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ON THE FUNCTIONAL EQUATION
 $F(mn)F((m, n)) = F(m)F(n)f((m, n))$

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Let f be a multiplicative arithmetic function, $f(1) = 1$. Necessary and sufficient conditions on f will be found so that the functional equation

$$F(mn)F((m, n)) = F(m)F(n)f((m, n))$$

will have a solution F with $F(1) \neq 0$ and all solution F will be determined. It will be shown that two different types of solutions may exist and that one of these requires that f have a property similar to complete multiplicativity.

The special case of the equation

$$(1) \quad F(mn)F((m, n)) = F(m)F(n)f((m, n))$$

with f completely multiplicative and $F(1) \neq 0$ was solved completely by Apostol and Zuckerman [1]. Specialized results were also given for the case $F(1) = 0$, but this case was not solved in general.

We note that if $f(1) = 0$ then f is identically zero and is thus completely multiplicative. This case was solved completely in [1] and will not be considered here. Thus in the sequel we will assume that f is multiplicative and is not identically zero (which implies that $f(1) = 1$).

1. If F is a solution of (1) with $F(1) \neq 0$ then any constant multiple of F is also a solution. Thus we may reduce the problem of solving (1) with $F(1) \neq 0$ to that of solving

$$(2) \quad F(mn)F((m, n)) = F(m)F(n)f((m, n)), \quad F(1) = 1.$$

The proof of Theorem 1 is essentially the same as that of Theorem 2 of [1] and will be omitted.

THEOREM 1. F is a solution of (2) if, and only if, F is a non-zero multiplicative function and for each prime p

$$(3) \quad F(p^{a+b})F(p^a) = F(p^b)F(p^a)f(p^a) \quad \text{if } b \geq a \geq 1.$$

The problem is thus reduced to that of determining the form of F on the powers of each prime p .

THEOREM 2. Let F be a solution of (2). If $F(p^m) \neq 0$ for some

$m \geq 1$ then $F(p^{k^{m+n}}) = F(p^{m+n})f(p^m)^{k-1}$ ($k = 2, 3, \dots$), ($n = 0, 1, 2, \dots$).

The proof follows easily from (3) by induction on k .

COROLLARY 2.1. *Let F be a solution of (2). If $F(p^m) \neq 0$ and $f(p^m) = 0$ for some $m \geq 1$, then $F(p^n) = 0$ for $n \geq 2m$.*

COROLLARY 2.2. *Let F be a solution of (2). If $F(p^m) \neq 0$ and $f(p^m) \neq 0$ for some $m \geq 1$, then $F(p^{km}) \neq 0$ ($k = 2, 3, \dots$).*

THEOREM 3. *Let F be a solution of (2). If $F(p^m) \neq 0$ and $f(p^m) \neq 0$ for some $m \geq 1$, then $f(p^n) \neq 0$ whenever $F(p^n) \neq 0$. If, for some $m \geq 1$, $F(p^m) \neq 0$ and $f(p^m) = 0$ then $f(p^n) = 0$ whenever $F(p^n) \neq 0$.*

Proof. To prove the first proposition we observe that if $F(p^n) \neq 0$ and $f(p^n) = 0$ then $F(p^t) = 0$ for $t \geq 2n$ which contradicts Corollary 2.2. The proof of the second proposition is similar.

We are now in a position to determine the form of the solution F on the powers of p if $F(p^m) \neq 0$ and $f(p^m) = 0$.

THEOREM 4. *Let m be a positive integer such that $f(p^m) = 0$. Let $m_1 = m, m_2, \dots, m_k$, be the integers t on the interval $[m, 2m)$ for which $f(p^t) = 0$. The function F is a solution of (3) with m the smallest positive integer such that $F(p^m) \neq 0$ if, and only if, $F(p^m) \neq 0$ and $F(p^n) = 0$ whenever $n \neq m_i$.*

Proof. If F is a solution of (3) and m is the smallest positive integer such that $F(p^m) \neq 0$ then by Corollary 2.1 we see that $F(p^n) = 0$ for $n \geq 2m$ and by Theorem 3 we see that $F(p^n) = 0$ if $n \neq m_i$ ($m < n < 2m$). To prove the converse we substitute in (3).

