AN INEQUALITY FOR OPERATORS IN A HILBERT SPACE

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Let $A$ be a self-adjoint operator on a Hilbert space $H$ satisfying $mI \leq A \leq MI$, $0 < m < M$. Set $q = M/m$. Let $j$ and $k$ be real numbers, $jk \neq 0$, $j < k$. Then

\[
(A^k x, x)^{1/k}(A^j x, x)^{1/j} 
\leq \frac{(j^{-1}(q^j - 1))^{-1/j}k^{-1}(q^k - 1))^{1/j}(k-j)^{-1}(q^k - q^j)(x, x)^{(1/k) - (1/j)}}{1/j}
\]

for all $x \in H(x \neq 0)$. Letting $j = -1$ and $k = 1$, this inequality reduces to $(Ax, x)(A^{-1}x, x) \leq [(M + m)^2/4mM](x, x)$, the well-known Kantorovich Inequality.

Preliminaries. We shall make use of the following four inequalities:

For $a > 0$, $b > 0$,

\[
(1) \quad a^b - a \leq a - (1 - a)b \quad \text{if } 0 < a < 1
\]

\[
(2) \quad a^b - a \geq a - (1 - a)b \quad \text{if } a < 0.
\]

For $j < k$, $1 \leq y \leq q$,

\[
(3) \quad (q^k - 1)y^j - (q^j - 1)y^k - (q^k - q^j) \geq 0 \quad \text{if } jk > 0
\]

\[
(4) \quad -(q^k - 1)y^j + (q^j - 1)y^k + (q^k - q^j) \geq 0 \quad \text{if } jk < 0.
\]

(1) is the well-known inequality between the arithmetic and geometric means. Simple proofs of (2), (3) and (4) can be found in a recent paper by Goldman [3].

Let $C$ be a self-adjoint operator on a Hilbert space $H$ satisfying

\[
I \leq C \leq qI
\]

where $I$ is the identity operator (and (5) is understood in the usual sense that $(x, x) \leq (Cx, x) \leq q(x, x)$ for all $x \in H$). To the real valued function $u(\lambda)$, defined and continuous on $[1, q]$, there is associated in a natural way a self-adjoint operator on $H$ denoted by $u(C)$ (see e.g. [6] pp. 265–273).

We shall make use of the following [loc. cit.]:

**Lemma.** If $u(\lambda) \geq 0$ for $1 \leq \lambda \leq q$, then $u(C) \geq 0$, i.e., $u(C)$ is a positive operator.
Results.

THEOREM 1. Let C be a self-adjoint operator on a Hilbert space \( H \) satisfying \( I \leq C \leq qI \). Let \( j \) and \( k \) be real numbers, \( j < k, jk \neq 0 \). The operator

\[
(q^k - 1)C^j - (q^j - 1)C^k - (q^k - q^j)I
\]

is positive if \( jk > 0 \); while the operator

\[
-(q^k - 1)C^j + (q^j - 1)C^k + (q^k - q^j)I
\]

is positive if \( jk < 0 \).

Proof. The theorem follows directly from (3) and (4) by virtue of the Lemma.

Letting \( j = -1 \) and \( k = 1 \), Theorem 1 yields an inequality that is equivalent to one given by Diaz and Metcalf [2].

The following theorem, which is the main result of this paper, is a Hilbert space generalization of Cargo and Shisha [1] and Mond [5].

THEOREM 2. Let A be self-adjoint operator on a Hilbert space \( H \) satisfying \( mI \leq A \leq MI, 0 < m < M \). Set \( q = M/m \). Let \( j \) and \( k \) be real numbers \( jk \neq 0, j < k \). Then

\[
(A^k x, x)^{(1/k)} \leq (q^k - 1)^{1/k} \{k^{-1}(q^k - 1)\}^{1/j} \{(k - j)^{-1}(q^k - q^j)(x, x)\}^{(1/k)-(1/j)}
\]

for all \( x \in H(x \neq 0) \).

Proof. Set \( C \equiv A/m \). It obviously suffices to prove

\[
(C^k x, x)^{(1/k)} \leq (q^k - 1)^{1/k} \{k^{-1}(q^k - 1)\}^{1/j} \{(k - j)^{-1}(q^k - q^j)(x, x)\}^{(1/k)-(1/j)}.
\]

Since \( C \) satisfies (5), by Theorem 1,

\[
(q^k - 1)(C^j x, x) - (q^j - 1)(C^k x, x) \geq (q^k - q^j)(x, x) \quad \text{if } jk > 0
\]

and

\[
(q^k - 1)(C^j x, x) - (q^j - 1)(C^k x, x) \leq (q^k - q^j)(x, x) \quad \text{if } jk < 0.
\]

Rewrite (10) as

\[
-(q^k - 1)(C^j x, x) + (q^j - 1)(C^k x, x) \geq (k - j)^{-1}(q^k - q^j)(x, x)
\]

if \( jk > 0 \), and (11) as

\[
-k(k - j)^{-1}(q^k - 1)(C^j x, x) + \{k(k - j)^{-1}\} \{(k - j)^{-1}(q^k - q^j)(x, x)\}
\]

if \( jk < 0 \).
\[
-j(k-j)^{-1}\{j^{-1}(q^i-1)(C^k x, x)\} + \{k(k-j)^{-1}\{k^{-1}(q^k-1)(C^j x, x)\}
\leq (k-j)^{-1}(q^k-q^j)(x, x)
\]

if \(jk < 0\).

Assume \(k > 0\). Set
\[
a = j^{-1}(q^i-1)(C^k x, x), b = k^{-1}(q^k-1)(C^j x, x), \alpha = -j(k-j)^{-1}.
\]

If \(j > 0\), applying (2) and combining with (12), we obtain
\[
\{j^{-1}(q^i-1)(C^k x, x)\}^{1/(k-j)}\{k^{-1}(q^k-1)(C^j x, x)\}^{1/(k-j)}
\geq (k-j)^{-1}(q^k-q^j)(x, x)
\]

which when raised to the power \((k-j)/(k-j)\) yields
\[
\{j^{-1}(q^i-1)(C^k x, x)\}^{1/(k-j)}\{k^{-1}(q^k-1)(C^j x, x)\}^{1/(k-j)}
\leq \{(k-j)^{-1}(q^k-q^j)(x, x)\}^{(1/(k-j))}
\]

If \(j < 0 (k > 0)\), applying (1) and combining with (13) yields the reverse of (14) which, when raised to the power \((k-j)/(k-j)\), yields (15).

Finally, if \(j < k < 0\), set
\[
a = k^{-1}(q^k-1)(C^j x, x), b = j^{-1}(q^i-1)(C^k x, x), \alpha = k(k-j)^{-1}.
\]

Applying (2) and combining with (12) yields (14) which, when raised to the power \((k-j)/(k-j)\) yields (15). In all cases, therefore, we have (15), a rearrangement of (9). (Compare the method of proof of Theorem 2 with Goldman [3].)

The well-known [4] Kantorovich inequality, \((Ax, x)(A^{-1}x, x) \leq [(m + M)^2/4mM](x, x)^2\), is the special case of Theorem 2 with \(j = -1\), \(k = 1\).

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