A CHARACTERIZATION OF UNIQUELY DIVISIBLE COMMUTATIVE SEMIGROUPS

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Let \((S, +)\) be a commutative semigroup. If, for each \(x \in S\), and for each positive integer \(n\), there exists an (unique) element \(y\) of \(S\) such that \(x = ny\), then \(S\) is (uniquely) divisible. In this note we present a more or less intrinsic characterization of uniquely divisible commutative semigroups and remark on a special sub-class of these semigroups in which it is possible to discern the fine structure of the addition.

2. The characterization. Let \(P\) represent the additive semigroup of positive rational numbers. By a cone of a rational vector space we mean a convex subset \(C\) such that \(PC \subseteq C\) and \(-PC \cap C = 0\). A commutative semigroup is separative if \(2a = a + b = 2b\) implies \(a = b\) for any \(a, b \in S\). Let \(L\) be the maximal (lower) semilattice homomorphic image of \(S\), and let \(h\) be the natural map of \(S\) onto \(L\). For \(e \in L\), let \(h^{-1}(e) = S_e\). The Hewitt-Zuckerman theorem [3; or 1, Th. 4.18] states that, if \(S\) is separative, then each \(S_e\) is cancellative, and \(S\) is isomorphically embeddable in a semilattice of groups, \(\{V_e\}\) in such a way that each \(V_e\) is the difference group of \(S_e\), and the semilattice is isomorphic to \(L\).

Since an uniquely divisible commutative semigroup is clearly separative, we have immediately that any such entity is isomorphic to a divisible subsemigroup of a semilattice of divisible groups. Indeed, each \(V_e\) must be uniquely divisible, and hence a rational vector space (see [4], for example). Furthermore, since each \(S_e\) is cancellative, it follows from Hancock’s theorem [2, Th. 7] that each \(S_e\) is the direct sum of a rational vector space and a cone of a rational vector space. We have now:

**Theorem 1.** Let \(S\) be an uniquely divisible commutative semigroup. Then \(S\) is a semilattice of subsemigroups \(S_e\), each of which is the direct sum of a rational vector space and a cone of a rational vector space. Furthermore, the addition in \(S\) is determined by semigroup homomorphisms between these subsemigroups which are restrictions of homomorphisms (linear maps) between their difference groups.

3. A special case. We now restrict our attention to the situation in which, for each \(e \in L\), \(S_e \cong P\). In this case, any \(x_e \in S_e\) satisfies
\[ P_{x_e} = S_e. \] By \( x_a \) we shall mean an element of \( S_a \).

**Lemma 1.** Let \( e, f \in L, e \leq f; \) let \( x_e + x_f = rx_e, r \in P \). Then \( r \geq 1 \), and for \( s, t \in P, sx_e + tx_f = [s + t(r - 1)]x_e. \)

**Proof.** Suppose \( r < 1 \) and let \( z = x_e + (1/(1 - r))x_f \). Then
\[
z = \left[ (x_e + x_f) + \left( \frac{r}{1 - r} \right)x_f \right] = \left[ rx_e + \left( \frac{r}{1 - r} \right)x_f \right] = rz.
\]
Hence, \( r = 1 \), which is a contradiction.

Now, consider \( S \) as embedded in a semilattice of rational vector spaces as in the proof of Theorem 1. We have
\[
sv_e + tv_f = (sv_e + 0_e) + tv_f \\
= sv_e + (0_e + tv_f) \\
= sv_e + (0_e + x_f) \\
= sv_e + t(0_e + x_f) \\
= sv_e + t([r - 1]x_e) \\
= (s + t[r - 1])x_e.
\]
The proof is now complete.

**Lemma 2.** Let \( e, f, g \in L, e \leq f \leq g \). Suppose \( x_e + x_f = ax_e, x_e + x_g = bx_e, x_f + x_g = cx_f, a, b, c \in P \). If any two of \( a, b, c \) equal 2, then \( a = b = c = 2 \).

**Proof.** Note \([a + (b - 1)]x_e = ax_e + x_e = (x_e + x_f) + x_g = x_e + (x_f + x_g) = x_e + cx_f = [1 + c(a - 1)]x_e. \) By the uniqueness of roots, \( a + b - 1 = 1 + c(a - 1) \), and proof is complete.

**Lemma 3.** Let \( e, f \in L \). If \( x_e + x_{ef} = x_f + x_{ef} = 2x_{ef} \), then \( x_e + x_f = 2x_{ef} \).

**Proof.** Let \( x_e + x_f = ax_{ef} \). Then \( 3x_{ef} = x_{ef} + (x_{ef} + x_e) = 2x_{ef} + x_e = (x_{ef} + x_f) + x_e = (1 + a)x_{ef}. \) Hence \( a = 2 \).

**Theorem 2.** Let \( S \) be an uniquely divisible commutative semigroup such that \( x + y \neq y, \) all \( x, y \in S \). Then \( S \cong P \times L \).

**Proof.** Fix \( e \in L, x_e \in S_e \). For each \( f \in L \), choose \( x_f \in S_f \) such that:
1. \( x_e + x_f = 2x_f \) if \( f \leq e \),
2. \( x_f + x_{ef} = 2x_{ef} \) otherwise.

