THE SUM OF TWO INDEPENDENT EXPONENTIAL-TYPE RANDOM VARIABLES

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Let $X_1, X_2$ be nondegenerate, independent, exponential-type random variables (r.v.) with probability density functions, (p.d.f.) $f_1(x_1; \theta)$, $f_2(x_2; \theta)$, (not necessarily with respect to the same measure), where $f_i(x_i; \theta) = \exp \{x_i p_i(\theta) + q_i(\theta)\}$ for $\theta \in (a, b)$ and $p_i(\theta)$ is an analytic function of $\theta$ (for $\text{Re } \theta \in (a, b)$) with $p'_i(\theta)$ never equal to zero on $(a, b)$. If $X_1, X_2$ are neither both normal nor both Poisson type r.v.'s, then $X_1 + X_2$ is an exponential-type r.v. if and only if $p'_1(\theta) = p'_2(\theta)$.

2. Lemmas. It follows from Patil's result ([3]) that a r.v. $X$ is of exponential type if and only if the cumulants, $\lambda_j(\theta)$, exist and satisfy

$$\lambda_j(\theta) = p'(\theta)\lambda_{j+1}(\theta)$$

for $j = 1, 2, 3, \ldots$. 

Lehmann ([2], p. 52) has shown that $q(\theta)$ and hence also $\lambda_j(\theta)$ are analytic functions of $p(\theta)$. Then $\lambda_j(\theta)$ is an analytic function of $\theta$ for $\text{Re } \theta \in (a, b)$.

Let $\lambda_{j,i}(\theta)$ be the $j^{th}$ cumulant of $X_i$ and $\lambda_j(\theta)$ the $j^{th}$ cumulant of $Y$. Then

$$\lambda_j(\theta) = \lambda_{j,1}(\theta) + \lambda_{j,2}(\theta)$$

(3)

$$\lambda'_{j,i}(\theta) = p'_i(\theta)\lambda_{j+1,i}(\theta)$$

for $j = 1, 2, 3, \ldots$.

Let $h_j(\theta) = \lambda_{j,1}(\theta)\lambda_{j,2}(\theta) - \lambda_{j,1}(\theta)\lambda_{j,2}(\theta)$ and $c(\theta) = \lambda_{2,3}(\theta)\lambda_{2,1}(\theta)$.

**Lemma 1.** If $h_3(\theta) \equiv 0$ and if $c'(\theta) \equiv 0$, then either $X_1$ and $X_2$ are both normal or $p'_1(\theta) \equiv p'_2(\theta)$.

**Proof.** Since $h_3(\theta) \equiv 0$,

$$\lambda_{3,2}(\theta) = c(\theta)\lambda_{3,1}(\theta) .$$

Since $c'(\theta) \equiv 0$,

$$\lambda'_{2,2}(\theta) = c(\theta)\lambda'_{2,1}(\theta) .$$

From (3), (4) and (5) it follows that

$$p'_1(\theta)\lambda_{3,2}(\theta) = c(\theta)p'_1(\theta)\lambda_{3,1}(\theta) = p'_1(\theta)\lambda_{3,2}(\theta) .$$
If \( \lambda_{s,1}(\theta) \equiv 0 \), then \( \lambda_{s,1}(\theta) \equiv 0 \) and \( X_1, X_2 \) are both normal. If there is a point \( \theta_0 \) such that \( \lambda_{s,1}(\theta) \neq 0 \), then there is a neighborhood, \( N(\theta_0) \), in which \( \lambda_{s,1}(\theta) \neq 0 \). For \( \theta \in N(\theta_0) \), \( p_1'(\theta) = p_1'(\theta) \). By analyticity, \( p_1'(\theta) = p_1'(\theta) \) for \( \theta \in (a, b) \).

**LEMMA 2.** If \( h_j(\theta) \equiv 0 \) for \( j > 2 \) and if \( c'(\theta) \neq 0 \), then \( X_i \) and \( X_2 \) are Poisson type r.v.'s.