2. The case $F(p^m) \neq 0$ and $f(p^m) \neq 0$. We will first show that in this case f cannot be defined arbitrarily on the powers of p if a solution of (2) is to exist.

THEOREM 5. *Let F be a solution of (2). If $F(p^m) \neq 0$ and $f(p^m) \neq 0$ for some positive integer m , then*

$$f(p^{m^k}) = f(p^m)^k \quad (k = 1, 2, \dots).$$

Proof. From Corollary 2.2 and Theorem 3 we see that $F(p^{km}) \neq 0$ and that $f(p^{km}) \neq 0$ ($k = 1, 2, \dots$). Taking $a = b = km$ in (3) and using Theorem 2 we obtain

$$F(p^{2km}) = F(p^{km})f(p^{km}) = F(p^m)f(p^m)^{k-1}f(p^{km}).$$

If we now take $n = 0$ and replace k by $2k$ in Theorem 2 we obtain

$$F(p^{2km}) = F(p^m)f(p^m)^{2k-1} .$$

Comparing the last two equations we see that $f(p^m)^k = f(p^{km})$.

THEOREM 6. *Let F be a solution of (2). Suppose $f(p^m) \neq 0$ and $F(p^m) \neq 0$ for some $m \geq 1$, and let d be the smallest positive integer such that $F(p^{m+d}) \neq 0$. Then $d \mid m$. Furthermore, if n is a positive integer then $F(p^{m+n}) \neq 0$ if and only if $n \equiv 0 \pmod{d}$.*

Proof. (A) Such an integer d must exist by Corollary 2.2. Suppose $d \nmid m$. Let t be the smallest positive integer such that $td > m$. We can write $m < td = m + j < m + d$. From Theorem 2

$$F(p^{t(m+d)}) = F(p^{(t+1)m+j}) = F(p^{m+j})f(p^m)^t = 0 .$$

since $0 < j < d$; similarly, by Theorem 2 and 3 we have

$$F(p^{t(m+d)}) = F(p^{m+d})f(p^{m+d})^{t-1} \neq 0$$

which is impossible. Thus $d \mid m$.

(B) If $n \not\equiv 0 \pmod{d}$ there exist positive integers K and L such that $Km < Ln = Km + j < Km + d$. By considering $F(p^{Ln})$ we obtain a contradiction similar to that in part (A) if we assume $F(p^{m+n}) \neq 0$.

(C) Suppose $n \equiv 0 \pmod{d}$, say $n = kd$. Applying Theorem 2 twice we see that

$$F(p^{km+n}) = F(p^{k(m+d)}) = F(p^{m+d})f(p^{m+d})^{k-1} \neq 0$$

and

$$F(p^{km+n}) = F(p^{m+n})f(p^m)^{k-1} .$$

Thus

$$(4) \quad F(p^{m+n}) = F(p^{m+kd}) = \frac{F(p^{m+d})f(p^{m+d})^{k-1}}{f(p^m)^{k-1}} \neq 0 .$$

We now extend the result of Theorem 5 to completely characterize the solutions of (2) with $f(p^m) \neq 0$ and $F(p^m) \neq 0$.

