FRATTINI SUBGROUPS AND Φ-CENTRAL GROUPS

HOMER FRANKLIN BECHTELL, JR.
0-central groups are introduced as a step in the direction of determining sufficiency conditions for a group to be the Frattini subgroup of some finite $p$-group and the related extension problem. The notion of $\phi$-centrality arises by uniting the concept of an $E$-group with the generalized central series of Kaloujnine. An $E$-group is defined as a finite group $G$ such that $\phi(N) \leq \phi(G)$ for each subgroup $N \leq G$. If $H$ is a group of automorphisms of a group $N$, $N$ has an $H$-central series $N = N_0 > N_1 > \cdots > N_r = 1$ if $x^{-1}x^a \in N_j$ for all $x \in N_{j-1}$, all $a \in H$, $x^a$ the image of $x$ under the automorphism $a \in H$, $j = 0, 1, \cdots, r - 1$.

Denote the automorphism group induced on $\phi(G)$ by transformation of elements of an $E$-group $G$ by $H$. Then $\phi(H) = \mathcal{I}(\phi(G))$, $\mathcal{I}(\phi(G))$ the inner automorphism group of $\phi(G)$. Furthermore if $G$ is nilpotent, then each subgroup $N \leq \phi(G)$, $N$ invariant under $H$, possess an $H$-central series. A class of nilpotent groups $N$ is defined as $\phi$-central provided that $N$ possesses at least one nilpotent group of automorphisms $H \neq 1$ such that $\phi(H) = \mathcal{I}(N)$ and $N$ possesses an $H$-central series. Several theorems develop results about $\phi$-central groups and the associated $H$-central series analogous to those between nilpotent groups and their associated central series. Then it is shown that in a $p$-group, $\phi$-central with respect to a $p$-group of automorphism $H$, a nonabelian subgroup invariant under $H$ cannot have a cyclic center. The paper concludes with the permissible types of nonabelian groups of order $p^4$ that can be $\phi$-central with respect to a nontrivial group of $p$-automorphisms.

Only finite groups will be considered and the notation and the definitions will follow that of the standard references, e.g. [6]. Additionally needed definitions and results will be as follows: The group $G$ is the reduced partial product (or reduced product) of its subgroups $A$ and $B$ if $A$ is normal in $G = AB$ and $B$ contains no subgroup $K$ such that $G = AK$. For a reduced product, $A \cap B \leq \phi(B)$, (see [2]). If $N$ is a normal subgroup of $G$ contained in $\phi(G)$, then $\phi(G/N) \cong \phi(G)/N$, (see [5]). An elementary group, i.e., an $E$-group having the identity for the Frattini subgroup, splits over each of its normal subgroups, (see [1]).

1. For a group $G$, $\phi(G) = \phi$, $G/\phi = F$ is $\phi$-free i.e., $\phi(F)$ is the identity. The elements of $G$ by transformation of $\phi$ induce auto-
morphisms $\mathcal{H}$ on $\Phi$. Denoting the centralizer of $\Phi$ in $G$ by $M$, $G/M \cong \mathcal{H} \leq \mathcal{A}(\Phi)$, $\mathcal{A}(\Phi)$ the automorphism group of $\Phi$. Then if $\mathcal{F}(\Phi)$ denotes the inner automorphisms of $\Phi$, one has $\mathcal{F}(\Phi) \leq \Phi(\mathcal{H})$ and by a result of Gaschütz [5, Satz 11], $\mathcal{F}(\Phi)$ normal in $\mathcal{A}(\Phi)$ implies that $\mathcal{F}(\Phi) \leq \Phi(\mathcal{A}(\Phi))$.

