THE KLEIN GROUP AS AN AUTOMORPHISM GROUP WITHOUT FIXED POINT

STEVEN FREDRICK BAUMAN
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S. F. Bauman

An automorphism group $V$ acting on a group $G$ is said to be without fixed points if for any $g \in G$, $v(g) = g$ for all $v \in V$ implies that $g = 1$. The structure of $V$ in this case has been shown to influence the structure of $G$. For example if $V$ is cyclic of order $p$ and $G$ finite then John Thompson has shown that $G$ must be nilpotent. Gorenstein and Herstein have shown that if $V$ is cyclic of order 4 then a finite group $G$ must be solvable of $p$-length 1 for all $p \mid |G|$ and $G$ must possess a nilpotent commutator subgroup.

In this paper we will consider the case where $G$ is finite and $V$ noncyclic of order 4. Since $V$ is a two group all the orbits of $G$ under $V$ save the identity have order a positive power of 2. Thus $G$ is of odd order and by the work of Feit-Thompson $G$ is solvable. We will show that $G$ has $p$-length 1 for all $p \mid |G|$ and $G$ must possess a nilpotent commutator subgroup.

REMARK. It would be interesting to have a direct proof of solvability without resorting to the work of Feit-Thompson.

From now on in this paper $G$ represents a finite group admitting $V$ as a noncyclic four group without fixed points. If $X$ is a group admitting an automorphism group $A$ then $Z(X), \Phi(X), X - A$ will be respectively the center of $X$, the Frattini subgroup of $X$ and the semi-direct product of $S$ by $A$ in the holomorph of $X$. All other notations are standard.

Suppose $V = \{v_1, v_2, v_3\}$ where the $v_i$ are the nonidentity elements of $V$. Denote by $G_i$ the set of elements which are left fixed by $v_i$. These are easily seen to be $V$-invariant subgroups of $G$ and by a result of Burnside ([1] p. 90) $G_i$ are Abelian and $v_j$ restricted to $G_i$ is the inverse map if $i \neq j$. These subgroups $G_i$ are in a sense the building blocks of $G$.

**Lemma 2.** ([4] p. 555)

(i) $|G| = |G_1||G_2||G_3|$

(ii) $G = G_1G_2G_3$

(iii) Every element $g \in G$ has a unique decomposition $g = g_1g_2g_3$, $f_i \in G_i$.

**Lemma 2.** If $|G| = hm$ where $(h, m) = 1$ then $G$ contains a unique $V$ invariant group $H$ such that $|H| = h$. 
Proof. Since $G$ is solvable by Hall ([5] p. 141) groups of order $h$ exist and are conjugate in $G$. Thus there exist an odd number of them permuted by $V$. Since all the orbits have order power of 2 at least one group say $H$ is $V$ invariant. By Lemma 1 $H = H_1H_2H_3$ where clearly $H_i = H \cap G_i$ and $(G : H) = (G_1 : H_1)(G_2 : H_2)(G_3 : H_3)$. Thus the $H_i$ are Hall subgroups of the Abelian $G_i$ and thus uniquely determined by $G_i$ rather than $H$.

The decomposition of a $V$ invariant group $X$ into $X_1X_2X_3$ will play an important role in what will follow. The $X_1 \subseteq G_i$ are always $V$ invariant and it is clear that if $|X_i| = 1$ for any $i$ then $X$ is Abelian. For example if $X = X_1X_2$ is $V$ invariant and normalized by a subgroup $K_1 \subseteq G_i$ then $K_1X = K_1X_1X_2$ is Abelian. Thus subgroups of the complex $G_1G_2$ are centralized by elements in $G_1G_2$ which normalize them. If $X = X_1$ then even a stronger statement is available.

Lemma 3. If $X \subseteq G_i$, then $N_o(X) = C_o(X)$.

