THE LACK OF SELF-ADJOINTNESS IN THREE-POINT BOUNDARY VALUE PROBLEMS

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Suppose that $a < c < b$, $C_{[a,b]}$ is the set of all real-valued continuous functions on $[a, b]$, each of $p$ and $q$ is in $C_{[a,b]}$, $p(x) > 0$ for all $x$ in $[a, b]$ and each of $P, Q$ and $S$ is a real $2 \times 2$ matrix. The assumption is made that the only member $f$ of $C_{[a,b]}$ so that $(pf')' - qf = 0$ and

\[
\begin{bmatrix}
 f(a) \\
 p(a)f'(a)
\end{bmatrix} + Q
\begin{bmatrix}
 f(c) \\
 p(c)f'(c)
\end{bmatrix} + S
\begin{bmatrix}
 f(b) \\
 p(b)f'(b)
\end{bmatrix} = 0
\]

is the zero function. It follows that there is a real-valued continuous function $K_{12}$ on $[a, b] \times [a, b]$ such that if $g$ is in $C_{[a,b]}$, then the only element $f$ of $C_{[a,b]}$ so that $(pf')' - qf = g$ and $(J)$ holds is given by

\[
f(x) = \int_a^b K_{12}(x, t)g(t)dt \quad \text{for all } x \in [a, b].
\]

In this note it is shown that if in addition it is specified that $Q$ is not the zero $2 \times 2$ matrix, then $K_{12}$ is not symmetric, i.e., it is not true that $K_{12}(x, t) = K_{12}(t, x)$ for all $x, t$ in $[a, b]$.

The union of $(a, c)$ and $(c, b)$ is denoted by $R$. The symbol $j$ denotes the identity function on $[a, b]$, i.e., $j(x) = x$ for all $x$ in $[a, b]$. If $V$ is a function from $[a, b] \times [a, b]$ and $x$ is in $[a, b]$, then $V(j, x)$ is the function $h$ such that $h(t) = V(t, x)$ for all $t$ in $[a, b]$. If each of $f$ and $(pf')' - qf$ is in $C_{[a,b]}$, then $(pf')' - qf$ is denoted by $L_f$.

Given an element $g$ of $C_{[a,b]}$, one has the problem of determining a function $f$ so that

\[
\begin{cases}
 Lf = g \\
 (J) \quad \text{holds}.
\end{cases}
\]

Denote \[
\begin{bmatrix}
 0 & 1/p \\
 \int_a^t q & 0
\end{bmatrix}
\]
by $F(t)$ and
\[
\begin{bmatrix}
 0 \\
 \int_a^t q
\end{bmatrix}
\]
by $G(t)$ for all $t$ in $[a, b]$. Then problem $(*)$ may be reformulated as follows: find a function $Y$ from $[a, b]$ to $E_2$ such that

\[
Y(t) = Y(x) + G(t) - G(x) + \int_x^t dF \cdot Y \quad \text{for all } t, x \in [a, b]
\]

This completes the proof of the theorem.
\[
\int_a^b dH \cdot Y = N \quad \text{where}
\]
\[
H(x) = \begin{cases} 0 & \text{if } x = a \\ P & \text{if } a < x \leq c \\ P + Q & \text{if } c < x < b \\ P + Q + S & \text{if } x = b. \end{cases}
\]

The assumption is made for the rest of this paper that only the function \(Y\) which is constant at \(N\) satisfies (**) if \(G\) is constant at \(N\). It follows that for each continuous function \(G\) from \([a, b]\) to \(E_t\), (**) has exactly one solution.

Consider the function \(M\) from \([a, b] \times [a, b]\) to the set of \(2 \times 2\) matrices which has the following property:

\[
M(t, x) = I + \int_a^t dF \cdot M(j, x) \quad \text{for all } t, x \in [a, b]
\]

where \(I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\). Using Theorem B of [2], one has that the unique solution \(Y\) of (**) is given by

\[
Y(t) = \int_a^b K(t, j)dG \quad \text{for all } t \in [a, b]
\]

where

\[
K(t, x) = \begin{cases} \left[\int_a^b dH \cdot M(j, t)\right]^{-1} \int_a^b dH \cdot M(j, x) + M(t, x) & \text{if } a \leq x \leq t \\ \left[\int_a^b dH \cdot M(j, t)\right]^{-1} \int_a^b dH \cdot M(j, x) & \text{if } t < x \leq b. \end{cases}
\]

That \(\left[\int_a^b dH \cdot M(j, t)\right]^{-1}\) exists for all \(t \in [a, b]\) follows from the assumption that was made above.

Some straightforward calculation gives that

\[
K(t, x) = \begin{cases} (M(t, b)U(x)M(b, x) + M(t, x) & \text{if } a \leq x \leq t \\ M(t, b)U(x)M(b, x) & \text{if } t < x \leq b \end{cases}
\]

where

\[
U(x) = \begin{bmatrix} u_{11}(x) & u_{12}(x) \\ u_{21}(x) & u_{22}(x) \end{bmatrix} = -\left[\int_a^b dH \cdot M(j, b)\right]^{-1} \int_a^b dH \cdot M(j, b)
\]

for all \(x \in [a, b]\).