Supposing first that $M \not\leq \Phi$, there exists a reduced product $G = MK$ such that $M \cap K \leq \Phi(K)$ and $M\Phi(K)/M \cong \Phi(G/M) \cong \Phi(\mathcal{H})$. Moreover $M\Phi/\Phi \cong A \leq F$. Thus $A$ is normal in $F$ and the elements in $A$ correspond to the identity transformation on $\Phi$. Thus $F/A \cong G/M\Phi$ corresponds to a subgroup of outer automorphisms of $\Phi$, namely $F/A \cong \mathcal{H}/\mathcal{F}(\Phi)$. Since $F$ is $\Phi$-free, there exists a reduced product $F = AB$ such that $A \cap B \leq \Phi(B)$ and $F/A \cong B/A \cap B$. By combining these latter statements, $\mathcal{H}/\mathcal{F}(\Phi) \cong B/A \cap B$. Moreover $\Phi(B/A \cap B) \cong \Phi(B)/A \cap B \cong \Phi(\mathcal{H}/\mathcal{F}(\Phi)) = \Phi(\mathcal{H})/\mathcal{F}(\Phi)$, i.e., $\Phi(B)/A \cap B \cong \Phi(\mathcal{H})/\mathcal{F}(\Phi)$. However note that if $\Phi(K) \leq \Phi(G)$, then $M\Phi(K)/M \leq M\Phi(G)/M \cong \mathcal{F}(\Phi) \leq \Phi(\mathcal{H})$. Thus $\mathcal{F}(\Phi) = \Phi(\mathcal{H})$.

Now suppose that $M \leq \Phi$. Then $\Phi(G/M) \cong \Phi(G)/M \cong \Phi(\mathcal{H})$. Since $M = Z$, $Z$ the center of $\Phi$, and $\Phi/Z \cong \mathcal{F}(\Phi)$ again it follows that $\mathcal{F}(\Phi) = \Phi(\mathcal{H})$.

**Lemma 1.** A necessary condition that a group $N$ be the Frattini subgroup of an $E$-group $G$ is that $\mathcal{A}(N)$ contains a subgroup $\mathcal{H}$ such that $\Phi(\mathcal{H}) = \mathcal{F}(\Phi)$.

**Corollary 1.1.** A necessary and sufficient condition that the centralizer of $\Phi$ in an $E$-group $G$ be the center of $\Phi$ is that $G/\Phi \cong \mathcal{H}/\mathcal{F}(\Phi)$.

Using the notation of the above, $G/M\Phi \cong \mathcal{H}/\mathcal{F}(\Phi) \cong T \leq F$. However $G/\Phi \cong F$ elementary implies $F = ST$, $S \cap T = 1$, $S$ normal in $F$ and $F/S = T$. Then:

**Theorem 1.** Necessary conditions that a nilpotent group $N$ be the Frattini subgroup of an $E$-group $G$ is that $\mathcal{A}(N)$ contains a subgroup $\mathcal{H}$ such that

\begin{enumerate}
  \item $\Phi(\mathcal{H}) = \mathcal{F}(N)$, and
  \item there exists an extension of $N$ to a group $M$ such that $M/N \cong \mathcal{H}/\mathcal{F}(N)$.
\end{enumerate}

A sufficiency condition may well be lacking since $M/N$ elementary only implies that $\Phi(M) \leq N$; equality is not implied.

Let $K$ denote a normal subgroup of an $E$-group $G$ such that $\Phi < K \leq G$ and that $M$ is the $G$-centralizer of $K$. If $M \not\leq \Phi$ but $M\Phi < K$ properly, $M\Phi/\Phi \not\cong \mathcal{F}(K)$. On the other hand $K < M\Phi$ implies $M\Phi/M \cong$
\[ \Phi/M \cap \Phi \cong \Phi/K \cap M \cong (K \cap M)\Phi/K \cap M \cong \mathcal{H}(K). \] Thus \( \Phi(G/M) \cong \Phi(\mathcal{H}) = \mathcal{H}(K) \), \( \mathcal{H} \) the group of automorphisms of \( K \) induced by transformation of elements in \( G \). Summarizing:

**Theorem 2.** If \( K \) is a subgroup normal in an \( E \)-group \( G \), \( \Phi < K \leq G \), and \( M \) is the \( G \)-centralizer of \( K \), then \( \Phi(\mathcal{H}) = \mathcal{H}(K) \), \( \mathcal{H} \) the group of automorphisms of \( K \) induced by transformation of elements of \( G \), if and only if \( K \leq M\Phi \), i.e., \( K = \Phi Z \), \( Z \) the center of \( K \).