Proof. Suppose $i = 1$. It is easy to see that $N = N_o(X)$ is $V$ invariant and thus $N = N_1N_2N_3$. By the above remark $XN_2$ and $XN_3$ are Abelian. Since $XN_1$ is Abelian, the result follows.

Before we continue to the main results, we must examine the inheritance properties of groups admitting automorphism groups without fixed points. If $G$ is such a group and $H$ is a $V$ invariant subgroup, then clearly $H$ is also such a group. If $K$ is a normal $V$ invariant subgroup of $G$, there exists the canonical way of inducing $V$ on $G/K$. This definition gives rise to an automorphism group $\tilde{V}$ acting on $G/K$.

Lemma 4. ([4] p. 556) In above situation
(i) $\tilde{V}$ is without fixed points on $G/K$.
(ii) $(G/K)_i = G_iK/K$.

Lemma 5. Suppose $V$ acts on $M$ and $A$ without fixed points. Suppose also that $A$ is an elementary Abelian $p$-group where $(p, |M|) = 1$ and $M$ is acting faithfully on $A$. If the complex $M_1M_2$ is a normal subgroup of $M$ and $A_i \neq \{1\} \ i = 1, 2, 3$ then $A$ is $M-V$ reducible.

Proof. By Maschke’s theorem it will suffice to show that some proper subgroup of $A$ is $M-V$ invariant. Now $C = C(A_1A_2A_3)$ is $M-V$ invariant and $C \neq A$ since $M$ acts faithfully. Hence if $|C| \neq 1$ we are done and so we may assume $|C| = 1$. Now $A_2A_3$ is the subset of $A$ inverted by $v_1$ and so is invariant under $M_1 - V$. Set $K = \cap m_2(A_2A_3)$ where the intersection is taken over all $m_2 \in M_2$. Since $M_1 - V$ normal-
izes \( M_2, K \) is \( M_1M_2 - V \) invariant. Furthermore, \( A_2 \subseteq K \) since \( M_2 \) centralizes \( A_2 \). If \( K = A_2 \), then \( M_1M_2 \) centralizes \( K \) so \( K \subseteq C = \{1\} \) contrary to the fact that \( |A_2| \neq 1 \). Thus we must have that \( |K \cap A_2| \neq 1 \). But then if \( R = \cap m_\alpha(K) \) where the intersection is taken over all \( m_\alpha \in M_2 \) we have that \( \{1\} \subseteq A \cap K_3 \subseteq R \subseteq A_1A_2 \subseteq A \). Since \( M_1M_2 \) is normal in \( M \) and \( M_3 \) is \( V \) invariant it follows that \( R \) is \( M - V \) invariant and proper in \( A \). This completes the proof of the lemma.

