

# Pacific Journal of Mathematics

**AN INEQUALITY FOR OPERATORS IN A HILBERT SPACE**

BERTRAM MOND

## AN INEQUALITY FOR OPERATORS IN A HILBERT SPACE

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Let  $A$  be a self-adjoint operator on a Hilbert space  $H$  satisfying  $mI \leq A \leq MI$ ,  $0 < m < M$ . Set  $q = M/m$ . Let  $j$  and  $k$  be real numbers,  $jk \neq 0$ ,  $j < k$ . Then

$$(A^k x, x)^{1/k} / (A^j x, x)^{1/j} \leq \{j^{-1}(q^j - 1)\}^{-1/k} \{k^{-1}(q^k - 1)\}^{1/j} \{(k - j)^{-1}(q^k - q^j)(x, x)\}^{(1/k) - (1/j)}$$

for all  $x \in H (x \neq 0)$ . Letting  $j = -1$  and  $k = 1$ , this inequality reduces to  $(Ax, x)(A^{-1}x, x) \leq [(M + m)^2/4mM](x, x)^2$ , the well-known Kantorovich Inequality.

**Preliminaries.** We shall make use of the following four inequalities:

For  $a > 0, b > 0$ ,

$$(1) \quad \alpha^a b^{1-\alpha} \leq \alpha a + (1 - \alpha)b \quad \text{if } 0 < \alpha < 1$$

$$(2) \quad \alpha^a b^{1-\alpha} \geq \alpha a + (1 - \alpha)b \quad \text{if } \alpha < 0.$$

For  $j < k, 1 \leq y \leq q$ ,

$$(3) \quad (q^k - 1)y^j - (q^j - 1)y^k - (q^k - q^j) \geq 0 \quad \text{if } jk > 0$$

$$(4) \quad -(q^k - 1)y^j + (q^j - 1)y^k + (q^k - q^j) \geq 0 \quad \text{if } jk < 0.$$

(1) is the well-known inequality between the arithmetic and geometric means. Simple proofs of (2), (3) and (4) can be found in a recent paper by Goldman [3].

Let  $C$  be a self-adjoint operator on a Hilbert space  $H$  satisfying

$$(5) \quad I \leq C \leq qI$$

where  $I$  is the identity operator (and (5) is understood in the usual sense that  $(x, x) \leq (Cx, x) \leq q(x, x)$  for all  $x \in H$ ). To the real valued function  $u(\lambda)$ , defined and continuous on  $[1, q]$ , there is associated in a natural way a self-adjoint operator on  $H$  denoted by  $u(C)$  (see e.g. [6] pp. 265-273).

We shall make use of the following [loc. cit.]:

**LEMMA.** *If  $u(\lambda) \geq 0$  for  $1 \leq \lambda \leq q$ , then  $u(C) \geq 0$ , i.e.,  $u(C)$  is a positive operator.*

**Results.**

**THEOREM 1.** *Let  $C$  be a self-adjoint operator on a Hilbert space  $H$  satisfying  $I \leq C \leq qI$ . Let  $j$  and  $k$  be real numbers,  $j < k$ ,  $jk \neq 0$ . The operator*

$$(6) \quad (q^k - 1)C^j - (q^j - 1)C^k - (q^k - q^j)I$$

*is positive if  $jk > 0$ ; while the operator*

$$(7) \quad -(q^k - 1)C^j + (q^j - 1)C^k + (q^k - q^j)I$$

*is positive if  $jk < 0$ .*

*Proof.* The theorem follows directly from (3) and (4) by virtue of the Lemma.

Letting  $j = -1$  and  $k = 1$ , Theorem 1 yields an inequality that is equivalent to one given by Diaz and Metcalf [2].

The following theorem, which is the main result of this paper, is a Hilbert space generalization of Cargo and Shisha [1] and Mond [5].

