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ON REAL NUMBERS HAVING NORMALITY OF ORDER k

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This paper contains three theorems concerning real numbers having normality of order k . The first theorem gives a simple construction of a periodic decimal having normality of order k to base r . After introducing the notion of c -uniform distribution modulo one, we prove in the second theorem that α has normality of order k to base r if and only if the function αr^x is r^k -uniformly distributed modulo one. In the third theorem we show that α has normality of order k to base r if and only if, for every integer b and every positive integer $t \leq k$,

$$\lim \frac{N(b, n)}{r} = r^{-t}$$

where $N(b, n)$ is the number of integers x with $1 \leq x \leq n$ for which

$$[\alpha r^x] \equiv b \pmod{r^t}.$$

Let α be a real number, $0 < \alpha < 1$. Let r be a positive integer greater than one and construct the "decimal" representation of α to base r . Suppose that a certain sequence of digits occurs $N(n)$ times among the first n digits in the representation of α . If $N(n)/n$ tends to a limit f as n tends to infinity, then f is called the *relative frequency* with which the sequence occurs in α . If the sequence has k digits and appears in α with relative frequency r^{-k} , then it is said to occur with *normal frequency*. If every sequence of k digits appears in α with normal frequency, then α is said to have *normality of order k* . If α has normality of order k for every integer $k \geq 1$ then it was proved by Niven and Zuckerman [7] and later by Cassels [2] that α is a normal number as defined by Borel [1]. Borel proved that almost all real numbers are normal. We also note that α has normality of order one if and only if it is *simply normal* to base r . This notion is also due to Borel.

The expression "normality of order k " is due to I. J. Good who gave a method [5] for constructing decimals of period r^k having normality of order k for any $k \geq 1$. The problem was also studied by Rees [8], de Bruijn [4] and Korobov [6] who gave a variety of methods of constructing such decimals. In Section 2 of this paper we give yet another construction for a periodic decimal having normality of order k . While the method does not yield a decimal of minimum period, it has the advantage of being extremely simple.

In addition to the problem of constructing numbers having normality of order k , it is of interest to ask what characteristic properties such numbers possess. For example, D. D. Wall [9] proved that a real number α is normal to base r if and only if the function αr^x is uniformly distributed modulo one. Wall also showed that α is normal to base r if and only if, for every positive integer c and every integer b , $[\alpha r^x] \equiv b \pmod{c}$ with relative frequency $1/c$ where $[\alpha r^x]$ denotes the largest integer less than or equal to αr^x . In Section 3 we introduce the notion of c -uniform distribution modulo one and show that a real number α has normality of order k if and only if αr^x is r^k -uniformly distributed modulo one. We also show that α has normality of order k if and only if for every integer b and every integer t with $0 < t \leq k$,

$$[\alpha r^{kt}] \equiv b \pmod{r^t}$$

with relative frequency r^{-t} .

2. Construction of a number having normality of order k . Perhaps the simplest example of a normal number was given by D. G. Champernowne [3] who showed that the decimal

$$\alpha = .12345678910111213 \dots$$

is normal to base 10 where α is formed by writing the decimal representations of the natural numbers in order after the decimal point. Analogously, we prove the following theorem.

THEOREM 1. *Let r and k be integers with $r \geq 2$ and $k \geq 1$. Working to base r^k form the periodic decimal*

$$\alpha = .\dot{0}12 \dots (r^k \dot{-} 1).$$

Written to base r , α has period kr^k and normality of order k .

Proof. Let Y_n denote the block $a_1 a_2 \dots a_n$ of the first n digits of the representation of α to base r and let $B_k = b_1 b_2 \dots b_k$ denote an arbitrary sequence of k digits to base r . Let C_i denote the i th digit in the representation of α to base r^k . We will also use C_i to denote the block of k digits in the representation of α to base r which corresponds to the digit C_i in the representation of α to base r^k . Thus, we use C_1 to denote 0 and also to denote the block of k zeros with which the representation of α to base r begins. In any given instance the intended meaning will be clear from the context.