**Theorem 1.** For all \( p ||G| \) \( G \) has \( p \)-length 1.

**Proof.** We prove the theorem by induction on \( |G| \). We may assume \( G \) has no normal \( p' \)-groups and \( P_0 \neq \{1\} \) is the maximal normal \( p \)-group of \( G \). By Hall ([5] p. 332) we have \( C_\theta(P_0) \subseteq P_0 \). By Lemma 2, the fact that \( P_0 \) is self centralizing and induction, we may assume \( G = PQ = QP \) where \( P \) and \( Q \) are \( V \) invariant \( p \) and \( q \) Sylow groups of \( G \). By induction we also get that \( QP_0 \triangleleft G, (P : P_0) = p \) and \( P_0 \) is elementary Abelian. By ([2] p. 795) \( Q \) possesses a characteristic subgroup \( C \) such that class \( (C) \leq 2, C/Z(C) \) is elementary Abelian and the only automorphisms of \( G \) which become the identity when restricted to \( C \) have order a power of \( q \). \( PC \) is then a \( V \) invariant group and by induction if \( C \neq \{1\} \), since \( P_0 \) is self centralizing we get \( P \triangleleft PC \). Thus \( PC/P_0 = P/P_0 \times C P_0/P_0 \). Since \( P/P_0 \) does not centralize \( QP_0/P_0 \) this contradicts the choice of \( C \). Thus \( Q = C \). Since \( P \) is normal in any proper \( V \) invariant subgroup containing it we get that \( (P/P_0 - V) \) is irreducible on \( QP_0/\Phi(Q)P_0 \). Thus either \( Q \) is Abelian or \( Z(Q) \subseteq \Phi(Q) \). Since \( Q/Z(Q) \) is elementary we get that \( Z(Q) = \Phi(Q) \). Thus either \( Q \) is Abelian or nonabelian of class 2 with \( Z(Q) = \Phi(Q) \). Since \( |P/P_0| = p \) we may suppose \( P/P_0 = (P/P_0)_3 \). By the irreducibility of \( P/P_0 - V \) on \( QP_0/\Phi(Q)P_0 \) we have that either \( Q_1Q_2 \subseteq \Phi(Q) \) or \( Q_3 \subseteq \Phi(Q) \). The first possibility implies that \( P/P_0 \) centralizes \( QP_0/\Phi(Q)P_0 \) and thus \( P \) would be normal in \( G \). Thus we have that \( Q_3 \subseteq Z(Q) \) and since \( Q/Q_3 \) is Abelian we have \( Q_1Q_2 \triangleleft Q \) and \( Q_2Q_3 \triangleleft Q \).

Since \( Q_1Q_2 \) does not centralize \( P_0 \), there exists an irreducible \( Q - V \) submodule \( \Lambda \) of \( P_0 \) which is not centralized by \( Q_1Q_2 \). Thus \( \Lambda \neq \{1\} \). Since \( QP_0/P_0 \triangleleft G_0/P_0 \) we have that \( \cap x(\Lambda) \) where \( x \) ranges through \( P/P_0 \) is a \( G/P_0 - V \) subspace of \( \Lambda \). Since \( P/P_0 = (P/P_0)_3 \) and \( \Lambda \neq \{1\} \) this space is not the identity space. By the irreducibility of \( \Lambda \) as a \( Q - V \) space we get that \( \Lambda \) is also \( G/P_0 - V \) irreducible. If \( \Lambda_i = \{1\} \) \( i = 1 \) or \( i = 2 \) we get that \( (P/P_0)_3 \subset \text{Ker} \theta \) where \( \theta \) maps \( G/P_0 \) into \( \text{Aut}(\Lambda) \). Since \( Q \) does not centralize \( \Lambda \) this mapping is not the identity and the result follows by induction. Thus \( \Lambda = \Lambda_1 \Lambda_2 \Lambda_3 \) where \( \Lambda_i \neq \{1\} \) \( i = 1, 2, 3 \).

We have that \( \Lambda \) admits \( G/P_0 \) and thus form the extension \( G^* = \Lambda \cdot G/P_0 \). \( G^* \) is \( V \) invariant and if \( |G^*| < |G| \) we may apply induction.
to \(G^*\). Let \(R/P_0\) be the maximal normal \(q\)-subgroup of \(G^*\). Since \(Q\) does not centralize \(A\) we have that \(R/P_0\) is a proper \(V\) invariant subgroup of \(Q P_0/P_0\). Since \(G^*\) has \(p\)-length 1, \(A(PR/P_0) \triangleleft G^*\). Thus \(PR/P_0 \triangleleft G/P_0\) and \(PR \neq G\). We are done by induction on \(PR\). We may assume that \(A = P_0\). But since \(A_i \neq \{1\} i = 1, 2, 3\) and \(Q - B\) is faithful irreducible on \(P_0\) we have a contradiction to Lemma 5. This completes the proof of Theorem 1.

