ON THE FUNCTIONAL EQUATION

\[ F(mn)F((m, n)) = F(m)F(n)f((m, n)) \]

James E. Shockley
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Let \( f \) be a multiplicative arithmetic function, \( f(1) = 1 \). Necessary and sufficient conditions on \( f \) will be found so that the functional equation
\[ F(mn)F((m, n)) = F(m)F(n)f((m, n)) \]
will have a solution \( F \) with \( F(1) \neq 0 \) and all solution \( F \) will be determined. It will be shown that two different types of solutions may exist and that one of these requires that \( f \) have a property similar to complete multiplicativity.

The special case of the equation
\[ F(mn)F((m, n)) = F(m)F(n)f((m, n)) \]
with \( f \) completely multiplicative and \( F(1) \neq 0 \) was solved completely by Apostol and Zuckerman [1]. Specialized results were also given for the case \( F(1) = 0 \), but this case was not solved in general.

We note that if \( f(1) = 0 \) then \( f \) is identically zero and is thus completely multiplicative. This case was solved completely in [1] and will not be considered here. Thus in the sequel we will assume that \( f \) is multiplicative and is not identically zero (which implies that \( f(1) = 1 \).

1. If \( F \) is a solution of (1) with \( F(1) \neq 0 \) then any constant multiple of \( F \) is also a solution. Thus we may reduce the problem of solving (1) with \( F(1) \neq 0 \) to that of solving
\[ F(mn)F((m, n)) = F(m)F(n)f((m, n)), \quad F(1) = 1. \]

The proof of Theorem 1 is essentially the same as that of Theorem 2 of [1] and will be omitted.

**Theorem 1.** \( F \) is a solution of (2) if, and only if, \( F \) is a non-zero multiplicative function and for each prime \( p \)
\[ F(p^{a+b})F(p^a) = F(p^b)F(p) f(p^a) \quad \text{if} \quad b \geq a \geq 1. \]

The problem is thus reduced to that of determining the form of \( F \) on the powers of each prime \( p \).

**Theorem 2.** Let \( F \) be a solution of (2). If \( F(p^a) \neq 0 \) for some
$m \geq 1$ then $F(p^{kn+n}) = F(p^{kn})f(p^n)^{k-1} (k = 2, 3, \ldots), (n = 0, 1, 2, \ldots)$.

The proof follows easily from (3) by induction on $k$.

**Corollary 2.1.** Let $F$ be a solution of (2). If $F(p^n) \neq 0$ and $f(p^n) = 0$ for some $m \geq 1$, then $F(p^n) = 0$ for $n \geq 2m$.

**Corollary 2.2.** Let $F$ be a solution of (2). If $F(p^n) \neq 0$ and $f(p^n) \neq 0$ for some $m \geq 1$, then $F(p^{kn}) \neq 0 (k = 2, 3, \ldots)$.

**Theorem 3.** Let $F$ be a solution of (2). If $F(p^n) \neq 0$ and $f(p^n) \neq 0$ for some $m \geq 1$, then $f(p^n) = 0$ whenever $F(p^n) \neq 0$. If, for some $m \geq 1$, $F(p^n) = 0$ and $f(p^n) = 0$ then $f(p^n) = 0$ whenever $F(p^n) \neq 0$.

**Proof.** To prove the first proposition we observe that if $F(p^n) \neq 0$ and $f(p^n) = 0$ then $F(p^n) = 0$ for $n \geq 2m$ which contradicts Corollary 2.2. The proof of the second proposition is similar.

We are now in a position to determine the form of the solution $F$ on the powers of $p$ if $F(p^n) \neq 0$ and $f(p^n) = 0$.

**Theorem 4.** Let $m$ be a positive integer such that $f(p^n) = 0$. Let $m_1 = m, m_2, \ldots, m_s$, be the integers on the interval $[m, 2m)$ for which $f(p^t) = 0$. The function $F$ is a solution of (3) with $m$ the smallest positive integer such that $F(p^n) \neq 0$ if, and only if, $F(p^n) \neq 0$ and $F(p^n) = 0$ whenever $n \neq m_i$.

