ON THE FUNCTIONAL EQUATION

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Let \( f \) be a multiplicative arithmetic function, \( f(1) = 1. \) Necessary and sufficient conditions on \( f \) will be found so that the functional equation
\[ F(mn)F((m, n)) = F(m)F(n)f((m, n)) \]
will have a solution \( F \) with \( F(1) \neq 0 \) and all solution \( F \) will be determined. It will be shown that two different types of solutions may exist and that one of these requires that \( f \) have a property similar to complete multiplicativity.

The special case of the equation
\[ F(mn)F((m, n)) = F(m)F(n)f((m, n)) \]
with \( f \) completely multiplicative and \( F(1) \neq 0 \) was solved completely by Apostol and Zuckerman [1]. Specialized results were also given for the case \( F(1) = 0 \), but this case was not solved in general.

We note that if \( f(1) = 0 \) then \( f \) is identically zero and is thus completely multiplicative. This case was solved completely in [1] and will not be considered here. Thus in the sequel we will assume that \( f \) is multiplicative and is not identically zero (which implies that \( f(1) = 1 \)).

1. If \( F \) is a solution of (1) with \( F(1) \neq 0 \) then any constant multiple of \( F \) is also a solution. Thus we may reduce the problem of solving (1) with \( F(1) \neq 0 \) to that of solving
\[ F(mn)F((m, n)) = F(m)F(n)f((m, n)) \]

The proof of Theorem 1 is essentially the same as that of Theorem 2 of [1] and will be omitted.

THEOREM 1. \( F \) is a solution of (2) if, and only if, \( F \) is a non-zero multiplicative function and for each prime \( p \)
\[ F(p^{a+b})F(p^{a}) = F(p^{a})F(p^{a})f(p^{a}) \text{ if } b \geq a \geq 1. \]

The problem is thus reduced to that of determining the form of \( F \) on the powers of each prime \( p \).

THEOREM 2. Let \( F \) be a solution of (2). If \( F(p^{n}) \neq 0 \) for some
\[ m \geq 1 \text{ then } F(p^{km+n}) = F(p^{m+n})f(p^m)^{k-1} (k = 2, 3, \ldots), (n = 0, 1, 2, \ldots). \]

The proof follows easily from (3) by induction on \( k \).

**COROLLARY 2.1.** Let \( F \) be a solution of (2). If \( F(p^m) \neq 0 \) and \( f(p^m) = 0 \) for some \( m \geq 1 \), then \( F(p^n) = 0 \) for \( n \geq 2m \).

**COROLLARY 2.2.** Let \( F \) be a solution of (2). If \( F(p^m) \neq 0 \) and \( f(p^m) = 0 \) for some \( m \geq 1 \), then \( F(p^{km}) \neq 0 \) for \( k = 2, 3, \ldots \).

**THEOREM 3.** Let \( F \) be a solution of (2). If \( F(p^m) \neq 0 \) and \( f(p^m) \neq 0 \) for some \( m \geq 1 \), then \( f(p^n) = 0 \) whenever \( F(p^n) \neq 0 \). If, for some \( m \geq 1 \), \( F(p^m) \neq 0 \) and \( f(p^m) = 0 \) then \( f(p^n) = 0 \) whenever \( F(p^n) \neq 0 \).

**Proof.** To prove the first proposition we observe that if \( F(p^n) \neq 0 \) and \( f(p^n) = 0 \) then \( F(p^t) = 0 \) for \( t \geq 2n \) which contradicts Corollary 2.2. The proof of the second proposition is similar.

We are now in a position to determine the form of the solution \( F \) on the powers of \( p \) if \( F(p^m) \neq 0 \) and \( f(p^m) = 0 \).

**THEOREM 4.** Let \( m \) be a positive integer such that \( f(p^m) = 0 \). Let \( m_1 = m, m_2, \ldots, m_k \), be the integers \( t \) on the interval \([m, 2m)\) for which \( f(p^t) = 0 \). The function \( F \) is a solution of (3) with \( m \) the smallest positive integer such that \( F(p^m) \neq 0 \) if, and only if, \( F(p^m) \neq 0 \) and \( F(p^n) = 0 \) whenever \( n \neq m \).