*Proof.* Since \( h_j(\theta) \equiv 0 \),

(6) \[ \lambda_{j+1,1}(\theta) = c(\theta)\lambda_{j,1}(\theta) . \]

Differentiating (6) and using (3), we get

\[ c(\theta)\lambda_{j+1,1}(\theta) + c'(\theta)\lambda_{j,1}(\theta) = p_2'(\theta)\lambda_{j+1,1}(\theta) . \]

Then,

(7) \[ c(\theta)p_1'(\theta)\lambda_{j+1,1}(\theta) + c'(\theta)\lambda_{j,1}(\theta) = p_2'(\theta)c(\theta)\lambda_{j+1,1}(\theta) . \]

In particular,

(8) \[ c(\theta)p_1'(\theta)\lambda_{2,1}(\theta) + c'(\theta)\lambda_{2,1}(\theta) = p_2'(\theta)c(\theta)\lambda_{3,1}(\theta) . \]

Multiplying (7) by \( \lambda_{2,1}(\theta) \) and (8) by \( \lambda_{j+1,1}(\theta) \), we find that

(9) \[ c'(\theta)[\lambda_{2,1}(\theta)\lambda_{j+1,1}(\theta) - \lambda_{3,1}(\theta)\lambda_{j,1}(\theta)] = 0 \quad \text{for } j \geq 2 . \]

Since \( c'(\theta) \neq 0 \), there is a sub-interval \( M \) of \( (a, b) \) in which \( c'(\theta) \neq 0 \). For \( \theta \in M \),

\[ \lambda_{s,1}(\theta)\lambda_{j+1,1}(\theta) - \lambda_{s,1}(\theta)\lambda_{j,1}(\theta) = 0 , \]

or

(10) \[ \lambda_{j+1,1}(\theta) = \frac{\lambda_{s,1}(\theta)}{\lambda_{s,1}(\theta)} \lambda_{j,1}(\theta) . \]

By analyticity, (10) is true for all \( \theta \in (a, b) \). Now let \( \alpha(\theta) = \lambda_{s,1}(\theta)/\lambda_{s,1}(\theta) \).

Then, by (3),

\[ p_1'(\theta)\lambda_{4,1}(\theta) = \lambda_{4,1}(\theta) = \alpha'(\theta)\lambda_{s,1}(\theta) + \alpha(\theta)\lambda_{s,1}(\theta) \]
\[ = \alpha'(\theta)\lambda_{s,1}(\theta) + \alpha(\theta)p_1'(\theta)\lambda_{s,1}(\theta) . \]

Since \( \lambda_{4,1}(\theta) = \alpha(\theta)\lambda_{s,1}(\theta) \), it follows that

\[ \alpha'(\theta)\lambda_{s,1}(\theta) = 0 . \]

So \( \alpha'(\theta) = 0 \) and \( \alpha(\theta) = d \). Then (10) becomes

(11) \[ \lambda_{j+1,1}(\theta) = d\lambda_{j,1}(\theta) \quad \text{for } j \geq 2 . \]
This implies

\[(12) \quad \lambda_{j,1}(\theta) = d^{i-\varepsilon}\lambda_{x,1}(\theta) \quad \text{for} \quad j \geq 2.\]

By (6),

\[(13) \quad \lambda_{j,2}(\theta) = d^{i-\varepsilon}c(\theta)\lambda_{x,1}(\theta) \quad \text{for} \quad j \geq 2.\]

Now,

\[
\begin{align*}
p'_i(\theta) &= \lambda'_{j,1}(\theta)/\lambda_{x,1}(\theta), \\
p'_i(\theta) &= \lambda'_{j,2}(\theta)/\lambda_{x,1}(\theta) = \lambda'_{j,1}(\theta)/d\lambda_{x,1}(\theta).
\end{align*}
\]

So

\[(14) \quad \lambda_{1,1}(\theta) = d^{-i}\lambda_{x,1}(\theta) + k_1.\]

Similarly,

\[(15) \quad \lambda_{1,2}(\theta) = d^{-i}c(\theta)\lambda_{x,1}(\theta) + k_2.\]

Using (12), (13), (14) and (15), we find that

\[
\begin{align*}
\log M_i(t; \theta) &= k_i t + d^{\varepsilon} \lambda_{x,1}(\theta)(e^{dt} - 1) \\
\log M_i(t; \theta) &= k_i t + d^{\varepsilon}c(\theta)\lambda_{x,1}(\theta)(e^{dt} - 1),
\end{align*}
\]

where \(M_i(t; \theta)\) is the moment generating function corresponding to \(f_i(x_i; \theta)\).

This concludes the proof of Lemma 2.

3. The sum of two independent exponential-type random variables.

**Theorem 1.** If \(X_1, X_2\) are neither both normal nor both Poisson type r.v.'s, then \(X_1 + X_2\) is an exponential-type r.v. if and only if \(p'_i(\theta) = p'_2(\theta)\).