**Theorem 2.** If \(G\) admits \(V\) without fixed points then \(G' = (G, G)\) is nilpotent.

**Proof.** Suppose \(G\) contains two distinct minimal normal \(V\) invariant subgroups \(N_1\) and \(N_2\). If \(N_1\) is disjoint from \(G'\) then by induction on \(G/N_1\) the theorem is proved. If \(N_1\) and \(N_2\) are in \(G'\) then by induction \(G'/N_1\) and \(G'/N_2\) are nilpotent. The minimality of \(N_i\) imply that the mapping of \(G'\) into \(G'/N_1 \times G'/N_2\) is an imbedding and thus again we are done. Therefore \(G\) contains a unique minimal normal \(V\) invariant group. It is an elementary Abelian \(p\)-group \(P_0\) which is characteristic. \(G\) must contain no normal \(p'\)-groups and by Theorem 1 we have that \(G\) has a normal \(p\)-Sylow group \(P\). Now \(C_\delta(P) = Z(P) \times K\) where \(K\) is a characteristic therefore \(V\) invariant \(p'\)-group of \(C_\delta(P)\). Since \(C_\delta(P)\) \(G\) we get that \(C_\delta(P) = Z(P) \subseteq P\). Consider \(G/\Phi(P)\). If induction applies \(G'\Phi(P)/\Phi(P)\) is a nilpotent group and since \(C_{G/\Phi(P)}(P/\Phi(P)) = P/\Phi(P)\) we must have that \(G'\Phi(P)/\Phi(P)\) is a \(p\)-group and therefore so is \(G'\). Thus we have that \(P\) is elementary Abelian. Let \(M\) be a \(V\) invariant complement to \(P\) in \(G\). By Maschke's theorem and the remark on the number of minimal \(V\) invariant normal subgroups of \(G\) we have that \(P = P_0\) and \(P\) is \((M - V)\) irreducible.

Consider any proper \(V\) invariant subgroup \(K\) of \(M\). Then \(PK \subset G\). By induction \(PK\) has a nilpotent commutator subgroup. Since \(C_{PK}(P) = P\) this must be a \(p\)-group and therefore contained in \(P\). Since \(PK/P \cong K\) we must have that \(K\) is Abelian. Thus every proper \(V\) invariant subgroup of \(M\) is Abelian. If \(M\) is Abelian then \(G' \subseteq P\) and we are done. We assume henceforth that \(M\) is not Abelian. Thus \(M = M_1 M_2 M_3\) where \(M_i \neq \{1\}\) for any \(i\). Since \(C_\delta(P) = P\), \(P = P_1 P_2 P_3\) where \(P_i \neq \{1\}\) for any \(i\).

If \(M\) contains two \(V\) invariant subgroups \(K\) and \(L\) of prime index, then since these are both Abelian we get that some \(M_i\) say \(M_i \subset Z(M)\). Thus \(M_1 M_2\) and \(M_3 M_4\) are normal in \(M\). \(M - V\) is faithful and irreducible on \(P\). This situation is in contradiction to Lemma 5. Since \(M\) is solvable and \(V\) invariant, we have a \(V\) invariant Sylow system. If more than two primes divide \(|M|\) then we would have \(M\) Abelian. If \(M\) is a \(q\)-group for some prime \(q\), we can get an \(M_2 M_3 M\). Thus to avoid this case we are forced to the following situation. \(R\) and \(S\)
are $V$ invariant $r$ and $s$ Sylow subgroups, each is Abelian and $M = RS = SR$. We may suppose that $M$ contains a $V$ invariant normal Abelian subgroup $K$ such that $(M: K) = s$. Thus $RM$ and $S$ is cyclic. Thus $S \subseteq M_i$ for some $i$. To be specific suppose $S \subseteq M_i$. Then by Maschke’s theorem applied to $S_i$ acting on $R_1R_2R_3/\Phi(R)$ we get that $R_1R_3 = M_1M_3$ is normalized by $S_i$ and thus $M_2M_3 M$. We have $(M - V)$ irreducible and faithful on $P = P_1P_2P_3$, and we again contradict Lemma 5. This completes the proof of Theorem 2.

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