On the other hand if \( K \) is a subgroup of \( \Phi \), the following decomposition of \( \Phi \) is obtained:

**Theorem 3.** If a subgroup \( K \) of \( \Phi \) normal in an \( E \)-group \( G \) has an automorphism group \( \mathcal{H} \) induced by transformation of elements of \( G \) with \( \Phi(\mathcal{H}) = \mathcal{H}(K) \), then \( \Phi = KB \), \( B \) the centralizer of \( K \) in \( \Phi \), and \( K \cap B \neq 1 \) unless \( K = 1 \). \( \mathcal{H} \) denotes the automorphism group of \( B \) induced by transformation of elements of \( G \), then \( \Phi(\mathcal{H}) = \mathcal{H}(B) \).

**Proof.** Denote the \( G \)-centralizer of \( K \) by \( M \). Then \( G/M \cong \mathcal{H} \) and \( MK/M \cong K/Z \cong \mathcal{H}(K) \), \( Z \) the center of \( K \). Since the homomorphic image of an \( E \)-group is an \( E \)-group, then \( \mathcal{H} \) is an \( E \)-group and \( \mathcal{H}/\mathcal{H}(K) \) is an elementary group. Hence \( G/KM \) is an elementary group which implies that \( \Phi \leq KM \) since \( G \) is an \( E \)-group. \( B = M \cap \Phi \) is normal in \( G \) and it follows that \( \Phi = KB \). Since \( K \) is nilpotent, the center of \( K \) exists properly unless \( G \) is an elementary group.

Symmetrically \( K \) is contained in the \( G \)-centralizer \( J \) of \( B \). Then as above \( JB/J \) is mapped into \( \Phi(\mathcal{H}) \) and since \( G/\mathcal{J} \) is elementary, the mapping is onto, i.e., \( JB/J \cong \Phi(\mathcal{H}) \cong B/J \cap B \cong \mathcal{H}(B) \).

**Remark 1.** Note that in Theorem 3, each subgroup \( K \) contained in the center of \( \Phi \) and normal in \( G \) satisfies the condition \( \Phi(\mathcal{H}) = \mathcal{H}(K) \) and so \( \mathcal{H} \) is an elementary group.

For normal subgroups \( N \) of a nilpotent group \( G \), transformation by elements of \( G \) on \( N \) induce a group of automorphisms \( \mathcal{H} \) for which a series of subgroups exist, \( N = N_0 > N_1 > \cdots > N_r = 1 \), such that \( x^{-a}x^a \in N_i \), \( a \in \mathcal{H} \), \( x \in N_{i-1} \). Following Kaloujnine [8], \( N \) is said to have an \( \mathcal{H} \)-central series. In general \( E \)-groups do not have this property on the normal subgroups except in the trivial case of \( \mathcal{H} \) the identity mapping. If \( N \) is nilpotent and \( \mathcal{H}(N) \leq \mathcal{H} \) then the series can be refined to a series for which \( |N_{i-1}/N_i| \) is a prime integer.

A group \( N \) does not necessarily have an \( \mathcal{H} \)-central series for each subgroup \( \mathcal{H} \leq \mathcal{H}(N) \) even if \( N \) is nilpotent. For example if \( N \) is
the quaternion group and $\mathcal{H}$ is $\mathcal{I}(N)$, $N$ has only one proper characteristic subgroup.

Combining Lemma 1 with the above one has:

**Theorem 4.** A necessary condition that a group $N$ be the Frattini subgroup of a nilpotent group $G$ is that $\mathcal{I}(N)$ contains a nilpotent subgroup $\mathcal{H}$ such that

1. $\Phi(\mathcal{H}) = \mathcal{I}(N)$ and
2. $N$ possesses an $\mathcal{H}$-central series.