**Proof.** If \( F \) is a solution of (3) and \( m \) is the smallest positive integer such that \( F(p^m) \neq 0 \) then by Corollary 2.1 we see that \( F(p^n) = 0 \) for \( n \geq 2m \) and by Theorem 3 we see that \( F(p^n) = 0 \) if \( n \neq m \), \( m < n < 2m \). To prove the converse we substitute in (3).

2. The case \( F(p^m) \neq 0 \) and \( f(p^m) \neq 0 \). We will first show that in this case \( f \) cannot be defined arbitrarily on the powers of \( p \) if a solution of (2) is to exist.

**THEOREM 5.** Let \( F \) be a solution of (2). If \( F(p^m) \neq 0 \) and \( f(p^m) \neq 0 \) for some positive integer \( m \), then

\[ f(p^{mk}) = f(p^m)^k \quad (k = 1, 2, \ldots). \]

**Proof.** From Corollary 2.2 and Theorem 3 we see that \( F(p^{km}) \neq 0 \) and that \( f(p^{km}) \neq 0 \) for \( k = 1, 2, \ldots \). Taking \( a = b = km \) in (3) and using Theorem 2 we obtain

\[ F(p^{km+n}) = F(p^{km})f(p^{km}) = F(p^m)f(p^m)^{k-1}f(p^{km}). \]
If we now take $n = 0$ and replace $k$ by $2k$ in Theorem 2 we obtain

$$F(p^{2km}) = F(p^m)f(p^m)^{2k - 1}.$$  

Comparing the last two equations we see that $f(p^m)^k = f(p^{km}).$

**Theorem 6.** Let $F$ be a solution of (2). Suppose $f(p^m) \neq 0$ and $F(p^m) \neq 0$ for some $m \geq 1$, and let $d$ be the smallest positive integer such that $F(p^{m+d}) \neq 0$. Then $d | m$. Furthermore, if $n$ is a positive integer then $F(p^{m+n}) \neq 0$ if and only if $n = 0 \pmod{d}$.

**Proof.** (A) Such an integer $d$ must exist by Corollary 2.2. Suppose $d \nmid m$. Let $t$ be the smallest positive integer such that $td > m$. We can write $m < td = m + j < m + d$. From Theorem 2

$$F(p^{(m+d)}) = F(p^{(m+j)m}) = F(p^m+j)f(p^m)^j = 0,$$

since $0 < j < d$; similarly, by Theorem 2 and 3 we have

$$F(p^{(m+d)}) = F(p^{m+d})f(p^{m+d})^{i-1} \neq 0,$$

which is impossible. Thus $d | m$.

(B) If $n \equiv 0 \pmod{d}$ there exist positive integers $K$ and $L$ such that $Km < Ln = Km + j < Km + d$. By considering $F(p^m)$ we obtain a contradiction similar to that in part (A) if we assume $F(p^{m+n}) \neq 0$.

(C) Suppose $n \equiv 0 \pmod{d}$, say $n = kd$. Applying Theorem 2 twice we see that

$$F(p^{km+n}) = F(p^{km+kd}) = F(p^{m+kd})f(p^{m+kd})^{k-1} \neq 0$$

and

$$F(p^{km+n}) = F(p^{m+n})f(p^m)^{k-1}.$$  

Thus

$$F(p^{m+n}) = F(p^{m+kd}) = \frac{F(p^{m+kd})f(p^{m+kd})^{k-1}}{f(p^m)^{k-1}} \neq 0.$$  

We now extend the result of Theorem 5 to completely characterize the solutions of (2) with $f(p^m) \neq 0$ and $F(p^m) \neq 0$.

