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FRATTINI SUBGROUPS AND Φ -CENTRAL GROUPS

HOMER FRANKLIN BECHTELL, JR.

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Φ -central groups are introduced as a step in the direction of determining sufficiency conditions for a group to be the Frattini subgroup of some finite p -group and the related extension problem. The notion of Φ -centrality arises by uniting the concept of an E -group with the generalized central series of Kaloujnine. An E -group is defined as a finite group G such that $\Phi(N) \leq \Phi(G)$ for each subgroup $N \leq G$. If \mathcal{H} is a group of automorphisms of a group N , N has an \mathcal{H} -central series $N = N_0 > N_1 > \cdots > N_r = 1$ if $x^{-1}x^a \in N_j$ for all $x \in N_{j-1}$, all $a \in \mathcal{H}$, x^a the image of x under the automorphism $a \in \mathcal{H}$, $j = 0, 1, \dots, r-1$.

Denote the automorphism group induced on $\Phi(G)$ by transformation of elements of an E -group G by \mathcal{H} . Then $\Phi(\mathcal{H}) = \mathcal{I}(\Phi(G))$, $\mathcal{I}(\Phi(G))$ the inner automorphism group of $\Phi(G)$. Furthermore if G is nilpotent, then each subgroup $N \leq \Phi(G)$, N invariant under \mathcal{H} , possess an \mathcal{H} -central series. A class of nilpotent groups N is defined as Φ -central provided that N possesses at least one nilpotent group of automorphisms $\mathcal{H} \neq 1$ such that $\Phi(\mathcal{H}) = \mathcal{I}(N)$ and N possesses an \mathcal{H} -central series. Several theorems develop results about Φ -central groups and the associated \mathcal{H} -central series analogous to those between nilpotent groups and their associated central series. Then it is shown that in a p -group, Φ -central with respect to a p -group of automorphism \mathcal{H} , a nonabelian subgroup invariant under \mathcal{H} cannot have a cyclic center. The paper concludes with the permissible types of nonabelian groups of order p^4 that can be Φ -central with respect to a nontrivial group of p -automorphisms.

Only finite groups will be considered and the notation and the definitions will follow that of the standard references, e.g. [6]. Additionally needed definitions and results will be as follows: The group G is the *reduced partial product* (or reduced product) of its subgroups A and B if A is normal in $G = AB$ and B contains no subgroup K such that $G = AK$. For a reduced product, $A \cap B \leq \Phi(B)$, (see [2]). If N is a normal subgroup of G contained in $\Phi(G)$, then $\Phi(G/N) \cong \Phi(G)/N$, (see [5]). An elementary group, i.e., an E -group having the identity for the Frattini subgroup, splits over each of its normal subgroups, (see [1]).

1. For a group G , $\Phi(G) = \Phi$, $G/\Phi = F$ is Φ -free i.e., $\Phi(F)$ is the identity. The elements of G by transformation of Φ induce auto-

morphisms \mathcal{H} on Φ . Denoting the centralizer of Φ in G by M , $G/M \cong \mathcal{H} \leq \mathcal{A}(\Phi)$, $\mathcal{A}(\Phi)$ the automorphism group of Φ . Then if $\mathcal{I}(\Phi)$ denotes the inner automorphisms of Φ , one has $\mathcal{I}(\Phi) \leq \Phi(\mathcal{H})$ and by a result of Gaschütz [5, Satz 11], $\mathcal{I}(\Phi)$ normal in $\mathcal{A}(\Phi)$ implies that $\mathcal{I}(\Phi) \leq \Phi(\mathcal{A}(\Phi))$.

