THE SUM OF TWO INDEPENDENT EXPONENTIAL-TYPE RANDOM VARIABLES

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VARIABLES

E. M. Bolger

Let \( X_1, X_2 \) be nondegenerate, independent, exponential-type random variables \((r.v.)\) with probability density functions, \((p.d.f.)\) \( f_i(x_i; \theta), f_2(x_2; \theta) \), \( \text{(not necessarily with respect to the same measure)} \), where \( f_i(x_i; \theta) = \exp \{ x_i p_i(\theta) + q_i(\theta) \} \) for \( \theta \in (a, b) \) and \( p_i(\theta) \) is an analytic function of \( \theta \) \( \text{(for Re} \theta \in (a, b) \text{)} \) with \( p_i(\theta) \) never equal to zero on \( (a, b) \). If \( X_1, X_2 \) are neither both normal nor both Poisson type r.v.'s, then \( X_1 + X_2 \) is an exponential-type r.v. if and only if \( p_i(\theta) = p_2(\theta) \).

2. Lemmas. It follows from Patil's result \([3]\) that a r.v. \( X \) is of exponential type if and only if the cumulants, \( \lambda_j(\theta) \), exist and satisfy

\[
\lambda_j'(\theta) = p'(\theta) \lambda_{j+1}(\theta) \quad \text{for} \quad j = 1, 2, 3, \ldots .
\]

Lehmann \([2]\), p. 52) has shown that \( q(\theta) \) and hence also \( \lambda_j(\theta) \) are analytic functions of \( p(\theta) \). Then \( \lambda_j(\theta) \) is an analytic function of \( \theta \) for \( \text{Re} \theta \in (a, b) \).

Let \( \lambda_{jk}(\theta) \) be the \( j^\text{th} \) cumulant of \( X_i \) and \( \lambda_j(\theta) \) the \( j^\text{th} \) cumulant of \( Y \). Then

\[
\lambda_j(\theta) = \lambda_{j,1}(\theta) + \lambda_{j,2}(\theta)
\]

\[
\lambda_j'(\theta) = p'(\theta) \lambda_{j+1,1}(\theta) \quad \text{for} \quad j = 1, 2, 3, \ldots .
\]

Let \( h_j(\theta) = \lambda_{j,1}(\theta) \lambda_{2,2}(\theta) - \lambda_{j,2}(\theta) \lambda_{2,1}(\theta) \) and \( c(\theta) = \lambda_{2,2}(\theta)/\lambda_{2,1}(\theta) \).

\textbf{Lemma 1.} \text{If} \( h_j(\theta) = 0 \) \text{and if} \( c'(\theta) = 0 \), \text{then either} \( X_1 \) \text{and} \( X_2 \) \text{are both normal or} \( p_i(\theta) = p_2(\theta) \).

\textbf{Proof.} \text{Since} \( h_j(\theta) = 0 \),

\[
\lambda_{2,4}(\theta) = c(\theta) \lambda_{3,1}(\theta)
\]

\text{Since} \( c'(\theta) = 0 \),

\[
\lambda_{2,4}(\theta) = c(\theta) \lambda_{3,1}(\theta)
\]

\text{From (3), (4) and (5) it follows that}

\[
p_i'(\theta) \lambda_{2,2}(\theta) = c(\theta) p_i'(\theta) \lambda_{3,1}(\theta) = p_i'(\theta) \lambda_{2,2}(\theta).
\]
If \( \lambda_{3,2}(\theta) \equiv 0 \), then \( \lambda_{3,1}(\theta) \equiv 0 \) and \( X_1, X_2 \) are both normal. If there is a point \( \theta_0 \) such that \( \lambda_{3,2}(\theta) \neq 0 \), then there is a neighborhood, \( N(\theta_0) \), in which \( \lambda_{3,2}(\theta) \neq 0 \). For \( \theta \in N(\theta_0) \), \( p'_i(\theta) = p'_2(\theta) \). By analyticity, \( p'_i(\theta) = p'_i(\theta) \) for \( \theta \in (a, b) \).

**Lemma 2.** If \( h_j(\theta) \equiv 0 \) for \( j > 2 \) and if \( c'(\theta) \neq 0 \), then \( X_1 \) and \( X_2 \) are Poisson type r.v.'s.