**Proof.** If $F$ is a solution of (3) and $m$ is the smallest positive integer such that $F(p^n) \neq 0$ then by Corollary 2.1 we see that $F(p^n) = 0$ for $n \geq 2m$ and by Theorem 3 we see that $F(p^n) = 0$ if $n \neq m_i (m < n < 2m)$. To prove the converse we substitute in (3).

2. The case $F(p^n) \neq 0$ and $f(p^n) \neq 0$. We will first show that in this case $f$ cannot be defined arbitrarily on the powers of $p$ if a solution of (2) is to exist.

**Theorem 5.** Let $F$ be a solution of (2). If $F(p^n) \neq 0$ and $f(p^n) \neq 0$ for some positive integer $m$, then

$$f(p^{kn}) = f(p^n)^k \quad (k = 1, 2, \ldots).$$

**Proof.** From Corollary 2.2 and Theorem 3 we see that $F(p^{kn}) \neq 0$ and that $f(p^{kn}) \neq 0 (k = 1, 2, \ldots)$. Taking $a = b = km$ in (3) and using Theorem 2 we obtain

$$F(p^{kn}) = F(p^{kn})f(p^{kn}) = F(p^n)f(p^n)^{k-1}f(p^{kn}).$$
If we now take $n = 0$ and replace $k$ by $2k$ in Theorem 2 we obtain

$$F(p^{2km}) = F(p^m)f(p^m)^{2k-1}. $$

Comparing the last two equations we see that $f(p^m)^k = f(p^{km})$.

**Theorem 6.** Let $F$ be a solution of (2). Suppose $f(p^m) \neq 0$ and $F(p^m) \neq 0$ for some $m \geq 1$, and let $d$ be the smallest positive integer such that $F(p^{m+d}) \neq 0$. Then $d \mid m$. Furthermore, if $n$ is a positive integer then $F(p^{m+n}) \neq 0$ if and only if $n \equiv 0 \pmod{d}$.

**Proof.** (A) Such an integer $d$ must exist by Corollary 2.2. Suppose $d \mid m$. Let $t$ be the smallest positive integer such that $td > m$. We can write $m < td = m + j < m + d$. From Theorem 2

$$F(p^{t(m+d)}) = F(p^{(t+1)m+j}) = F(p^{m+j})f(p^m)^t = 0,$$

since $0 < j < d$; similarly, by Theorem 2 and 3 we have

$$F(p^{t(m+d)}) = F(p^{m+d})f(p^{m+d})^{t-1} = 0,$$

which is impossible. Thus $d \mid m$.

(B) If $n \not\equiv 0 \pmod{d}$ there exist positive integers $K$ and $L$ such that $Km < Ln = Km + j < Km + d$. By considering $F(p^m)$ we obtain a contradiction similar to that in part (A) if we assume $F(p^{m+n}) \neq 0$.

(C) Suppose $n \equiv 0 \pmod{d}$, say $n = kd$. Applying Theorem 2 twice we see that

$$F(p^{km+n}) = F(p^{km+d}) = F(p^{m+d})f(p^{m+d})^{k-1} \neq 0$$

and

$$F(p^{km+n}) = F(p^{m+n})f(p^m)^{k-1}. $$

Thus

$$F(p^{m+n}) = F(p^{m+kd}) = \frac{F(p^{m+kd})f(p^{m+kd})^{k-1}}{f(p^m)^{k-1}} \neq 0. $$

We now extend the result of Theorem 5 to completely characterize the solutions of (2) with $f(p^m) \neq 0$ and $F(p^m) \neq 0$.