The dihedral group $N$ of order eight has an $\mathcal{H}$-central series for $\mathcal{H} = \mathcal{I}(N)$, however $|\Phi(\mathcal{H})| = 2$ and $|\mathcal{I}(N)| = 4$. There are Abelian groups which trivially satisfy (1) but not (2). So both conditions are necessary.

A $\Phi$-central group $N$ will be defined as a nilpotent group possessing at least one nilpotent group of automorphisms $\mathcal{H} \neq 1$ such that

1. $\Phi(\mathcal{H}) = \mathcal{I}(N)$ and
2. $N$ possesses an $\mathcal{H}$-central series.

$\Phi$-central groups have the following properties:

**Theorem 5.**

1. If $N$ is $\Phi$-central with respect to an automorphism group $\mathcal{H}$, $M$ a subgroup of $N$ invariant under $\mathcal{H}$, and $S$ a subgroup of $N/M$ invariant under $\mathcal{H}^*$, $\mathcal{H}^*$ the group of automorphisms induced on $N/M$ by $\mathcal{H}$, then there exists a subgroup $K$ of $N$ containing $M$, invariant under $\mathcal{H}$, with $K/M \cong S$. Moreover $\mathcal{H}^* \cong \mathcal{H}/M$, $M$ the set of all $a \in \mathcal{H}$ such that $x^{-1}ax \in M$.

2. If $N$ is $\Phi$-central with respect to an automorphism group $\mathcal{H}$ and $M$ is a member of the $\mathcal{H}$-central series, then $N/M$ is $\Phi$-central with respect to $\mathcal{H}^*$, $\mathcal{H}^*$ the group of automorphisms induced on $N/M$ by $\mathcal{H}$.

**Proof.** The proof of (1) relies on the fact that the groups considered are nilpotent and $\mathcal{I}(N) \leq \mathcal{H}$. The only additional comment necessary for (2) is that under a homomorphic mapping of a nilpotent group the Frattini subgroup goes onto the Frattini subgroup of the image (see [2]).

**Theorem 6.** Let $N$ be a group $\Phi$-central under an automorphism group $\mathcal{H}$. If $M$ is a subgroup of $N$ invariant under $\mathcal{H}$ then

1. $M$ possesses an $\mathcal{H}$-central series,
2. $M$ possesses a proper subgroup of fixed points under $\mathcal{H}$, and
3. $M$ can be included as a member of an $\mathcal{H}$-central series of $N$. 


Proof. As Kaloujnine [8] has introduced, a descending $\mathcal{H}$-central chain can be defined by $N = N_0 \supseteq N_1 \supseteq \cdots \supseteq N_j \supseteq \cdots$ for $N_j = [N_{j-1}, \mathcal{H}], [N_{j-1}, \mathcal{H}]$ the set of $x^{-1}x^a$ for all $x \in N_{j-1}, a \in \mathcal{H}$. A series occurs if for some integer $r, N_r = 1$. Analogous to the corresponding proofs for nilpotent groups, a group possessing an $\mathcal{H}$-central series, possesses a descending $\mathcal{H}$-central series, $M$ possesses a proper subgroup of fixed points under, $\mathcal{H}$ (the set corresponds to a generalized center of $N$ relative to $\mathcal{H}$) and $M$ can be included as a member of an $\mathcal{H}$-central series of $N$. However $M$ may not necessarily be a $\Phi$-central group.

Even though the notion of $\Phi$-centrality is derived from the properties of the Frattini subgroup of a nilpotent group, it is not a sufficient condition for group extension purposes e.g., consider the extension of cyclic group of order three to the symmetric group on three symbols.

Since $\Phi(K)$ for a nilpotent group $K$ is the direct product of the Frattini subgroup of the Sylow $p$-subgroups of $K$ (see Gaschütz [5, Satz 6]), then the determination of the nilpotent groups $N$ which can be the Frattini subgroup of some nilpotent group $G$ reduces to the consideration of $G$ as a $p$-group. The next section discusses several properties of $\Phi$-central $p$-groups.