Supposing first that $M \not\leq \Phi$, there exists a reduced product $G = MK$ such that $M \cap K \leq \Phi(K)$ and $M\Phi(K)/M \cong \Phi(G/M) \cong \Phi(\mathcal{H})$. Moreover $M\Phi/\Phi \cong A \leq F$. Thus A is normal in F and the elements in A correspond to the identity transformation on Φ . Thus $F/A \cong G/M\Phi$ corresponds to a subgroup of outer automorphisms of Φ , namely $F/A \cong \mathcal{H}/\mathcal{I}(\Phi)$. Since F is Φ -free, there exists a reduced product $F = AB$ such that $A \cap B \leq \Phi(B)$ and $F/A \cong B/A \cap B$. By combining these latter statements, $\mathcal{H}/\mathcal{I}(\Phi) \cong B/A \cap B$. Moreover $\Phi(B/A \cap B) \cong \Phi(B)/A \cap B \cong \Phi(\mathcal{H}/\mathcal{I}(\Phi)) = \Phi(\mathcal{H})/\mathcal{I}(\Phi)$, i.e., $\Phi(B)/A \cap B \cong \Phi(\mathcal{H})/\mathcal{I}(\Phi)$. However note that if $\Phi(K) \leq \Phi(G)$, then $M\Phi(K)/M \leq M\Phi(G)/M \cong \mathcal{I}(\Phi) \leq \Phi(\mathcal{H})$. Thus $\mathcal{I}(\Phi) = \Phi(\mathcal{H})$.

Now suppose that $M \leq \Phi$. Then $\Phi(G/M) \cong \Phi(G)/M \cong \Phi(\mathcal{H})$. Since $M = Z$, Z the center of Φ , and $\Phi/Z \cong \mathcal{I}(\Phi)$ again it follows that $\mathcal{I}(\Phi) = \Phi(\mathcal{H})$.

LEMMA 1. *A necessary condition that a group N be the Frattini subgroup of an E -group G is that $\mathcal{A}(N)$ contains a subgroup \mathcal{H} such that $\Phi(\mathcal{H}) = \mathcal{I}(\Phi)$.*

COROLLARY 1.1. *A necessary and sufficient condition that the centralizer of Φ in an E -group G be the center of Φ is that $G/\Phi \cong \mathcal{H}/\mathcal{I}(\Phi)$.*

Using the notation of the above, $G/M\Phi \cong \mathcal{H}/\mathcal{I}(\Phi) \cong T \leq F$. However $G/\Phi \cong F$ elementary implies $F = ST$, $S \cap T = 1$, S normal in F and $F/S = T$. Then:

THEOREM 1. *Necessary conditions that a nilpotent group N be the Frattini subgroup of an E -group G is that $\mathcal{A}(N)$ contains a subgroup \mathcal{H} such that*

- (1) $\Phi(\mathcal{H}) = \mathcal{I}(N)$, and
- (2) *there exists an extension of N to a group M such that $M/N \cong \mathcal{H}/\mathcal{I}(N)$.*

A sufficiency condition may well be lacking since M/N elementary only implies that $\Phi(M) \leq N$; equality is not implied.

Let K denote a normal subgroup of an E -group G such that $\Phi < K \leq G$ and that M is the G -centralizer of K . If $M \not\leq \Phi$ but $M\Phi < K$ properly, $M\Phi/\Phi \not\cong \mathcal{I}(K)$. On the other hand $K < M\Phi$ implies $M\Phi/M \cong$

$\Phi/M \cap \Phi \cong \Phi/K \cap M \cong (K \cap M)\Phi/K \cap M \cong \mathcal{I}(K)$. Thus $\Phi(G/M) \cong \Phi(\mathcal{H}) = \mathcal{I}(K)$, \mathcal{H} the group of automorphisms of K induced by transformation of elements in G . Summarizing:

THEOREM 2. *If K is a subgroup normal in an E -group G , $\Phi < K \leq G$, and M is the G -centralizer of K , then $\Phi(\mathcal{H}) = \mathcal{I}(K)$, \mathcal{H} the group of automorphisms of K induced by transformation of elements of G , if and only if $K \leq M\Phi$, i.e., $K = \Phi Z$, Z the center of K .*

On the other hand if K is a subgroup of Φ , the following decomposition of Φ is obtained:

THEOREM 3. *If a subgroup K of Φ normal in an E -group G has an automorphism group \mathcal{H} induced by transformation of elements of G with $\Phi(\mathcal{H}) = \mathcal{I}(K)$, then $\Phi = KB$, B the centralizer of K in Φ , and $K \cap B \neq 1$ unless $K = 1$. \mathcal{K} denotes the automorphism group of B induced by transformation of elements of G , then $\Phi(\mathcal{K}) = \mathcal{I}(B)$.*