Note that \(Y = \begin{bmatrix} f \\ Pf' \end{bmatrix}\) where \(f\) is the unique solution to (*). Denote \(K\) by \(\begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix}\). It follows that
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\[ f(t) = \int_{a}^{b} K_{12}(t, j) g dj \text{ for all } t \text{ in } [a, b]. \]

**THEOREM A.** If \( Q \) is not the 0-matrix (i.e., (*) is a three-point problem) then it is not true that \( K_{12}(t, x) = K_{12}(x, t) \) for all \( x \) and \( t \) in \( R \).

**Proof.** Denote \( M \) by \( \begin{bmatrix} A & B \\ C & D \end{bmatrix} \). From [2] one has the following identities:

\[
B(t, x) = A(t, b)B(b, x) + B(t, b)D(b, x) \quad \text{if } x \text{ and } t \text{ are in } [a, b] \\
(\text{since } M(t, b)M(b, x) = M(t, x) \text{ for all } t, x \text{ in } [a, b]), \\
A(t, x)D(t, x) - B(t, x)C(t, x) = 1 \quad \text{(i.e., } \det M(t, x) = 1) , \\
A(t, x) = D(x, t) , \\
B(t, x) = -B(x, t) , \quad \text{and} \\
C(t, x) = -C(x, t) \quad \text{if } x \text{ and } t \text{ are in } [a, b]. \\
\]

Note that \( LA(j, x) = LB(j, x) = 0 \) if \( x \) is in \( [a, b] \).

Suppose that \( K_{12}(t, x) = K_{12}(x, t) \) for all \( x \) and \( t \) in \( R \).

If \( a < x < t < b \), then

\[
K_{12}(t, x) = [A(t, b)u_{11}(x) + B(t, b)u_{21}(x)]B(b, x) \\
+ [A(t, b)u_{12}(x) + B(t, b)u_{22}(x)]D(b, x) + B(t, x)
\]

and

\[
K_{12}(x, t) = [A(x, b)u_{11}(t) + B(x, b)u_{21}(t)]B(b, t) \\
+ [A(x, b)u_{12}(t) + B(x, b)u_{22}(t)]D(b, t) .
\]

Using the identities listed above,

\[
A(t, b)[-u_{11}(x)B(x, b) + u_{12}(x)A(x, b) - B(x, b)] \\
+ B(t, b)[-u_{21}(x)B(x, b) + u_{22}(x)A(x, b) + A(x, b)] \\
= A(t, b)[u_{12}(t)A(x, b) + u_{22}(t)B(x, b)] \\
- B(t, b)[u_{11}(t)A(x, b) + u_{21}(t)B(x, b)] .
\]

An examination of this expression yields the fact that it remains true if \( x \) and \( t \) are interchanged or \( x \) is set equal to \( t \).

Denote by \( x \) a number in \( R \). Since \( u_{11}, u_{11}, u_{22}, u_{22} \) are constant on \( (a, c) \) and \( (c, b) \) and \( A(j, b) \) and \( B(j, c) \) are independent solutions \( v \) of \( L v = 0 \), it follows that

\[-u_{11}(x)B(x, b) + u_{12}(x)A(x, b) - B(x, b) = u_{12}(t)A(x, b) + u_{22}(t)B(x, b) \]
and
\[-u_{21}(x)B(x, b) + u_{22}(x)A(x, b) + A(x, b) = -u_{11}(t)A(x, b) -u_{12}(t)B(x, b)\]
for all \(x\) and \(t\) in \(R\).

Similarly, it follows that
(i) \(-u_{11}(x) - 1 = u_{22}(t)\),
(ii) \(u_{12}(x) = u_{12}(t)\),
(iii) \(u_{21}(x) = u_{21}(t)\) and
(iv) \(u_{22}(x) + 1 = -u_{11}(t)\) for all \(x\) and \(t\) in \(R\).

(ii) and (iii) imply that \(u_{12}\) and \(u_{21}\) are constant on \(R\). (i) and (iv) give the same information so that only (i) need be considered. Denote \(u_{11}(c -)\) by \(c_1\), \(u_{22}(c -)\) by \(c_3\), \(u_{11}(c +)\) by \(c_3\) and \(u_{22}(c +)\) by \(c_4\). Hence (i) gives that \(c_1 + c_2 = -1, c_1 + c_4 = -1, c_3 + c_4 = -1\) and \(c_3 + c_2 = -1\).

But these equations imply that \(c_2 = c_4\) and \(c_1 = c_3\), i.e., that \(u_{11}\) and \(u_{22}\) are constant on \(R\). Hence, \(U\) is constant on \(R\). If \(t\) is in \((a, c)\) and \(x\) is in \((c, b)\), then
\[
\left[\int_a^b dH \cdot M(j, b)\right]^{-1} \int_t^z dH \cdot M(j, b) = U(x) - U(t) = 0
\]
so that
\[
QM(c, b) = \int_t^z dH \cdot M(j, b) = 0,
\]
i.e., \(Q = 0\), a contradiction. Hence the theorem is established.

If \(n\) is an integer greater than 3, this theorem can be extended to \(n\) point boundary value problems. This is the case in which \(H\) is a step function with \(n\) discontinuities (with one at \(a\) and another at \(b\)). What happens when \(H\) has points of change other than discontinuities is not at all clear to this author.

\section*{References}

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