sigma, \xi) \in \mathcal{L}_1$. If every function in $A$ is nonnegative at $\xi$, then, by Lemma 2(b), $A \subseteq A_{\sigma, \xi}$, so that $(\sigma, \xi) \in \mathcal{F}_1$ and $F \subseteq A_{\sigma, \xi} \subseteq L_{\sigma, \xi}$. If some function in $A$ is negative at $\xi$, then there is a decomposition $\{M_0, M_1\}$ of $S(\sigma)$ satisfying the conditions of Lemma 2(c). Let $f \in F$. Then $f$ belongs to $C_N$ and so vanishes on $M_0 \subseteq N$. Also, if $\eta \in M_1$, then $A \subseteq B_{\xi, \eta}$, so that $(\xi, \eta) \in \mathcal{F}_2$ and $f \in B_{\xi, \eta}$, i.e., $f(\eta) = f(\xi)$. Since $\sigma(M_1) = 1$, this yields $f(\xi) = f(\xi) \sigma(M_1) = \sigma(f)$, so that $f \in L_{\sigma, \xi}$. In either case, $F \subseteq L_{\sigma, \xi}$. Hence

\[ A \subseteq F \subseteq \left[ \bigcap \{ L_{\sigma, \xi} : (\sigma, \xi) \in \mathcal{L}_1 \} \right] \cap \left[ \bigcap \{ L_\sigma : \sigma \in \mathcal{L}_2 \} \right] = A. \]

**Remark.** The result is valid if any of $\mathcal{F}_1$, $\mathcal{F}_2$ and $N$ are void. If $\mathcal{F}_1$ is void, $A$ is a lattice, so that the property of being a lower semi-lattice but not a lattice forces all the functions in $A$ to be nonnegative at least on a nonvoid subset of $E$.

**Consequences of Theorem 2.** (a) Since all sets of the forms $A_{\sigma, \xi}$, $B_{\xi, \zeta}$ and $C_N$ are semi-algebras, the wedge $A$ satisfying the conditions of Theorem 2 is automatically a semi-algebra.

(b) Theorem 2 holds if the condition (ii) is strengthened to "$A$ is a semi-algebra".

(c) Any wedge $A$ contained in $C^+(E)$ which satisfies the conditions of Theorem 2 is an ideal of some semi-algebra $T$ which has the type 1 property. For $(\sigma, \xi) \in \mathcal{F}_1$, let

\[ T_{\sigma, \xi} = \{ f : f \in C^+(E), f(\eta) \leq f(\xi) (\forall \eta \in S(\sigma)) \}. \]

Then $A_{\sigma, \xi} \cap C^+(E)$ is an ideal of $T_{\sigma, \xi}$, so that $T$ may be taken to be
(d) Any wedge $A$ contained in $C^+(E)$ which satisfies the conditions of Theorem 2 and in addition contains the function 1 has the type 1 property, and hence is a lattice.

3. Semi-algebras which are lattices. In this section, let $A$ be a uniformly closed semi-algebra contained in $E$ which is a lattice. Since $A$ is in particular a lower semi-lattice, the representation (I) given in Theorem 2 is valid, and, in fact, when $\mathcal{F}_1$ is void, expresses $A$ as an intersection of lattices. However if $\mathcal{F}_1$ is nonvoid, then (I) is unsatisfactory since semi-algebras of the form $A_{\sigma,\xi}$ are not lattices unless $\sigma$ is either the zero measure or has all of its mass concentrated at one point. This section will be concerned with modifying the family $\mathcal{F}_1$ so that $A$ is given as the intersection of certain elementary lattices.

Suppose $\mathcal{F}_1$ contains the pair $(\sigma, \xi)$ with $S(\sigma)$ containing at least two points. For $\eta \in S(\sigma)$, define the function $p = p_{\sigma,\xi}$ by

$$p(\eta) = \sup \{ f(\eta) : f \in A, f(\xi) = 1 \}.$$

(There is no loss of generality in supposing that the supremum is taken over a nonvoid set, for otherwise $S(\sigma) \cup \{\xi\}$ would be a subset of $N$, defined as in Theorem 2.) Note that $0 \leq p(\eta) \leq 1$ for each point $\eta$ of $S(\sigma)$ and that $p(\eta) = 0$ if and only if $\eta \in N$. The set

$$P_{\sigma,\xi} = \{ f : f \in C(E), p(\eta) f(\xi) \geq f(\eta) \geq 0 (\forall \eta \in S(\sigma)) \} = C_N \cap S(\sigma) \cap \{ A_{\sigma,\xi} : \rho = p(\eta)^{-1} \delta_\eta, \eta \in S(\sigma) \setminus N \}$$

is a uniformly closed lattice semi-algebra which contains $A$. We show that $P_{\sigma,\xi} \subseteq A_{\sigma,\xi}$, so that

$$A = \bigcap \{ P_{\sigma,\xi} : (\sigma, \xi) \in \mathcal{F}_1 \} \cap \bigcap \{ B_{\xi,\zeta} : (\xi, \zeta) \in \mathcal{F}_2 \} \cap C_N = A.$$