Proof. Denote the G -centralizer of K by M . Then $G/M \cong \mathcal{H}$ and $MK/M \cong K/Z \cong \mathcal{I}(K)$, Z the center of K . Since the homomorphic image of an E -group is an E -group, then \mathcal{H} is an E -group and $\mathcal{H}/\mathcal{I}(K)$ is an elementary group. Hence G/KM is an elementary group which implies that $\Phi \leq KM$ since G is an E -group. $B = M \cap \Phi$ is normal in G and it follows that $\Phi = KB$. Since K is nilpotent, the center of K exists properly unless G is an elementary group.

Symmetrically K is contained in the G -centralizer J of B . Then as above JB/J is mapped into $\Phi(\mathcal{K})$ and since G/JB is elementary, the mapping is onto, i.e., $JB/J \cong \Phi(\mathcal{K}) \cong B/J \cap B \cong \mathcal{I}(B)$.

REMARK 1. Note that in Theorem 3, each subgroup K contained in the center of Φ and normal in G satisfies the condition $\Phi(\mathcal{H}) = \mathcal{I}(K)$ and so \mathcal{H} is an elementary group.

For normal subgroups N of a nilpotent group G , transformation by elements of G on N induce a group of automorphisms \mathcal{H} for which a series of subgroups exist, $N = N_0 > N_1 > \dots > N_r = 1$, such that $x^{-1}x^a \in N_i$, $a \in \mathcal{H}$, $x \in N_{i-1}$. Following Kaloujnine [8], N is said to have an \mathcal{H} -central series. In general E -groups do not have this property on the normal subgroups except in the trivial case of \mathcal{H} the identity mapping. If N is nilpotent and $\mathcal{I}(N) \leq \mathcal{H}$ then the series can be refined to a series for which $|N_{i-1}/N_i|$ is a prime integer.

A group N does not necessarily have an \mathcal{H} -central series for each subgroup $\mathcal{H} \leq \mathcal{A}(N)$ even if N is nilpotent. For example if N is

the quaternion group and \mathcal{H} is $\mathcal{A}(N)$, N has only one proper characteristic subgroup.

Combining Lemma 1 with the above one has:

THEOREM 4. *A necessary condition that a group N be the Frattini subgroup of a nilpotent group G is that $\mathcal{A}(N)$ contains a nilpotent subgroup \mathcal{H} such that*

- (1) $\Phi(\mathcal{H}) = \mathcal{F}(N)$ and
- (2) N possesses an \mathcal{H} -central series.

The dihedral group N of order eight has an \mathcal{H} -central series for $\mathcal{H} = \mathcal{A}(N)$, however $|\Phi(\mathcal{H})| = 2$ and $|\mathcal{F}(N)| = 4$. There are Abelian groups which trivially satisfy (1) but not (2). So both conditions are necessary.

A Φ -central group N will be defined as a nilpotent group possessing at least one nilpotent group of automorphisms $\mathcal{H} \neq 1$ such that

- (1) $\Phi(\mathcal{H}) = \mathcal{F}(N)$ and
- (2) N possesses an \mathcal{H} -central series.

Φ -central groups have the following properties:

THEOREM 5.

(1) *If N is Φ -central with respect to an automorphism group \mathcal{H} , M a subgroup of N invariant under \mathcal{H} , and S a subgroup of N/M invariant under \mathcal{H}^* , \mathcal{H}^* the group of automorphisms induced on N/M by \mathcal{H} , then there exists a subgroup K of N containing M , invariant under \mathcal{H} , with $K/M \cong S$. Moreover $\mathcal{H}^* \cong \mathcal{H}|_M$, \mathcal{M} the set of all $a \in \mathcal{H}$ such that $x^{-1}x^a \in M$.*

(2) *If N is Φ -central with respect to an automorphism group \mathcal{H} and M is a member of the \mathcal{H} -central series, then N/M is Φ -central with respect to \mathcal{H}^* , \mathcal{H}^* the group of automorphisms induced on N/M by \mathcal{H} .*

Proof. The proof of (1) relies on the fact that the groups considered are nilpotent and $\mathcal{F}(N) \leq \mathcal{H}$. The only additional comment necessary for (2) is that under a homomorphic mapping of a nilpotent group the Frattini subgroup goes onto the Frattini subgroup of the image (see [2]).