**Theorem 7.** Let $F$ and $f$ be multiplicative functions. Suppose $p$ is a prime, that $m$ and $d$ are the smallest positive integers such that $F(p^m) \neq 0$, $f(p^m) \neq 0$ and $F(p^{m+d}) \neq 0$. Then $F$ is a solution of (3) if, and only if,

(5) $d \mid m$,

(6) $f(p^{m+kd}) = f(p^m)^{1-kd/m}$ ($k = 1, 2, 3, \cdots$).
\begin{align*}
(7) \quad F(p^{m+kd}) &= F(p^m)f(p^{kd/m}) = \frac{F(p^m)}{f(p^m)} f(p^{m+kd}), \\
(8) \quad F(p^n) &= 0 \text{ if } 0 < n < m \text{ or } \text{if } n \equiv 0 \pmod{d}.
\end{align*}

**Proof.** (5) and (8) were established in Theorem 6. To prove (6) we let \( k_1 = 2k, k_2 = 2, m_1 = m + d, m_2 = m + kd, n_2 = 2km - 2m, \) so that \( k_1 m_1 = k_2 m_2 + n_2. \) From Theorem 2 we obtain

\begin{equation}
F(p^{k_1 m_1}) = F(p^{m+d}) f(p^{m+d})^{2k-1}.
\end{equation}

From Theorem 2 and equations (3) and (4) we obtain

\begin{align*}
F(p^{k_2 m_2 + n_2}) &= F(p^{(2k-1) m + kd}) f(p^{m+kd}) \\
&= F(p^{m+kd}) f(p^m)^{2k-2} f(p^{m+kd}) \\
&= \frac{F(p^{m+d}) f(p^{m+d})^{k-1}}{f(p^m)^{k-1}} f(p^m)^{2k-2} f(p^{m+kd}).
\end{align*}

Equating the last expression with (9) we obtain

\begin{equation}
f(p^{m+kd}) = \frac{f(p^{m+d})^k}{f(p^m)^{k-1}}.
\end{equation}

For the special case \( k = m/d \) we obtain from Theorem 5 and (10)

\begin{equation*}
f(p^{m}) = f(p^m)^{m/d} = \frac{f(p^m)^{m/d}}{f(p^m)^{m/d-1}}
\end{equation*}

so that \( f(p^{m+d}) = f(p^m)^{1+d/m}. \)

Substituting this in (10) we obtain (6).

To prove (7) we apply Theorem 2 twice obtaining

\begin{align*}
F(p^{(m+kd) m/d}) &= F(p^{m+kd}) f(p^{m+kd})^{m/d-1} \\
&= F(p^{m+kd})^{m/d-1} = F(p^m) f(p^{m})^{m/d-1}.
\end{align*}

From this relation, along with (6) used twice we find

\begin{align*}
F(p^{m+kd}) &= \frac{F(p^m) f(p^m)^{m/d-1}}{f(p^m)^{m/d-1}} = \frac{F(p^m) f(p^m)^{m+d/k-1}}{f(p^m)^{m+d/k-1}} \\
&= F(p^m)^{m/d} f(p^{m+kd}).
\end{align*}

The other part of (7) follows from (6).

To prove the converse we substitute in (3).

3. **Summary.** We have reduced the problem of solving (2) to that of finding a multiplicative function \( F \) (not identically zero) that satisfies (3) for the powers of each prime \( p. \) Such a function \( F \) will exist if, and only if, one of the following holds for \( f \) and \( F \) on the
powers of $p$:

1. There is a positive integer $m$ such that $f(p^m) = 0$ and $F(p^n) = 0$ except possibly for the integers $n$ in the interval $[m, 2m)$ for which $f(p^n) = 0$.

2. There is a positive integer $m$, a positive divisor $d$ of $m$ and a complex number $C$ such that

$$f(p^{m+kd}) = f(p^m)^{1+kd/m} 
eq 0 \quad (k = 0, 1, 2, \ldots)$$

and $F$ has the defining properties

$$F(p^{m+kd}) = Cf(p^{m+kd}) \quad (k = 0, 1, 2, \ldots)$$

$$F(p^n) = 0 \text{ if } n \neq m + kd \text{ for some nonnegative integer } k.$$

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