2. Only $p$-groups and their $p$-groups of automorphisms will be considered.

Lemma 2. (Blackburn [3].) If $M$ is a group invariant under a group of automorphisms $\mathcal{H}$ and $N$ is a subgroup of $M$ of order $p^2$ invariant under $\mathcal{H}$, then $\mathcal{H}$ possesses a subgroup $\mathcal{M}$ of index at most $p$ under which $N$ is a fixed-point set.

Proof. $\mathcal{H}$ is homomorphic to a $p$-group of $\mathcal{A}(N)$ and $|\mathcal{A}(N)| = p(p - 1)$ since $\mathcal{H}$ is a $p$-group. The kernel has index at most $p$.

Lemma 3. A group $N$, $\Phi$-central under the automorphism group $\mathcal{H}$, can contain no nonabelian subgroup $M$ of order $p^3$ and invariant under $\mathcal{H}$.

Proof. If $M$ is invariant under $\mathcal{H}$, then $M$ contains a subgroup $K$ of order $p^3$ invariant under $\mathcal{H}$ by Theorem 6. By Lemma 2, $\mathcal{H}$ possesses a subgroup $\mathcal{M}$ of index at most $p$ under which $K$ is a fixed-point set. Since $\mathcal{H}$ contains $\mathcal{A}(N)$, $K \leq Z(N)$, $Z(N)$ the center of $N$. Consequently $K \leq Z(M)$, $M$ must be Abelian, and so a contradiction.

Corollary 3.1. (Hobby [7].) No nonabelian $p$-group of order $p^3$ can be the Frattini subgroup of a $p$-group.
Proof. Denote the induced group of automorphisms on \( \Phi(G) \) by the elements of a \( p \)-group \( G \) by \( \mathcal{H} \). Then \( \Phi(G) \) is \( \Phi \)-central under \( \mathcal{H} \).

**Corollary 3.2.** Each Frattini subgroup of order greater than \( p^2 \) of a \( p \)-group \( G \) contains an Abelian subgroup \( N \) of order \( p^2 \) and normal in \( G \).

**Lemma 4.** Let \( N \) be a group \( \Phi \)-central under an automorphism group \( \mathcal{H} \). A noncyclic Abelian subgroup \( M \) of \( N \), invariant under \( \mathcal{H} \) and having order \( p^2 \) contains an elementary Abelian subgroup \( K \) of order \( p^2 \), invariant under \( \mathcal{H} \) and a fixed-point set for \( \mathcal{I}(N) \).

Proof. If \( M \) is invariant under \( \mathcal{H} \) and elementary Abelian, \( M \) contains an elementary Abelian subgroup \( K \) of order \( p^2 \) and invariant under \( \mathcal{H} \) by Theorem 6. On the other hand if \( M \) is invariant under \( \mathcal{H} \) and of the form \( \{x, y \mid x^{p^2} = y^p = 1\} \), the characteristic subgroup \( K = \{x^p, y\} \) in \( M \) has order \( p^2 \) and is invariant under \( \mathcal{H} \). In either case \( K \) is invariant under a subgroup \( \mathcal{M} \) of index at most \( p \) by Lemma 2. The result follows since \( \Phi(\mathcal{H}) = \mathcal{I}(N) \leq \mathcal{M} \).

**Corollary 4.1.** A noncyclic Abelian normal subgroup \( M \) of a \( p \)-group \( G \), \( |M| = p^3 \), and \( M \leq \Phi(G) \), contains an elementary Abelian subgroup \( N \) of order \( p^2 \), normal in \( G \), and contained in the center of \( \Phi(G) \).

**Theorem 7.** Let \( N \) denote a group \( \Phi \)-central under an automorphism group \( \mathcal{H} \). Each nonabelian subgroup \( M \) of \( N \), invariant under \( \mathcal{H} \), contains an elementary Abelian subgroup \( K \) of order \( p^2 \) which is invariant under \( \mathcal{H} \) and is a fixed-point set under \( \mathcal{I}(N) \).