THEOREM 6. *Let N be a group Φ -central under an automorphism group \mathcal{H} . If M is a subgroup of N invariant under \mathcal{H} then*

- (1) M possesses an \mathcal{H} -central series,
- (2) M possesses a proper subgroup of fixed points under \mathcal{H} ,
and
- (3) M can be included as a member of an \mathcal{H} -central series of N .

Proof. As Kaloujnine [8] has introduced, a *descending \mathcal{H} -central chain* can be defined by $N = N_0 \geq N_1 \geq \dots \geq N_j \geq \dots$ for $N_j = [N_{j-1}, \mathcal{H}]$, $[N_{j-1}, \mathcal{H}]$ the set of $x^{-1}x^a$ for all $x \in N_{j-1}$, $a \in \mathcal{H}$. A series occurs if for some integer r , $N_r = 1$. Analogous to the corresponding proofs for nilpotent groups, a group possessing an \mathcal{H} -central series, possesses a descending \mathcal{H} -central series, M possesses a proper subgroup of fixed points under, \mathcal{H} (the set corresponds to a generalized center of N relative to \mathcal{H}) and M can be included as a member of an \mathcal{H} -central series of N . However M may not necessarily be a Φ -central group.

Even though the notion of Φ -centrality is derived from the properties of the Frattini subgroup of a nilpotent group, it is not a sufficient condition for group extension purposes e.g., consider the extension of cyclic group of order three to the symmetric group on three symbols.

Since $\Phi(K)$ for a nilpotent group K is the direct product of the Frattini subgroup of the Sylow p -subgroups of K (see Gaschütz [5, Satz 6]), then the determination of the nilpotent groups N which can be the Frattini subgroup of some nilpotent group G reduces to the consideration of G as a p -group. The next section discusses several properties of Φ -central p -groups.

2. *Only p -groups and their p -groups of automorphisms will be considered.*

LEMMA 2. (Blackburn [3].) *If M is a group invariant under a group of automorphisms \mathcal{H} and N is a subgroup of M of order p^2 invariant under \mathcal{H} , then \mathcal{H} possesses a subgroup \mathcal{M} of index at most p under which N is a fixed-point set.*

Proof. \mathcal{H} is homomorphic to a p -group of $\mathcal{A}(N)$ and $|\mathcal{A}(N)| = p(p-1)$ since \mathcal{H} is a p -group. The kernel has index at most p .

LEMMA 3. *A group N , Φ -central under the automorphism group \mathcal{H} , can contain no nonabelian subgroup M of order p^3 and invariant under \mathcal{H} .*

Proof. If M is invariant under \mathcal{H} , then M contains a subgroup K of order p^2 invariant under \mathcal{H} by Theorem 6. By Lemma 2, \mathcal{H} possesses a subgroup \mathcal{M} of index at most p under which K is a fixed-point set. Since \mathcal{H} contains $\mathcal{S}(N)$, $K \leq Z(N)$, $Z(N)$ the center of N . Consequently $K \leq Z(M)$, M must be Abelian, and so a contradiction.

COROLLARY 3.1. (Hobby [7].) *No nonabelian p -group of order p^3 can be the Frattini subgroup of a p -group.*

Proof. Denote the induced group of automorphisms on $\Phi(G)$ by the elements of a p -group G by \mathcal{H} . Then $\Phi(G)$ is Φ -central under \mathcal{H} .