Let $u \in P_{\sigma,\xi}$. If $u(\xi) = 0$, then $u(\eta) = 0$ for each $\eta$ belonging to $S(\sigma)$ so that $\sigma(u) = 0 = u(\xi)$ and $u \in A_{\sigma,\xi}$. If $u(\xi) \neq 0$, suppose, with no loss of generality, that $u(\xi) = 1$. Since $u(\eta) \leq p(\eta)$ for $\eta \in S(\sigma)$ and since $u$ is continuous, for given positive $\epsilon$ and given point $\zeta \in S(\sigma)$, there exists a function $f_\zeta \in A$ and an open subset $V_\zeta$ of $E$ such that $\zeta \in V_\zeta$, $f_\zeta(\xi) = 1$ and $f_\zeta(\eta) > u(\eta) - \epsilon$ for each point $\eta$ of $V_\zeta \cap S(\sigma)$. Because $S(\sigma)$ is compact, there exists a finite set $\zeta_1, \zeta_2, \ldots, \zeta_k$ of points of $S(\sigma)$ such that

$$S(\sigma) \subseteq \cup \{ V_{\zeta_i} : i = 1, 2, \ldots, k \}.$$

The function $f = f_{\zeta_1} \cup f_{\zeta_2} \cup \cdots \cup f_{\zeta_k}$ belongs to $A$ and $f(\xi) = 1$, $f(\eta) > u(\eta) - \epsilon$ for each point $\eta$ of $S(\sigma)$. Hence
\[ \sigma(u) \leq \sigma(f + \varepsilon) = \sigma(f) + \sigma(\varepsilon) \leq f(\xi) + \sigma(\varepsilon) = 1 + \sigma(\varepsilon). \]

Since \( \sigma(u) \leq 1 + \sigma(\varepsilon) \) for each positive \( \varepsilon \), \( \sigma(u) \leq 1 \). It is deduced that for any function \( u \) in \( P_{\sigma,\xi} \), \( u \) belongs to \( A_{\sigma,\xi} \).

We can now obtain the following result.

**Theorem 3.** Let \( A \) be a uniformly closed subwedge of \( C(E) \) which is a lattice and closed under squaring. Let \( \mathcal{F}_{1} \) be the family of all pairs \((\sigma, \xi)\), with \( \xi \) a point of \( E \) and \( \sigma \) a positive Radon measure which either is the zero measure or has total mass at least unity all concentrated at a point distinct from \( \xi \), such that \( A \subseteq A_{\sigma,\xi} \); let \( \mathcal{F}_{2} \) and \( N \) be defined as in Theorem 2. Then the equation

\[
A = \left[ \bigcap \{ A_{\sigma,\xi} : (\sigma, \xi) \in \mathcal{F}_{1} \} \right] \cap \left[ \bigcap \{ B_{\xi,\eta} : (\xi, \eta) \in \mathcal{F}_{2} \} \right] \cap C_{N}
\]

expresses \( A \) as an intersection of uniformly closed lattice semi-algebras.

**Remark.** If the wedge \( A \) is contained in \( C^{+}(E) \), then a simpler representation for \( A \) is possible. Define for \( 0 \leq \alpha \leq 1 \) and points \( \xi, \eta \) of \( E \) the set

\[
Q_{\alpha,\xi,\eta} = \{ f : f \in C^{+}(E), \alpha f(\xi) \geq f(\eta) \}.
\]

Then \( A \) can be expressed as an intersection of semi-algebras of the form \( Q_{\alpha,\xi,\eta} \). (Observe that \( C^{+}(E) \subseteq A_{0,\xi} \), that \( B_{\xi,\xi} \cap C^{+}(E) = Q_{1,\xi,\xi} \cap Q_{1,\xi,\xi} \) and that \( C_{N} \cap C^{+}(E) = \cap \{ Q_{0,\xi,\eta} : \xi \in E, \eta \in N \} \).)

**Bibliography**


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