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THE SUM OF TWO INDEPENDENT EXPONENTIAL-TYPE RANDOM VARIABLES

EDWARD MARTIN BOLGER

THE SUM OF TWO INDEPENDENT EXPONENTIAL-TYPE RANDOM VARIABLES

E. M. BOLGER

Let X_1, X_2 be nondegenerate, independent, exponential-type random variables (r.v.) with probability density functions, (p.d.f.) $f_1(x_1; \theta), f_2(x_2; \theta)$, (not necessarily with respect to the same measure), where $f_i(x_i; \theta) = \exp \{x_i p_i(\theta) + q_i(\theta)\}$ for $\theta \in (a, b)$ and $p_i(\theta)$ is an analytic function of θ (for $Re \theta \in (a, b)$) with $p_i'(\theta)$ never equal to zero on (a, b) . If X_1, X_2 are neither both normal nor both Poisson type r.v.'s, then $X_1 + X_2$ is an exponential-type r.v. if and only if $p_1'(\theta) = p_2'(\theta)$.

2. Lemmas. It follows from Patil's result ([3]) that a r.v. X is of exponential type if and only if the cumulants, $\lambda_j(\theta)$, exist and satisfy

$$(1) \quad \lambda_j'(\theta) = p'(\theta)\lambda_{j+1}(\theta) \quad \text{for } j = 1, 2, 3, \dots$$

Lehmann ([2], p. 52) has shown that $q(\theta)$ and hence also $\lambda_j(\theta)$ are analytic functions of $p(\theta)$. Then $\lambda_j(\theta)$ is an analytic function of θ for $Re \theta \in (a, b)$.

Let $\lambda_{j,i}(\theta)$ be the j^{th} cumulant of X_i and $\lambda_j(\theta)$ the j^{th} cumulant of Y . Then

$$(2) \quad \lambda_j(\theta) = \lambda_{j,1}(\theta) + \lambda_{j,2}(\theta)$$

$$(3) \quad \lambda_{j,i}'(\theta) = p_i'(\theta)\lambda_{j+1,i}(\theta) \quad \text{for } j = 1, 2, 3, \dots$$

Let $h_j(\theta) = \lambda_{j,1}(\theta)\lambda_{j,2}(\theta) - \lambda_{j,2}(\theta)\lambda_{j,1}(\theta)$ and $c(\theta) \equiv \lambda_{2,2}(\theta)/\lambda_{2,1}(\theta)$.

LEMMA 1. If $h_3(\theta) \equiv 0$ and if $c'(\theta) \equiv 0$, then either X_1 and X_2 are both normal or $p_1'(\theta) \equiv p_2'(\theta)$.

Proof. Since $h_3(\theta) \equiv 0$,

$$(4) \quad \lambda_{3,2}(\theta) = c(\theta)\lambda_{3,1}(\theta).$$

Since $c'(\theta) \equiv 0$,

$$(5) \quad \lambda_{2,2}'(\theta) = c(\theta)\lambda_{2,1}'(\theta).$$

From (3), (4) and (5) it follows that

$$p_2'(\theta)\lambda_{3,2}(\theta) = c(\theta)p_1'(\theta)\lambda_{3,1}(\theta) = p_1'(\theta)\lambda_{3,2}(\theta).$$

If $\lambda_{3,2}(\theta) \equiv 0$, then $\lambda_{3,1}(\theta) \equiv 0$ and X_1, X_2 are both normal. If there is a point θ_0 such that $\lambda_{3,2}(\theta) \neq 0$, then there is a neighborhood, $N(\theta_0)$, in which $\lambda_{3,2}(\theta) \neq 0$. For $\theta \in N(\theta_0)$, $p_1'(\theta) = p_2'(\theta)$. By analyticity, $p_1'(\theta) = p_2'(\theta)$ for $\theta \in (a, b)$.