COROLLARY 3.2. *Each Frattini subgroup of order greater than p^3 of a p -group G contains an Abelian subgroup N of order p^3 and normal in G .*

LEMMA 4. *Let N be a group Φ -central under an automorphism group \mathcal{H} . A noncyclic Abelian subgroup M of N , invariant under \mathcal{H} and having order p^3 contains an elementary Abelian subgroup K of order p^2 , invariant under \mathcal{H} and a fixed-point set for $\mathcal{F}(N)$.*

Proof. If M is invariant under \mathcal{H} and elementary Abelian, M contains an elementary Abelian subgroup K of order p^2 and invariant under \mathcal{H} by Theorem 6. On the other hand if M is invariant under \mathcal{H} and of the form $\{x, y \mid x^{p^2} = x^p = 1\}$, the characteristic subgroup $K = \{x^p, y\}$ in M has order p^2 and is invariant under \mathcal{H} . In either case K is invariant under a subgroup \mathcal{M} of index at most p by Lemma 2. The result follows since $\Phi(\mathcal{H}) = \mathcal{F}(N) \leq \mathcal{M}$.

COROLLARY 4.1. *A noncyclic Abelian normal subgroup M of a p -group G , $|M| = p^3$, and $M \leq \Phi(G)$, contains an elementary Abelian subgroup N of order p^2 , normal in G , and contained in the center of $\Phi(G)$.*

THEOREM 7. *Let N denote a group Φ -central under an automorphism group \mathcal{H} . Each nonabelian subgroup M of N , invariant under \mathcal{H} , contains an elementary Abelian subgroup K of order p^2 which is invariant under \mathcal{H} and is a fixed-point set under $\mathcal{F}(N)$.*

Proof. Suppose M is a nonabelian subgroup of least order for which the theorem is not valid. By Lemma 3, $|M| \geq p^4$. Since $\Phi(M) \neq 1$, denote by P the cyclic subgroup of order p , consisting of fixed-points under \mathcal{H} and contained in $\Phi(M)$. One such subgroup always exists by Theorem 6. Then $M/P \leq N/P$, both are invariant under \mathcal{H}^* , and N/P is Φ -central under \mathcal{H}^* , \mathcal{H}^* the induced automorphisms on N/P by \mathcal{H} .

If M/P is Abelian, then M/P not cyclic implies that the elements of order p in M/P form a characteristic subgroup K/P , invariant under \mathcal{H}^* , which is elementary Abelian and $|K/P| \geq p^2$. Thus K/P contains a subgroup L/P of order p^2 and invariant under \mathcal{H}^* . This implies that L is a noncyclic commutative subgroup invariant under \mathcal{H} by Lemma 3.

For M/P nonabelian, M/P contains an elementary Abelian subgroup

L/P of order p^2 invariant under \mathcal{H}^* by the induction hypothesis. Again Lemma 3 implies that L of order p^3 is a noncyclic commutative subgroup invariant under \mathcal{H} .

By Lemma 4, K exists for L and hence for M in both cases.

COROLLARY 7.1. *A nonabelian subgroup invariant under \mathcal{H} of a group N , ϕ -central under an automorphism group \mathcal{H} , cannot have a cyclic center.*

COROLLARY 7.2. *A nonabelian normal (characteristic) subgroup of a p -group G that is contained in $\Phi(G)$ cannot have a cyclic center.*

REMARK 2. Corollary 7.2 is stronger than the results of Hobby [7, Theorem 1, Remark 1] and includes a theorem of Burnside [2] that no nonabelian group whose center is cyclic can be the derived group of a p -group. Together with Lemma 5, the results, as necessary conditions, prove useful in determining whether or not a p -group could be the Frattini subgroup of a given p -group.

LEMMA 5. *Let N denote a group ϕ -central under an automorphism group \mathcal{H} . An Abelian subgroup $M \leq N$ of type $(2, 1)$ and invariant under \mathcal{H} , is contained in the center of N .*

Proof. The result holds for N Abelian so consider the case of N nonabelian. If $M = \{x, y \mid x^{p^2} = y^p = 1\}$, then as in Lemma 4, $\{x^p, y\}$ is invariant under \mathcal{H} and is contained in the center of N . Since M contains only p cyclical subgroups of order p^2 and $x^a \neq x^j$ for an integer j and $a \in \mathcal{H}$, it follows that x^a has at most p images under \mathcal{H} . Therefore the subgroup \mathcal{M} of \mathcal{H} having x as a fixed point has index at most p in \mathcal{H} . Since $\Phi(\mathcal{H}) = \mathcal{I}(N) \leq \mathcal{M}$, then x is fixed by $\mathcal{I}(N)$ i.e., x is in the center of N .