**Proof.** If \(p'_i(\theta) = p'_2(\theta)\), then if follows from (2) and (3) that

\[
\begin{align*}
\lambda_{j+1}(\theta) &= \lambda_{j+1,1}(\theta) + \lambda_{j+1,2}(\theta) \\
&= [p'_i(\theta)]^{-1}\lambda'_{j,1}(\theta) + [p'_2(\theta)]^{-1}\lambda'_{j,2}(\theta) \\
&= [p'_i(\theta)]^{-1}\lambda'_{j}(\theta).
\end{align*}
\]

Conversely, assume \(X_1 + X_2\) is an exponential-type r.v.. Then, using (1), (2), and (3), we find that

\[(16) \quad p'_i(\theta)[\lambda_{j,1}(\theta) + \lambda_{j,2}(\theta)] = p'_i(\theta)\lambda_{j,1}(\theta) + p'_2(\theta)\lambda_{j,2}(\theta).\]

In particular,
Multiplying (16) by \( \lambda_{j,1}(\theta) \) and (17) by \( \lambda_{j,2}(\theta) \) and then subtracting, we get

\[
[p'(\theta) - p'_i(\theta)]h_j(\theta) = 0 \quad \text{for } j \geq 2.
\]

Now, if for some \( j_0 \geq 2, \ h_{j_0}(\theta) \neq 0 \), then there is a subinterval, \( M \), of \( (a, b) \) in which \( h_{j_0}(\theta) \neq 0 \). Then, for \( \theta \in M, \ p_i(\theta) = p'(\theta) \). By analyticity, \( p'_i(\theta) = p'(\theta) \) for all \( \theta \in (a, b) \). Substitution in (16) yields \( p'_i(\theta) = p'(\theta) \) for \( \theta \in (a, b) \). If, on the other hand, \( h_{j}(\theta) \equiv 0 \), for \( j \geq 2 \), the result follows from Lemmas 1 and 2 since we assumed that \( X_1, X_2 \) are neither both normal nor both Poisson type r.v.’s.

It should be noted that Girshick and Savage [1] proved that if \( X_1 \) and \( X_2 \) are independent identically distributed r.v.’s such that their sum is of exponential-type, then \( X_1 \) and \( X_2 \) are also of exponential-type.

The following theorem gives necessary and sufficient conditions for the sum of two Poisson-type r.v.’s to be exponential-type.

**Theorem 2.** If \( \log M_i(t; \theta) = C_i t + A_i(\theta)[V^{*i} - 1] \), then \( X_1 + X_2 \) is an exponential-type r.v. if and only if either \( b_1 = b_2 \) or \( p'_i(\theta) = p'_i(\theta) \).

**Proof.** If \( X_1 + X_2 \) is an exponential-type r.v., then, as in the proof of the preceding theorem,

\[
[p'(\theta) - p'_i(\theta)]h_j(\theta) = 0 \quad \text{for } j \geq 2.
\]

Equivalently,

\[
[\lambda_{j,1}(\theta)\lambda_{j,2}(\theta) - \lambda_{j,2}(\theta)\lambda_{j,1}(\theta)]
= p'_i(\theta)[p'(\theta)]^{-1}[\lambda_{j,1}(\theta)\lambda_{j,2}(\theta) - \lambda_{j,2}(\theta)\lambda_{j,1}(\theta)] \quad \text{for } j \geq 2.
\]

Since, for \( j \geq 2, \lambda_{j,i}(\theta) = b_{i}A_i(\theta), \) (19) becomes

\[
[b_{i}b_{i}^2 - b_{i}b_{i}^2]A_i(\theta)A_2(\theta) = p'_i(\theta)[p'(\theta)]^{-1}[b_{i}b_{i}^2 - b_{i}b_{i}^2]A_i(\theta)A_2(\theta).
\]

But \( A_i(\theta)A_2(\theta) > 0 \), so that

\[
[b_{i}b_{i}^2 - b_{i}b_{i}^2] = p'_i(\theta)[p'(\theta)]^{-1}[b_{i}b_{i}^2 - b_{i}b_{i}^2].
\]

Now, if \( b_{i}b_{i}^2 = b_{i}b_{i}^2 \) for all \( j \geq 2, \) then \( b_{i}b_{i}^2 = b_{i}b_{i}^2 \), so that \( b_1 = b_2 \). On the other hand, if, for some \( j_0, b_{i_0}b_{i_0}^2 - b_{i_0}b_{i_0}^2 \neq 0, \) then \( p'_i(\theta) = p'(\theta) \) and it follows that \( p'_i(\theta) = p'_i(\theta) \).

Conversely, if \( p'_i(\theta) = p'(\theta) \), then \( X_1 + X_2 \) is an exponential-type r.v. since (1) is satisfied. If \( b_1 = b_2 \), let
\[ p'(\theta) = [A'_1(\theta) + A'_2(\theta)]/b_1[A_1(\theta) + A_2(\theta)] . \]

It is easy to see that (1) is again satisfied.

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