LEMMA 2. *If $h_j(\theta) \equiv 0$ for $j > 2$ and if $c'(\theta) \not\equiv 0$, then X_1 and X_2 are Poisson type r.v.'s.*

Proof. Since $h_j(\theta) \equiv 0$,

$$(6) \quad \lambda_{j,2}(\theta) = c(\theta)\lambda_{j,1}(\theta).$$

Differentiating (6) and using (3), we get

$$c(\theta)\lambda_{j,1}'(\theta) + c'(\theta)\lambda_{j,1}(\theta) = p_2'(\theta)\lambda_{j+1,2}(\theta).$$

Then,

$$(7) \quad c(\theta)p_1'(\theta)\lambda_{j+1,1}(\theta) + c'(\theta)\lambda_{j,1}(\theta) = p_2'(\theta)c(\theta)\lambda_{j+1,1}(\theta).$$

In particular,

$$(8) \quad c(\theta)p_1'(\theta)\lambda_{3,1}(\theta) + c'(\theta)\lambda_{2,1}(\theta) = p_2'(\theta)c(\theta)\lambda_{3,1}(\theta).$$

Multiplying (7) by $\lambda_{3,1}(\theta)$ and (8) by $\lambda_{j+1,1}(\theta)$, we find that

$$(9) \quad c'(\theta)[\lambda_{2,1}(\theta)\lambda_{j+1,1}(\theta) - \lambda_{3,1}(\theta)\lambda_{j,1}(\theta)] = 0 \quad \text{for } j \geq 2.$$

Since $c'(\theta) \not\equiv 0$, there is a sub-interval M of (a, b) in which $c'(\theta) \neq 0$. For $\theta \in M$,

$$\lambda_{2,1}(\theta)\lambda_{j+1,1}(\theta) - \lambda_{3,1}(\theta)\lambda_{j,1}(\theta) = 0,$$

or

$$(10) \quad \lambda_{j+1,1}(\theta) = \frac{\lambda_{3,1}(\theta)}{\lambda_{2,1}(\theta)}\lambda_{j,1}(\theta).$$

By analyticity, (10) is true for all $\theta \in (a, b)$. Now let $\alpha(\theta) = \lambda_{3,1}(\theta)/\lambda_{2,1}(\theta)$. Then, by (3),

$$\begin{aligned} p_1'(\theta)\lambda_{4,1}(\theta) &= \lambda_{3,1}'(\theta) = \alpha'(\theta)\lambda_{2,1}(\theta) + \alpha(\theta)\lambda_{2,1}'(\theta) \\ &= \alpha'(\theta)\lambda_{2,1}(\theta) + \alpha(\theta)p_1'(\theta)\lambda_{3,1}(\theta). \end{aligned}$$

Since $\lambda_{4,1}(\theta) = \alpha(\theta)\lambda_{3,1}(\theta)$, it follows that

$$\alpha'(\theta)\lambda_{2,1}(\theta) = 0.$$

So $\alpha'(\theta) = 0$ and $\alpha(\theta) = d$. Then (10) becomes

$$(11) \quad \lambda_{j+1,1}(\theta) = d\lambda_{j,1}(\theta) \quad \text{for } j \geq 2.$$

This implies

$$(12) \quad \lambda_{j,1}(\theta) = d^{j-2}\lambda_{2,1}(\theta) \quad \text{for } j \geq 2 .$$

By (6),

$$(13) \quad \lambda_{j,2}(\theta) = d^{j-2}c(\theta)\lambda_{2,1}(\theta) \quad \text{for } j \geq 2 .$$

Now,

$$\begin{aligned} p'_1(\theta) &= \lambda'_{1,1}(\theta)/\lambda_{2,1}(\theta) , \\ p'_i(\theta) &= \lambda'_{2,1}(\theta)/\lambda_{3,1}(\theta) = \lambda'_{2,1}(\theta)/d\lambda_{2,1}(\theta) . \end{aligned}$$

So

$$(14) \quad \lambda_{1,1}(\theta) = d^{-1}\lambda_{2,1}(\theta) + k_1 .$$

Similarly,

$$(15) \quad \lambda_{1,2}(\theta) = d^{-1}c(\theta)\lambda_{2,1}(\theta) + k_2 .$$

Using (12), (13), (14) and (15), we find that

$$\begin{aligned} \log M_1(t; \theta) &= k_1 t + d^{-2}\lambda_{2,1}(\theta)(e^{dt} - 1) \\ \log M_2(t; \theta) &= k_2 t + d^{-2}c(\theta)\lambda_{2,1}(\theta)(e^{dt} - 1) , \end{aligned}$$

where $M_i(t; \theta)$ is the moment generating function corresponding to $f_i(x_i; \theta)$.