Proof. Suppose \( M \) is a nonabelian subgroup of least order for which the theorem is not valid. By Lemma 3, \( |M| \geq p^4 \). Since \( \Phi(M) \neq 1 \), denote by \( P \) the cyclic subgroup of order \( p \), consisting of fixed-points under \( \mathcal{H} \) and contained in \( \Phi(M) \). One such subgroup always exists by Theorem 6. Then \( M/P \leq N/P \), both are invariant under \( \mathcal{H}^* \), and \( N/P \) is \( \Phi \)-central under \( \mathcal{H}^* \), \( \mathcal{H}^* \) the induced automorphisms on \( N/P \) by \( \mathcal{H} \).

If \( M/P \) is Abelian, then \( M/P \) not cyclic implies that the elements of order \( p \) in \( M/P \) form a characteristic subgroup \( K/P \), invariant under \( \mathcal{H}^* \), which is elementary Abelian and \( |K/P| \geq p^2 \). Thus \( K/P \) contains a subgroup \( L/P \) of order \( p^2 \) and invariant under \( \mathcal{H}^* \). This implies that \( L \) is a noncyclic commutative subgroup invariant under \( \mathcal{H} \) by Lemma 3.

For \( M/P \) nonabelian, \( M/P \) contains an elementary Abelian subgroup
L/P of order $p^2$ invariant under $\mathcal{H}^*$ by the induction hypothesis. Again Lemma 3 implies that $L$ of order $p^3$ is a noncyclic commutative subgroup invariant under $\mathcal{H}$.

By Lemma 4, $K$ exists for $L$ and hence for $M$ in both cases.

**Corollary 7.1.** A nonabelian subgroup invariant under $\mathcal{H}$ of a group $N$, $\Phi$-central under an automorphism group $\mathcal{H}$, cannot have a cyclic center.

**Corollary 7.2.** A nonabelian normal (characteristic) subgroup of a $p$-group $G$ that is contained in $\Phi(G)$ cannot have a cyclic center.

**Remark 2.** Corollary 7.2 is stronger than the results of Hobby [7, Theorem 1, Remark 1] and includes a theorem of Burnside [2] that no nonabelian group whose center is cyclic can be the derived group of a $p$-group. Together with Lemma 5, the results, as necessary conditions, prove useful in determining whether or not a $p$-group could be the Frattini subgroup of a given $p$-group.

**Lemma 5.** Let $N$ denote a group $\Phi$-central under an automorphism group $\mathcal{H}$. An Abelian subgroup $M \leq N$ of type $(2,1)$ and invariant under $\mathcal{H}$, is contained in the center of $N$.

**Proof.** The result holds for $N$ Abelian so consider the case of $N$ nonabelian. If $M = \{x, y \mid x^{p^2} = y^p = 1\}$, then as in Lemma 4, $\{x^p, y\}$ is invariant under $\mathcal{H}$ and is contained in the center of $N$. Since $M$ contains only $p$ cyclical subgroups of order $p^2$ and $x^a \neq x^j$ for an integer $j$ and $a \in \mathcal{H}$, it follows that $x^a$ has at most $p$ images under $\mathcal{H}$. Therefore the subgroup $M$ of $\mathcal{H}$ having $x$ as a fixed point has index at most $p$ in $\mathcal{H}$. Since $\Phi(\mathcal{H}) = \mathcal{I}(N) \leq \mathcal{H}$, then $x$ is fixed by $\mathcal{I}(N)$ i.e., $x$ is in the center of $N$.

**Corollary 5.1.** An Abelian subgroup $M$ of type $(2,1)$, normal in a $p$-group $G$, and contained in $\Phi(G)$ is contained in the center of $\Phi(G)$.