Since the representation of α is periodic, it clearly suffices to show

that every B_k appears precisely k times starting in Y_{kr^k} . We note that B_k appears precisely once starting in Y_{kr^k} as one of the C_i ; i.e., starting in Y_{kr^k} in a position congruent to one modulo k . The problem is to determine how many times B_k appears starting in Y_{kr^k} in a position congruent to $k - j + 1$ for each $j = 1, 2, \dots, k - 1$. This is equivalent to asking how many times B_k appears with the mid-point of two adjacent C_i 's coming between the j th and $(j + 1)$ st digits of B_k for each j . And this occurs when and only when, for some i ,

$$C_i = c_1c_2 \cdots c_{k-j}b_1b_2 \cdots b_j$$

and

$$C_{i+1} = b_{j+1}b_{j+2} \cdots b_kd_1d_2 \cdots d_j .$$

Case 1. Suppose that at least one of b_1, b_2, \dots, b_j is different from $r - 1$. Then, for some i ,

$$C_i = b_{j+1}b_{j+2} \cdots b_kb_1b_2 \cdots b_j$$

and

$$C_{i+1} = b_{j+1}b_{j+2} \cdots b_kd_1d_2 \cdots d_j$$

where $d_1d_2 \cdots d_j$ is the successor to $b_1b_2 \cdots b_j$ in the sequence of j -tuples

$$(1) \quad 00 \cdots 0, 00 \cdots 01, \dots, (r - 1) \cdots (r - 1) .$$

Thus, in this case, B_k does appear starting in Y_{kr^k} in a position congruent to $k - j + 1$ and this is the only way it can appear in this position.

Case 2. Suppose that $b_1 = b_2 = \cdots = b_j = r - 1$ and that at least one of $b_{j+1}, b_{j+2}, \dots, b_k$ is different from zero. If $d_{j+1}d_{j+2} \cdots d_k$ is the predecessor of $b_{j+1}b_{j+2} \cdots b_k$ in the sequence of $(k - j)$ -tuples

$$(2) \quad 00 \cdots 0, 00 \cdots 01, \dots, (r - 1) \cdots (r - 1) ,$$

then, for some i ,

$$C_i = d_{j+1}d_{j+2} \cdots d_kb_1b_2 \cdots b_j$$

and

$$C_{i+1} = b_{j+1}b_{j+2} \cdots b_k 00 \cdots 0 .$$

Thus, in this case, B_k again appears starting in Y_{kr^k} in a position congruent to $k - j + 1$ modulo k and this is the only way it can appear in this position.

Case 3. Finally, suppose that $b_1 = b_2 = \dots = b_j = r - 1$ and that $b_{j+1} = b_{j+2} = \dots = b_k = 0$. The only way such a B_k can appear starting in Y_{kr^k} in a position congruent to $k - j + 1$ is for

$$b_{j+1}b_{j+2} \dots b_k = 00 \dots 0$$

to have a predecessor in the sequence (2). Thus, in this case, B_k cannot appear in the desired position entirely contained in Y_{kr^k} . However, it clearly does appear starting in a position congruent to $k - j + 1$ modulo k in Y_{kr^k} and overlapping the mid-point between Y_{kr^k} and the next sequence of kr^k digits in the representation of α to base r .

Therefore, for each $j = 1, 2, \dots, k$, B_k occurs in the representation of α to base r starting in Y_{kr^k} in a position congruent to $k - j + 1$ modulo k precisely once. Since B_k was arbitrary, it follows that each sequence of k digits to base r appears in the representation of α to base r equally often. Thus, α has normality of order k as claimed.

Since the α of the preceding theorem is simply normal to base r^k , it is natural to ask if normality of order k to base r is implied by simple normality to base r^k . However, since $\beta = .\dot{1}02\dot{3}$ is simply normal to base 4 but does not have normality of order 2 to base 2, this is clearly not the case.

3. Properties of numbers having normality of order k . Let $(\alpha) = \alpha - [\alpha]$ denote the fractional part of the real number α . A real valued function $f(x)$ is said to be uniformly distributed modulo one if, for every real λ with $0 \leq \lambda \leq 1$, $\lim n_\lambda/n = \lambda$ where n_λ denotes the number of values of $x = 1, 2, \dots, n$ for which $(f(x)) < \lambda$. Analogously, for any integer $c > 1$, we say that $f(x)$ is *c-uniformly distributed modulo one* if the preceding definition holds for all λ 's which are positive rational fractions with denominator c . It then follows that $f(x)$ is uniformly distributed modulo one if and only if it is *c-uniformly distributed modulo one* for every integer $c > 1$. We also have the following result concerning numbers having normality of order k .

THEOREM 2. *The real number α has normality of order k to base r if and only if the function αr^x is r^k -uniformly distributed modulo one.*

Proof. Let αr^x be r^k -uniformly distributed modulo one. Let $b_1 b_2 \dots b_k$ denote an arbitrary sequence of digits to base r and let

$$\varepsilon = b_1 r^{-1} + b_2 r^{-2} + \dots + b_k r^{-k}.$$

It then follows that $\varepsilon \leq (\alpha r^x) < \varepsilon + r^{-k}$ with relative frequency r^{-k} .

But this simply says that the sequence $b_1 b_2 \cdots b_k$ appears in the representation of α to base r with normal frequency so that α has normality of order k .