**THEOREM 2.** *Let  $A$  be self-adjoint operator on a Hilbert space  $H$  satisfying  $mI \leq A \leq MI$ ,  $0 < m < M$ . Set  $q = M/m$ . Let  $j$  and  $k$  be real numbers  $jk \neq 0$ ,  $j < k$ . Then*

$$(8) \quad \begin{aligned} & (A^k x, x)^{1/k} / (A^j x, x)^{1/j} \\ & \leq \{j^{-1}(q^j - 1)\}^{-1/k} \{k^{-1}(q^k - 1)\}^{1/j} \{(k - j)^{-1}(q^k - q^j)(x, x)\}^{(1/k) - (1/j)} \end{aligned}$$

*for all  $x \in H$  ( $x \neq 0$ ).*

*Proof.* Set  $C \equiv A/m$ . It obviously suffices to prove

$$(9) \quad \begin{aligned} & (C^k x, x)^{1/k} / (C^j x, x)^{1/j} \\ & \leq \{j^{-1}(q^j - 1)\}^{-1/k} \{k^{-1}(q^k - 1)\}^{1/j} \{(k - j)^{-1}(q^k - q^j)(x, x)\}^{(1/k) - (1/j)}. \end{aligned}$$

Since  $C$  satisfies (5), by Theorem 1,

$$(10) \quad (q^k - 1)(C^j x, x) - (q^j - 1)(C^k x, x) \geq (q^k - q^j)(x, x) \quad \text{if } jk > 0$$

and

$$(11) \quad (q^k - 1)(C^j x, x) - (q^j - 1)(C^k x, x) \leq (q^k - q^j)(x, x) \quad \text{if } jk < 0.$$

Rewrite (10) as

$$(12) \quad \begin{aligned} & \{-j(k - j)^{-1}\} \{j^{-1}(q^j - 1)(C^k x, x)\} + \{k(k - j)^{-1}\} \{k^{-1}(q^k - 1)(C^j x, x)\} \\ & \geq (k - j)^{-1}(q^k - q^j)(x, x) \end{aligned}$$

if  $jk > 0$ , and (11) as

$$(13) \quad \{-j(k-j)^{-1}\}\{j^{-1}(q^j-1)(C^kx, x)\} + \{k(k-j)^{-1}\}\{k^{-1}(q^k-1)(C^jx, x)\} \\ \leq (k-j)^{-1}(q^k - q^j)(x, x)$$

if  $jk < 0$ .

Assume  $k > 0$ . Set

$$a = j^{-1}(q^j - 1)(C^kx, x), b = k^{-1}(q^k - 1)(C^jx, x), \alpha = -j(k-j)^{-1}.$$

If  $j > 0$ , applying (2) and combining with (12), we obtain

$$(14) \quad \{j^{-1}(q^j - 1)(C^kx, x)\}^{-j/(k-j)}\{k^{-1}(q^k - 1)(C^jx, x)\}^{k/(k-j)} \\ \geq (k-j)^{-1}(q^k - q^j)(x, x)$$

which when raised to the power  $(k-j)/(-kj)$  yields

$$(15) \quad \{j^{-1}(q^j - 1)(C^kx, x)\}^{1/k}\{k^{-1}(q^k - 1)(C^jx, x)\}^{-1/j} \\ \leq \{(k-j)^{-1}(q^k - q^j)(x, x)\}^{(1/k) - (1/j)}.$$

If  $j < 0$  ( $k > 0$ ), applying (1) and combining with (13) yields the reverse of (14) which, when raised to the power  $(k-j)/(-kj)$ , yields (15).

Finally, if  $j < k < 0$ , set

$$a = k^{-1}(q^k - 1)(C^jx, x), b = j^{-1}(q^j - 1)(C^kx, x), \alpha = k(k-j)^{-1}.$$

Applying (2) and combining with (12) yields (14) which, when raised to the power  $(k-j)/(-kj)$  yields (15). In all cases, therefore, we have (15), a rearrangement of (9). (Compare the method of proof of Theorem 2 with Goldman [3].)

The well-known [4] Kantorovich inequality,  $(Ax, x)(A^{-1}x, x) \leq [(m+M)^2/4mM](x, x)^2$ , is the special case of Theorem 2 with  $j = -1$ ,  $k = 1$ .

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