THEOREM 7. *Let F and f be multiplicative functions. Suppose p is a prime, that m and d are the smallest positive integers such that $F(p^m) \neq 0$, $f(p^m) \neq 0$ and $F(p^{m+d}) \neq 0$. Then F is a solution of (3) if, and only if,*

$$(5) \quad d \mid m ,$$

$$(6) \quad f(p^{m+kd}) = f(p^m)^{1+kd/m} \quad (k = 1, 2, 3, \dots) ,$$

$$(7) \quad F(p^{m+kd}) = F(p^m)f(p^m)^{kd/m} = \frac{F(p^m)}{f(p^m)} f(p^{m+kd}),$$

$$(8) \quad F(p^n) = 0 \text{ if } 0 < n < m \text{ or if } n \not\equiv 0 \pmod{d}.$$

Proof. (5) and (8) were established in Theorem 6. To prove (6) we let $k_1 = 2k$, $k_2 = 2$, $m_1 = m + d$, $m_2 = m + kd$, $n_2 = 2km - 2m$, so that $k_1m_1 = k_2m_2 + n_2$. From Theorem 2 we obtain

$$(9) \quad F(p^{k_1m_1}) = F(p^{m+d})f(p^{m+d})^{2k-1}.$$

From Theorem 2 and equations (3) and (4) we obtain

$$\begin{aligned} F(p^{k_2m_2+n_2}) &= F(p^{(2k-1)m+kd})f(p^{m+kd}) \\ &= F(p^{m+kd})f(p^m)^{2k-2}f(p^{m+kd}) \\ &= \frac{F(p^{m+d})f(p^{m+d})^{k-1}}{f(p^m)^{k-1}} f(p^m)^{2k-2}f(p^{m+kd}). \end{aligned}$$

Equating the last expression with (9) we obtain

$$(10) \quad f(p^{m+kd}) = \frac{f(p^{m+d})^k}{f(p^m)^{k-1}}.$$

For the special case $k = m/d$ we obtain from Theorem 5 and (10)

$$f(p^{2m}) = f(p^m)^2 = \frac{f(p^{m+d})^{m/d}}{f(p^m)^{m/d-1}}$$

so that $f(p^{m+d}) = f(p^m)^{1+d/m}$.

Substituting this in (10) we obtain (6).

To prove (7) we apply Theorem 2 twice obtaining

$$\begin{aligned} F(p^{(m+kd)m/d}) &= F(p^{m+kd})f(p^{m+kd})^{m/d-1} \\ &= F(p^{(m/d+k)m}) = F(p^m)f(p^m)^{m/d+k-1}. \end{aligned}$$

From this relation, along with (6) used twice we find

$$\begin{aligned} F(p^{m+kd}) &= \frac{F(p^m)f(p^m)^{m/d+k-1}}{f(p^{m+kd})^{m/d-1}} = \frac{F(p^m)f(p^m)^{m/d+k-1}}{f(p^m)^{m/d+k-1-kd/m}} \\ &= F(p^m)f(p^m)^{kd/m}. \end{aligned}$$

The other part of (7) follows from (6).

To prove the converse we substitute in (3).

3. Summary. We have reduced the problem of solving (2) to that of finding a multiplicative function F (not identically zero) that satisfies (3) for the powers of each prime p . Such a function F will exist if, and only if, one of the following holds for f and F on the

powers of p :

1. There is a positive integer m such that $f(p^m) = 0$ and $F(p^n) = 0$ except possibly for the integers n in the interval $[m, 2m)$ for which $f(p^n) = 0$.

2. There is a positive integer m , a positive divisor d of m and a complex number C such that

$$f(p^{m+kd}) = f(p^m)^{1+kd/m} \neq 0 \quad (k = 0, 1, 2, \dots)$$

and F has the defining properties

$$F(p^{m+kd}) = Cf(p^{m+kd}) \quad (k = 0, 1, 2, \dots)$$

$F(p^n) = 0$ if $n \neq m + kd$ for some nonnegative integer k .

REFERENCES

1. Tom M. Apostol and Herbert S. Zuckerman, *On the functional equation $F(mn)F((m, n)) = F(m)F(n)f((m, n))$* , Pacific J. Math. **14** (1964), 377-384.
2. P. Comment, *Sur l'equation fonctionnelle $F(mn)F((m, n)) = F(n)F(m)f((m, n))$* , Bull. Res. Council of Israel, Sect. F7F (1957/58), 14-20.

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