Lemma 1 assures the availability of such elements; there is no
ambiguity involved provided (1) is accomplished before (2). Fix \( f, g \in L \); we shall show \( x_f + x_{ef} = x_g + x_{ef} = 2x_{ef} \). To this end, note that \( x_e + x_{ef} = 2x_{ef} \) and \( x_e + x_{ef} = 2x_{ef} \) by (1) above. Hence, by Lemma 2, \( x_{ef} + x_{ef} = 2x_{ef} \). Since \( x_e + x_f = x_{ef} \), we have \( x_f + 2x_{ef} = 3x_{ef} \); by cancellation in \( S_{ef} \), it follows that \( x_f + x_{ef} = 2x_{ef} \). By applying Lemma 2 again, we have \( x_f + x_{ef} = 2x_{ef} \). By an argument identical to the one involving \( f \) and \( ef \) above, \( x_{ef} + x_{ef} = 2x_{ef} \). Finally, applying Lemma 2 for the final time, we have \( x_f + x_{ef} = 2x_{ef} \). Similarly, \( x + x_{ef} = 2x_{ef} \); by Lemma 3 it follows that \( x_f + x_g = 2x_{ef} \). Finally, if, say \( s \geq t \), then \( s x_f + tx_g = t(x_f + x_g) + (s - t)x_f = 2tx_{ef} + (s - t)x_f = (s + t)x_{ef} \) by Lemma 1. The function \( \phi: S \to P \times L \) defined by \( \phi(rx_f) = (r, f) \) is now clearly an isomorphism.

Next, let \( L \) be any semilattice, and let \( \phi \) be a homomorphism of \( L \) onto a chain \( B \). For each \( \beta \in B \), let \( L_\beta = \phi^{-1}(\beta) \). For each \( \beta \), let \( S_\beta = P \times L_\beta \), and let \( S = \cup \{ S_\beta : \beta \in B \} \). Define an addition in \( S \) by

\[
(r, e) + (s, f) = \begin{cases} 
(r + s, ef) & \text{if } e, f \in L_\beta, \\
(r, ef) & \text{if } e \in L_\beta, f \in L_\gamma, \beta < \gamma, \\
(s, ef) & \text{if } e \in L_\beta, f \in L_\gamma, \gamma < \beta.
\end{cases}
\]

With this addition, \( S \) is an uniquely divisible commutative semigroup with maximal semilattice image \( L \) and with each \( S_\beta \cong P \). The class of semigroups thus defined will be referred to as being of type \( \mathscr{E} \).

**Theorem 3.** Let \( S \) be an uniquely divisible commutative semigroup such that each \( S_\beta \) is isomorphic to \( P \). Then \( S \) is isomorphic to a semigroup of type \( \mathscr{E} \).

**Proof.** Define a relation \( \sim \) on \( S \) by \( x \sim y \) if and only if \( x + (x + y) \neq x + y \neq y + (x + y) \). To check transitivity, let \( x \sim y, y \sim z \). In particular, let \( x + (x + y) = r(x + y) \), \( y + (y + z) = s(y + z) \), with \( r, s > 1 \). Then \( x + (x + y + z) = r(x + y) + z = rx + (r - 1)y + (y + z) = rx + s(y + z) \neq x + y + z \). Hence \( x + (x + z) \neq x + z \).

It follows by arguments similar to the above that \( \sim \) is a congruence on \( S \) and that \( S/\sim \) is a chain. Let \( j \) be the natural map of \( S \) onto \( S/\sim \); note that \( j \) factors into the composition of \( h \) and an induced map from \( L \) to \( S/\sim \). For \( \beta \in S/\sim, j^{-1}(\beta) \) satisfies the conditions of Theorem 2. Specifically, \( j^{-1}(\beta) \cong P \times hj^{-1}(\beta) \). Thus any \( x \in j^{-1}(\beta) \) has an unique representation, \( x = rx_e \), with \( e \in hj^{-1}(\beta), r \in P \), and \( x_e \) selected from \( h^{-1}(e) \) in line with the proof of Theorem 2. Suppose \( \beta, \gamma \in S/\sim, \beta < \gamma \), and let \( rx_e \in j^{-1}(\beta), sx_f \in j^{-1}(\gamma) \). Then \( x_e + x_f \in j^{-1}(\beta) \) and \( x_f + (x_e + x_f) = x_e + x_f \). Let \( x_e + x_f = tx_{ef} \). By Lemma 1, \( x_e + x_{ef} = x_{ef} \); since \( x_e, x_{ef} \in j^{-1}(\beta), x_e + x_{ef} = 2x_{ef} \). Hence \((1 + t)x_{ef} = x_{ef} + \)
\[(x_e + x_f) = (x_{ef} + x_e) + x_f = 2x_{ef} + x_f = 2x_{ef};\] hence \(t = 1\). Now, if, say \(r \leq s\), then \(rx_e + sx_f = r(x_e + x_f) + (s-r)x_f = rx_{ef} + (s-r)x_f = rx_{ef}\) by Lemma 1. If, on the other hand, \(s < r\), then \(rx_e + sx_f = s(x_e + x_f) + (r-s)x_e = sx_{ef} + (r-s)x_e = rx_{ef}\) by Lemma 1. We have now shown that the addition of \(S\) satisfies:

\[
rx_e + sx_f = \begin{cases} 
(r+s)x_{ef} & \text{if } jh^{-1}(e) = jh^{-1}(f), \\
rx_{ef} & \text{if } jh^{-1}(e) < jh^{-1}(f), \\
sx_{ef} & \text{if } jh^{-1}(f) < jh^{-1}(e).
\end{cases}
\]

The mapping \(rx_e \rightarrow (r,e)\) now establishes that \(S\) is isomorphic to a semigroup of type \(\mathcal{S}\).

In closing, we remark that the relations used in proving Theorems 2 and 3 can be reformulated in terms of the homomorphisms guaranteed by Theorem 1. In Theorem 3 in particular, if \(e \leq f\), then \(x_e \sim x_f\) if and only if the addition homomorphism is an isomorphism. Furthermore, if \(x_e\) and \(x_f\) are not equivalent, then the addition homomorphism is the zero mapping.

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