**Proof.** Since \( h_j(\theta) \equiv 0 \),

\[
(6) \quad \lambda_{j,2}(\theta) = c(\theta)\lambda_{j,1}(\theta) .
\]

Differentiating (6) and using (3), we get

\[
c(\theta)\lambda_{j,1}(\theta) + c'(\theta)\lambda_{j,1}(\theta) = p'_2(\theta)\lambda_{j,1}(\theta) .
\]

Then,

\[
(7) \quad c(\theta)p'_2(\theta)\lambda_{j,1}(\theta) + c'(\theta)\lambda_{j,1}(\theta) = p'_2(\theta)c(\theta)\lambda_{j,1}(\theta) .
\]

In particular,

\[
(8) \quad c(\theta)p'_2(\theta)\lambda_{2,1}(\theta) + c'(\theta)\lambda_{2,1}(\theta) = p'_2(\theta)c(\theta)\lambda_{2,1}(\theta) .
\]

Multiplying (7) by \( \lambda_{2,1}(\theta) \) and (8) by \( \lambda_{j+1,1}(\theta) \), we find that

\[
(9) \quad c'(\theta)[\lambda_{2,1}(\theta)\lambda_{j+1,1}(\theta) - \lambda_{3,1}(\theta)\lambda_{2,1}(\theta)] = 0
\]

for \( j \geq 2 \).

Since \( c'(\theta) \neq 0 \), there is a sub-interval \( M \) of \((a, b)\) in which \( c'(\theta) \neq 0 \). For \( \theta \in M \),

\[
\lambda_{2,1}(\theta)\lambda_{j+1,1}(\theta) - \lambda_{3,1}(\theta)\lambda_{j,1}(\theta) = 0 ,
\]

or

\[
(10) \quad \lambda_{j+1,1}(\theta) = \frac{\lambda_{2,1}(\theta)}{\lambda_{3,1}(\theta)} \lambda_{j,1}(\theta) .
\]

By analyticity, (10) is true for all \( \theta \in (a, b) \). Now let \( a(\theta) = \lambda_{2,1}(\theta)/\lambda_{3,1}(\theta) \).

Then, by (3),

\[
p'_2(\theta)\lambda_{2,1}(\theta) = \lambda_{2,1}(\theta) = a'(\theta)\lambda_{2,1}(\theta) + a(\theta)\lambda_{2,1}(\theta) = a'(\theta)\lambda_{2,1}(\theta) + a(\theta)p'_2(\theta)\lambda_{3,1}(\theta) .
\]

Since \( \lambda_{2,1}(\theta) = a(\theta)\lambda_{3,1}(\theta) \), it follows that

\[
a'(\theta)\lambda_{3,1}(\theta) = 0 .
\]

So \( a'(\theta) = 0 \) and \( a(\theta) = d \). Then (10) becomes

\[
(11) \quad \lambda_{j+1,1}(\theta) = d\lambda_{j,1}(\theta) \quad \text{for } j \geq 2 .
\]
This implies
\begin{equation}
\lambda_{j,2}(\theta) = d^{j-2}\lambda_{2,1}(\theta) \quad \text{for } j \geq 2 .
\end{equation}

By (6),
\begin{equation}
\lambda_{j,3}(\theta) = d^{j-3}c(\theta)\lambda_{2,1}(\theta) \quad \text{for } j \geq 2 .
\end{equation}

Now,
\begin{equation}
p_i(\theta) = \lambda_{i,1}(\theta)/\lambda_{2,1}(\theta) ,
\end{equation}
\begin{equation}
p_j(\theta) = \lambda_{2,1}(\theta)/\lambda_{2,1}(\theta) = \lambda_{j,1}(\theta)/d\lambda_{2,1}(\theta) .
\end{equation}

So
\begin{equation}
\lambda_{1,1}(\theta) = d^{-1}\lambda_{2,1}(\theta) + k_1 .
\end{equation}

Similarly,
\begin{equation}
\lambda_{1,2}(\theta) = d^{-1}c(\theta)\lambda_{2,1}(\theta) + k_2 .
\end{equation}

Using (12), (13), (14) and (15), we find that
\begin{align*}
\log M_2(t; \theta) &= k_1t + d^{-2}\lambda_{2,1}(\theta)(e^{dt} - 1) \\
\log M_3(t; \theta) &= k_2t + d^{-2}c(\theta)\lambda_{2,1}(\theta)(e^{dt} - 1) ,
\end{align*}
where $M_i(t; \theta)$ is the moment generating function corresponding to $f_i(x; \theta)$.

This concludes the proof of Lemma 2.

3. The sum of two independent exponential-type random variables.

**Theorem 1.** If $X_1, X_2$ are neither both normal nor both Poisson type r.v.'s, then $X_1 + X_2$ is an exponential-type r.v. if and only if $p'_i(\theta) = p'_j(\theta)$.