**Theorem 7.** Let $F$ and $f$ be multiplicative functions. Suppose $p$ is a prime, that $m$ and $d$ are the smallest positive integers such that $F(p^m) \neq 0$, $f(p^m) \neq 0$ and $F(p^{m+d}) \neq 0$. Then $F$ is a solution of (3) if, and only if,

$$d | m,$$

$$f(p^{m+kd}) = f(p^m)^{k+kd/m} \quad (k = 1, 2, 3, \ldots).$$
(7) \[ F(p^{m+kd}) = F(p^m)f(p^m)^{kd/m} = \frac{F(p^m)}{f(p^m)} f(p^{m+kd}) , \]

(8) \[ F(p^n) = 0 \text{ if } 0 < n < m \text{ or } n \equiv 0 \pmod{d} . \]

**Proof.** (5) and (8) were established in Theorem 6. To prove (6) we let \( k_1 = 2k, k_2 = 2, m_1 = m + d, m_2 = m + kd, n_2 = 2km - 2m, \) so that \( k_1m_1 = k_2m_2 + n_2. \) From Theorem 2 we obtain

(9) \[ F(p^{k_1m_1}) = F(p^{m+d})f(p^{m+d})^{2k-1} . \]

From Theorem 2 and equations (3) and (4) we obtain

\[
F(p^{k_2m_2+n_2}) = F(p^{(2k-1)m+kd})f(p^{m+kd}) \\
= F(p^{m+kd})f(p^m)^{2k-2}f(p^{m+kd}) \\
= \frac{F(p^{m+kd})f(p^m)^{2k-2}}{f(p^m)^{k-1}} f(p^m)^{2k-2}f(p^{m+kd}) .
\]

Equating the last expression with (9) we obtain

(10) \[ f(p^{m+kd}) = \frac{f(p^{m+kd})^k}{f(p^m)^{k-1}} . \]

For the special case \( k = m/d \) we obtain from Theorem 5 and (10)

\[ f(p^{zm}) = f(p^m)^z = \frac{f(p^{m+d})^{m/d}}{f(p^m)^{m/d-1}} , \]

so that \( f(p^{m+d}) = f(p^m)^{1+d/m} . \)

Substituting this in (10) we obtain (6).

To prove (7) we apply Theorem 2 twice obtaining

\[
F(p^{(m+kd)m/d}) = F(p^{m+kd})f(p^{m+kd})^{m/d-1} \\
= F(p^{m/d+k}) = F(p^m)f(p^m)^{m/d-1} .
\]

From this relation, along with (6) used twice we find

\[
F(p^{m+kd}) = \frac{F(p^m)f(p^m)^{m/d+k-1}}{f(p^{m+kd})^{m/d-1}} = \frac{F(p^m)f(p^m)^{m/d+k-1}}{f(p^m)^{m/d+k-1-kd/m}} \\
= F(p^m)f(p^m)^{kd/m} .
\]

The other part of (7) follows from (6).

To prove the converse we substitute in (3).

3. Summary. We have reduced the problem of solving (2) to that of finding a multiplicative function \( F' \) (not identically zero) that satisfies (3) for the powers of each prime \( p. \) Such a function \( F' \) will exist if, and only if, one of the following holds for \( f \) and \( F \) on the
powers of $p$:

1. There is a positive integer $m$ such that $f(p^m) = 0$ and $F(p^n) = 0$ except possibly for the integers $n$ in the interval $[m, 2m)$ for which $f(p^n) = 0$.

2. There is a positive integer $m$, a positive divisor $d$ of $m$ and a complex number $C$ such that

$$f(p^{m+kd}) = f(p^{m+kd/m}) 
eq 0 \quad (k = 0, 1, 2, \cdots)$$

and $F$ has the defining properties

$$F(p^{m+kd}) = Cf(p^{m+kd}) \quad (k = 0, 1, 2, \cdots)$$

$$F(p^n) = 0 \text{ if } n \neq m + kd \text{ for some nonnegative integer } k.$$

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Received October 23, 1964, and in revised form February 24, 1965.

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