Conversely, suppose that α has normality of order k to base r . Let $\lambda = br^{-k}$ where b is an integer and $0 < b < r^k$. Then λ can be written in the form

$$\lambda = b_1 r^{-1} + b_2 r^{-2} + \cdots + b_k r^{-k}, \quad 0 \leq b_i < r$$

and $(\alpha r^x) < \lambda$ if and only if

$$a_{1+x} r^{-1} + a_{2+x} r^{-2} + \cdots + a_{k+x} r^{-k} < b_1 r^{-1} + b_2 r^{-2} + \cdots + b_k r^{-k}.$$

This inequality is equivalent to

$$a = a_{1+x} r^{k-1} + \cdots + a_{k+x} < b_1 r^{k-1} + \cdots + b_k = b$$

and it follows that $(\alpha r^x) < \lambda$ if and only if $a < b$. Clearly there are just b nonnegative integers a having this property and, by hypothesis, each k -tuple corresponding to such an a appears in the representation of α to base r with frequency r^{-k} . Therefore, $(\alpha r^x) < \lambda$ with frequency $br^{-k} = \lambda$ and α is r^k -uniformly distributed modulo one.

As noted above the following theorem is also analogous to a result of Wall.

THEOREM 3. *The real number α has normality of order k to base r if and only if, for every positive integer $t \leq k$ and every integer b , we have $[\alpha r^x] \equiv b \pmod{r^t}$ with relative frequency r^{-t} .*

Proof. There is no loss in generality in assuming that $0 \leq b < r^t$.

Suppose first that α has normality of order k to base r . Then αr^{-t} also has normality of order k . Therefore, by Theorem 2, αr^{x-t} is r^k -uniformly distributed modulo one and it follows that

$$br^{-t} \leq (\alpha r^{x-t}) < (b + 1)r^{-t}$$

with relative frequency $r^{-t} = r^{k-t} r^{-k}$. Thus, there exist positive integers n_x with relative frequency r^{-t} such that

$$n_x + br^{-t} \leq \alpha r^{x-t} < n_x + (b + 1)r^{-t}$$

or, equivalently, such that

$$n_x r^t + b \leq \alpha r^x < n_x r^t + b + 1.$$

But this says that

$$[\alpha r^x] \equiv b \pmod{r^t}$$

with relative frequency r^{-t} .

To prove the converse, we simply reverse the preceding argument reading k for t at each step.

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Edward Joseph Barbeau, <i>Semi-algebras that are lower semi-lattices</i>	1
Steven Fredrick Bauman, <i>The Klein group as an automorphism group without fixed point</i>	9
Homer Franklin Bechtell, Jr., <i>Frattini subgroups and Φ-central groups</i>	15
Edward Kenneth Blum, <i>A convergent gradient procedure in prehilbert spaces</i>	25
Edward Martin Bolger, <i>The sum of two independent exponential-type random variables</i>	31
David Wilson Bressler and A. P. Morse, <i>Images of measurable sets</i>	37
Dennison Robert Brown and J. G. LaTorre, <i>A characterization of uniquely divisible commutative semigroups</i>	57
Selwyn Ross Caradus, <i>Operators of Riesz type</i>	61
Jeffrey Davis and Isidore Isaac Hirschman, Jr., <i>Toeplitz forms and ultraspherical polynomials</i>	73
Lorraine L. Foster, <i>On the characteristic roots of the product of certain rational integral matrices of order two</i>	97
Alfred Gray and S. M. Shah, <i>Asymptotic values of a holomorphic function with respect to its maximum term</i>	111
Sidney (Denny) L. Gulick, <i>Commutativity and ideals in the biduals of topological algebras</i>	121
G. J. Kurowski, <i>Further results in the theory of monodiffrie functions</i>	139
Lawrence S. Levy, <i>Commutative rings whose homomorphic images are self-injective</i>	149
Calvin T. Long, <i>On real numbers having normality of order k</i>	155
Bertram Mond, <i>An inequality for operators in a Hilbert space</i>	161
John William Neuberger, <i>The lack of self-adjointness in three-point boundary value problems</i>	165
C. A. Persinger, <i>Subsets of n-books in E^3</i>	169
Oscar S. Rothaus and John Griggs Thompson, <i>A combinatorial problem in the symmetric group</i>	175
Rodolfo DeSapio, <i>Unknotting spheres via Smale</i>	179
James E. Shockley, <i>On the functional equation $F(mn)F((m, n)) = F(m)F(n)f((m, n))$</i>	185
Kenneth Edward Whipple, <i>Cauchy sequences in Moore spaces</i>	191