This concludes the proof of Lemma 2.

3. The sum of two independent exponential-type random variables.

THEOREM 1. *If X_1, X_2 are neither both normal nor both Poisson type r.v.'s, then $X_1 + X_2$ is an exponential-type r.v. if and only if $p'_1(\theta) = p'_2(\theta)$.*

Proof. If $p'_1(\theta) = p'_2(\theta)$, then it follows from (2) and (3) that

$$\begin{aligned} \lambda_{j+1}(\theta) &= \lambda_{j+1,1}(\theta) + \lambda_{j+1,2}(\theta) \\ &= [p'_1(\theta)]^{-1}\lambda'_{j,1}(\theta) + [p'_1(\theta)]^{-1}\lambda'_{j,2}(\theta) \\ &= [p'_1(\theta)]^{-1}\lambda'_j(\theta) . \end{aligned}$$

Conversely, assume $X_1 + X_2$ is an exponential-type r.v.. Then, using (1), (2), and (3), we find that

$$(16) \quad p'(\theta)[\lambda_{j,1}(\theta) + \lambda_{j,2}(\theta)] = p'_1(\theta)\lambda_{j,1}(\theta) + p'_2(\theta)\lambda_{j,2}(\theta) .$$

In particular,

$$(17) \quad p'(\theta)[\lambda_{2,1}(\theta) + \lambda_{2,2}(\theta)] = p'_1(\theta)\lambda_{2,1}(\theta) + p'_2(\theta)\lambda_{2,2}(\theta).$$

Multiplying (16) by $\lambda_{2,1}(\theta)$ and (17) by $\lambda_{j,1}(\theta)$ and then subtracting, we get

$$(18) \quad [p'(\theta) - p'_2(\theta)]h_j(\theta) \equiv 0 \quad \text{for } j \geq 2.$$

Now, if for some $j_0 \geq 2$, $h_{j_0}(\theta) \not\equiv 0$, then there is a subinterval, M , of (a, b) in which $h_{j_0}(\theta) \neq 0$. Then, for $\theta \in M$, $p'_2(\theta) = p'(\theta)$. By analyticity, $p'_2(\theta) = p'(\theta)$ for all $\theta \in (a, b)$. Substitution in (16) yields $p'_1(\theta) = p'(\theta)$ for $\theta \in (a, b)$. If, on the other hand, $h_j(\theta) \equiv 0$, for $j \geq 2$, the result follows from Lemmas 1 and 2 since we assumed that X_1, X_2 are neither both normal nor both Poisson type r.v.'s.

It should be noted that Girshick and Savage [1] proved that if X_1 and X_2 are independent identically distributed r.v.'s such that their sum is of exponential-type, then X_1 and X_2 are also of exponential-type.

The following theorem gives necessary and sufficient conditions for the sum of two Poisson-type r.v.'s to be exponential-type.

THEOREM 2. *If $\log M_i(t; \theta) = C_i t + A_i(\theta)[e^{b_i t} - 1]$, then $X_1 + X_2$ is an exponential-type r.v. if and only if either $b_1 = b_2$ or $p'_1(\theta) = p'_2(\theta)$.*

Proof. If $X_1 + X_2$ is an exponential-type r.v., then, as in the proof of the preceding theorem,

$$[p'(\theta) - p'_2(\theta)]h_j(\theta) \equiv 0 \quad \text{for } j \geq 2.$$

Equivalently,

$$(19) \quad \begin{aligned} & [\lambda_{j,1}(\theta)\lambda_{2,2}(\theta) - \lambda_{j,2}(\theta)\lambda_{2,1}(\theta)] \\ & = p'_2(\theta)[p'(\theta)]^{-1}[\lambda_{j,1}(\theta)\lambda_{2,2}(\theta) - \lambda_{j,2}(\theta)\lambda_{2,1}(\theta)] \end{aligned} \quad \text{for } j \geq 2.$$