COROLLARY 5.1. *An Abelian subgroup M of type $(2, 1)$, normal in a p -group G , and contained in $\Phi(G)$ is contained in the center of $\Phi(G)$.*

THEOREM 8. *The following two types of nonabelian groups of order p^4 cannot be ϕ -central groups with respect to a nontrivial p -group of automorphisms \mathcal{H} :*

- (1) $A = \{x, y, z \mid x^{p^2} = y^p = z^p = 1, [x, z] = y, [x, y] = [y, z] = 1\}$.
- (2) $B = \{x, y \mid x^{p^2} = y^{p^2} = 1, [x, y] = x^p\}$.

Proof. Consider (1) and note that each element of order p^2 is of

the form $z^a x^b y^c$ for $b \neq 0 \pmod{p^2}$. Then $(z^a x^b y^c)^p = x^{pb} y^{pc+ab(1+2+\cdots+(p-1))} = x^{pb} y^{abp(p-1)/2} = x^{pb}$ for $(b, p) = 1$. Thus $\{x^p\} = A^p$ is characteristic in A of order p . If A was Φ -central with respect to an automorphism group \mathcal{H} then A/A^p would be Φ -central with respect to the automorphism group \mathcal{H}^* induced on A/A^p by \mathcal{H} . This contradicts Lemma 3 if \mathcal{H} is nontrivial and so A cannot be Φ -central with respect to a nontrivial p -group of automorphisms \mathcal{H} .

Each maximal subgroup in (2) is Abelian, of order p^3 , and type (2, 1). If B was Φ -central under a nontrivial p -group of automorphisms \mathcal{H} then one of these maximal subgroups, say M , is invariant under \mathcal{H} . By Lemma 5, M is contained in the center of B and thus B is Abelian. So B cannot be Φ -central with respect to a nontrivial p -group of automorphisms \mathcal{H} .

COROLLARY 8.1. *The types (1) and (2) of p -groups of Theorem 8 cannot be Frattini subgroups of p -groups.*

REMARK 3. The remaining two types of nonabelian p -groups are of the forms

$$(3) \quad \{x, y, z \mid x^{p^2} = y^p = z^p = 1, [x, z] = x^p, [y, x] = [y, z] = 1\} \text{ and}$$

$$(4) \quad \{x, y, z, w \mid x^p = y^p = z^p = w^p = 1, [z, w] = x, [y, w] = [x, w] = 1\}.$$

Without attempting a classification it is sufficient to show the existence of p -groups G having $\Phi(G)$ of form (3) or (4). For $p > 5$, the group $G = \{x, y, z, w \mid x^{p^2} = y^{p^2} = z^p = w^p = 1, [y^p, z] = [y^p, x] = [x, w] = 1, [y^p, w] = [x, z] = [z, w] = x^p, [z, y] = y^p, [x, y] = z, [w, y] = x\}$, $|G| = p^6$, and $\Phi(G)$ is of the form (3). Then for $p = 5$, $G = \{u, v, w, x, y, z \mid u^p = v^p = w^p = x^p = y^p = z^p = 1, [v, w] = [v, x] = [v, z] = [x, y] = 1, [v, y] = [x, w] = [w, y] = u, [w, z] = v, [x, z] = w, [y, z] = x\}$, $|G| = p^6$, and $\Phi(G)$ is of type (4).

Groups G of order p^6 other than those given in Remark 3 exist having nonabelian $\Phi(G)$. However for all such cases $\Phi(G)$ contains a characteristic subgroup N of order p^2 such that G/N is not of form (3) nor (4) i.e., G cannot be the Frattini subgroup of any p -group. Remark 3 provides a ready source of examples of p -groups which are Φ -central, or in particular are Frattini subgroups of some p -group. This offsets the conjecture that such a source consisted of p -groups of relatively "large" order. The examples raise the following question: If the group F is the Frattini subgroup of a group G , does there always exist a group G^* such that $\Phi(G^*) \cong F$ and the centralizer of $\Phi(G^*)$ in G^* is the center of $\Phi(G^*)$?

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