**Theorem 8.** The following two types of nonabelian groups of order $p^4$ cannot be $\Phi$-central groups with respect to a nontrivial $p$-group of automorphisms $\mathcal{H}$:

1) $A = \{x, y, z \mid x^{p^2} = y^p = z^2 = 1, [x, z] = y, [x, y] = [y, z] = 1\}.$

2) $B = \{x, y \mid x^{p^2} = y^{p^2} = 1, [x, y] = x^p\}.$

**Proof.** Consider (1) and note that each element of order $p^2$ is of
the form $z^ax^by^c$ for $b \neq 0 \mod p^2$. Then $(z^ax^by^c)^p = x^{pb}y^{pb + ab(1 + 2 + \cdots + (p-1))} = x^{pb}y^{a(p+1)/2} = x^{pb}$ for $(b, p) = 1$. Thus $\{x^p\} = A^p$ is characteristic in $A$ of order $p$. If $A$ was $\Phi$-central with respect to an automorphism group $\mathcal{H}$ then $A/A^p$ would be $\Phi$-central with respect to the automorphism group $\mathcal{H}^*$ induced on $A/A^p$ by $\mathcal{H}$. This contradicts Lemma 3 if $\mathcal{H}$ is nontrivial and so $A$ cannot be $\Phi$-central with respect to a nontrivial $p$-group of automorphisms $\mathcal{H}$.

Each maximal subgroup in (2) is Abelian, of order $p^3$, and type (2,1). If $B$ was $\Phi$-central under a nontrivial $p$-group of automorphisms $\mathcal{H}$ then one of these maximal subgroups, say $M$, is invariant under $\mathcal{H}$. By Lemma 5, $M$ is contained in the center of $B$ and thus $B$ is Abelian. So $B$ cannot be $\Phi$-central with respect to a nontrivial $p$-group of automorphisms $\mathcal{H}$.

**Corollary 8.1.** The types (1) and (2) of $p$-groups of Theorem 8 cannot be Frattini subgroups of $p$-groups.

**Remark 3.** The remaining two types of nonabelian $p$-groups are of the forms

(3) \[ \{x, y, z \mid x^{p^2} = y^p = z^p = 1, [x, z] = x^p, [y, x] = [y, z] = 1 \} \]

(4) \[ \{x, y, z, w \mid x^p = y^p = z^p = w^p = 1, [z, w] = x, [y, w] = [x, w] = 1 \} . \]

Without attempting a classification it is sufficient to show the existence of $p$-groups $G$ having $\Phi(G)$ of form (3) or (4). For $p > 5$, the group $G = \{x, y, z, w \mid x^{p^2} = y^2 = z^p = w^p = 1, [y^p, z] = [y^p, x] = [x, w] = 1, [y^p, w] = [x, z] = [z, w] = x, [z, y] = y^p, [x, y] = z, [w, y] = x, |G| = p^6, and $\Phi(G)$ is of the form (3). Then for $p = 5$, $G = \{u, v, w, x, y, z \mid u^p = v^p = w^p = x^p = y^p = z^p = 1, [v, w] = [v, x] = [v, z] = [x, y] = 1, [v, y] = [x, w] = [w, y] = u, [w, z] = v, [x, z] = w, |G| = p^6, and $\Phi(G)$ is of type (4).

Groups $G$ of order $p^6$ other than those given in Remark 3 exist having nonabelian $\Phi(G)$. However for all such cases $\Phi(G)$ contains a characteristic subgroup $N$ of order $p^2$ such that $G/N$ is not of form (3) nor (4) i.e., $G$ cannot be the Frattini subgroup of any $p$-group. Remark 3 provides a ready source of examples of $p$-groups which are $\Phi$-central, or in particular are Frattini subgroups of some $p$-group. This offsets the conjecture that such a source consisted of $p$-groups of relatively "large" order. The examples raise the following question: If the group $F$ is the Frattini subgroup of a group $G$, does there always exist a group $G^*$ such that $\Phi(G^*) \cong F$ and the centralizer of $\Phi(G^*)$ in $G^*$ is the center of $\Phi(G^*)$?
BIBLIOGRAPHY


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Lewisburg, Pennsylvania
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