Since, for $j \geq 2$, $\lambda_{j,i}(\theta) = b_i^j A_i(\theta)$, (19) becomes

$$[b_1^j b_2^j - b_2^j b_1^j] A_1(\theta) A_2(\theta) = p'_2(\theta) [p'(\theta)]^{-1} [b_1^j b_2^j - b_2^j b_1^j] A_1(\theta) A_2(\theta).$$

But $A_1(\theta) A_2(\theta) > 0$, so that

$$[b_1^j b_2^j - b_2^j b_1^j] = p'_2(\theta) [p'(\theta)]^{-1} [b_1^j b_2^j - b_2^j b_1^j].$$

Now, if $b_1^j b_2^j = b_2^j b_1^j$ for all $j \geq 2$, then $b_1^2 b_2^2 = b_2^2 b_1^2$, so that $b_1 = b_2$. On the other hand, if, for some j_0 , $b_1^{j_0} b_2^{j_0} - b_2^{j_0} b_1^{j_0} \neq 0$, then $p'_2(\theta) = p'(\theta)$ and it follows that $p'_1(\theta) = p'_2(\theta)$.

Conversely, if $p'_1(\theta) = p'_2(\theta)$, then $X_1 + X_2$ is an exponential-type r.v. since (1) is satisfied. If $b_1 = b_2$, let

$$p'(\theta) = [A_1'(\theta) + A_2'(\theta)]/b_1[A_1(\theta) + A_2(\theta)] .$$

It is easy to see that (1) is again satisfied.

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TRW SYSTEMS
NAVAL ORDNANCE TEST STATION

Edward Joseph Barbeau, <i>Semi-algebras that are lower semi-lattices</i>	1
Steven Fredrick Bauman, <i>The Klein group as an automorphism group without fixed point</i>	9
Homer Franklin Bechtell, Jr., <i>Frattni subgroups and Φ-central groups</i>	15
Edward Kenneth Blum, <i>A convergent gradient procedure in prehilbert spaces</i>	25
Edward Martin Bolger, <i>The sum of two independent exponential-type random variables</i>	31
David Wilson Bressler and A. P. Morse, <i>Images of measurable sets</i>	37
Dennison Robert Brown and J. G. LaTorre, <i>A characterization of uniquely divisible commutative semigroups</i>	57
Selwyn Ross Caradus, <i>Operators of Riesz type</i>	61
Jeffrey Davis and Isidore Isaac Hirschman, Jr., <i>Toeplitz forms and ultraspherical polynomials</i>	73
Lorraine L. Foster, <i>On the characteristic roots of the product of certain rational integral matrices of order two</i>	97
Alfred Gray and S. M. Shah, <i>Asymptotic values of a holomorphic function with respect to its maximum term</i>	111
Sidney (Denny) L. Gulick, <i>Commutativity and ideals in the biduals of topological algebras</i>	121
G. J. Kurowski, <i>Further results in the theory of monodiffrie functions</i>	139
Lawrence S. Levy, <i>Commutative rings whose homomorphic images are self-injective</i>	149
Calvin T. Long, <i>On real numbers having normality of order k</i>	155
Bertram Mond, <i>An inequality for operators in a Hilbert space</i>	161
John William Neuberger, <i>The lack of self-adjointness in three-point boundary value problems</i>	165
C. A. Persinger, <i>Subsets of n-books in E^3</i>	169
Oscar S. Rothaus and John Griggs Thompson, <i>A combinatorial problem in the symmetric group</i>	175
Rodolfo DeSapio, <i>Unknotting spheres via Smale</i>	179
James E. Shockley, <i>On the functional equation $F(mn)F((m, n)) = F(m)F(n)f((m, n))$</i>	185
Kenneth Edward Whipple, <i>Cauchy sequences in Moore spaces</i>	191