**Proof.** If $p'_i(\theta) = p'_j(\theta)$, then it follows from (2) and (3) that
\begin{align*}
\lambda_{j+1,1}(\theta) &= \lambda_{j+1,1}(\theta) + \lambda_{j+1,2}(\theta) \\
&= [p'_i(\theta)]^{-1}\lambda_{j,1}(\theta) + [p'_i(\theta)]^{-1}\lambda_{j,2}(\theta) \\
&= [p'_i(\theta)]^{-1}\lambda_j(\theta) .
\end{align*}

Conversely, assume $X_1 + X_2$ is an exponential-type r.v. Then, using (1), (2), and (3), we find that
\begin{equation}
p'(\theta)[\lambda_{j,1}(\theta) + \lambda_{j,2}(\theta)] = p'_i(\theta)\lambda_{j,1}(\theta) + p'_i(\theta)\lambda_{j,2}(\theta) .
\end{equation}

In particular,
Multiplying (16) by $\lambda_{j;i}(\theta)$ and (17) by $\lambda_{j;i}(\theta)$ and then subtracting, we get

\begin{equation}
[p'(\theta) - p_2'(\theta)]h_j(\theta) \equiv 0 \quad \text{for } j \geq 2.
\end{equation}

Now, if for some $j_0 \geq 2$, $h_{j_0}(\theta) \neq 0$, then there is a subinterval, $M$, of $(a, b)$ in which $h_{j_0}(\theta) \neq 0$. Then, for $\theta \in M$, $p'_1(\theta) = p'_2(\theta)$. By analyticity, $p'_1(\theta) = p'_2(\theta)$ for all $\theta \in (a, b)$. Substitution in (16) yields $p'_1(\theta) = p'_2(\theta)$ for $\theta \in (a, b)$. If, on the other hand, $h_j(\theta) \equiv 0$, for $j \geq 2$, the result follows from Lemmas 1 and 2 since we assumed that $X_1, X_2$ are neither both normal nor both Poisson type r.v.'s.

It should be noted that Girshick and Savage [1] proved that if $X_1$ and $X_2$ are independent identically distributed r.v.'s such that their sum is of exponential-type, then $X_1$ and $X_2$ are also of exponential-type.

The following theorem gives necessary and sufficient conditions for the sum of two Poisson-type r.v.'s to be exponential-type.

**Theorem 2.** If $\log M(t; \theta) = C t + A(\theta)[e^{ut} - 1]$, then $X_1 + X_2$ is an exponential-type r.v. if and only if either $b_1 = b_2$ or $p'(\theta) = p_2'(\theta)$.

**Proof.** If $X_1 + X_2$ is an exponential-type r.v., then, as in the proof of the preceding theorem,

\begin{equation}
[p'(\theta) - p_2'(\theta)]h_j(\theta) \equiv 0 \quad \text{for } j \geq 2.
\end{equation}

Equivalently,

\begin{equation}
[\lambda_{j;i}(\theta)\lambda_{j;i}(\theta) - \lambda_{j;i}(\theta)\lambda_{j;i}(\theta)]
= p'_1(\theta)[p'(\theta)]^{-1}[\lambda_{j;i}(\theta)\lambda_{j;i}(\theta) - \lambda_{j;i}(\theta)\lambda_{j;i}(\theta)] \quad \text{for } j \geq 2.
\end{equation}

Since, for $j \geq 2$, $\lambda_{j;i}(\theta) = b_i^j A_i(\theta)$, (19) becomes

\begin{equation}
[b_i^j b_i^j - b_i^j b_i^j]A_i(\theta)A_i(\theta) = p'_1(\theta)[p'(\theta)]^{-1}[b_i^j b_i^j - b_i^j b_i^j]A_i(\theta)A_i(\theta).
\end{equation}

But $A_i(\theta)A_i(\theta) > 0$, so that

\begin{equation}
[b_i^j b_i^j - b_i^j b_i^j] = p'_1(\theta)[p'(\theta)]^{-1}[b_i^j b_i^j - b_i^j b_i^j] = 0.
\end{equation}

Now, if $b_i^j = b_i^j$ for all $j \geq 2$, then $b_i^j b_i^j = b_i^j b_i^j$, so that $b_i = b_i$. On the other hand, if, for some $j_0$, $b_i^j b_i^j - b_i^j b_i^j = 0$, then $p'_1(\theta) = p'_2(\theta)$ and it follows that $p'_1(\theta) = p'_2(\theta)$.

Conversely, if $p'_1(\theta) = p'_2(\theta)$, then $X_1 + X_2$ is an exponential-type r.v. since (1) is satisfied. If $b_i = b_i$, let...
\[ p'(\theta) = \frac{A_1'(\theta) + A_2'(\theta)}{b_1[A_1(\theta) + A_2(\theta)]} . \]

It is easy to see that (1) is again satisfied.

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