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**ON CLOSED MAPPINGS, BICOMPACT SPACES, AND A  
PROBLEM OF P. ALEKSANDROV**

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# ON CLOSED MAPPINGS, BICOMPACT SPACES, AND A PROBLEM OF P. ALEKSANDROV

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**The purpose of this paper is to show, under very general circumstances, that if  $f: X \rightarrow Y$  is a closed map, then  $f^{-1}y$  must be bicomact for "most"  $y \in Y$ . Two theorems of this sort are obtained, one of which is then used to answer a question of P. Alexandroff on the effect of closed maps on countable-dimensional spaces.**

If  $f: X \rightarrow Y$  is a closed map, then it is known that, under suitable assumptions,  $f^{-1}y$  has a bicomact boundary for all  $y \in Y$  (see I. Vainštein [19], A. H. Stone [18], and K. Morita and S. Hanai [12]), and  $f^{-1}y$  itself is bicomact for "most"  $y \in Y$  (see K. Morita [11] and the author [4]). In §§1 and 2 of this paper, we prove two theorems of the latter kind, whose main feature is that they require minimal restrictions on  $X$  and no restriction at all (other than being  $T_1$ ) on  $Y$ .

In §3, we give some applications of the results from §2. The most interesting among them is the following, which gives a complete answer to a question of P. Alexandroff, (Terminology is defined below).

**THEOREM (3.1).** *Let  $X$  be a countable-dimensional space with a countable net, and let  $f: X \rightarrow Y$  be a closed mapping of  $X$  onto some uncountable-dimensional space  $Y$ . Then  $Y_1 = \{y \in Y \mid \text{card}(f^{-1}y) \geq \mathfrak{c}\}$  is uncountable-dimensional.*

Observe that Theorem 3.1 is new even in case  $X$  is compact metric. In that case, E. Skljarenko [15] has shown that  $Y_1$  is not void, but his proof gives no further information about  $Y_1$ . Our proof is based on entirely different ideas.

Let me say here that I am very grateful to P. Alexandroff for valuable discussions about this question and to E. Michael for helping with the translation of this paper.

*Notation and terminology.* All spaces are completely regular (often is it sufficient to suppose  $T_1$ ); all mappings are continuous, and all coverings are open. We call a family  $\gamma = \{S\}$  of sets  $S \subseteq X$  a *net* in  $X$ , if, for every  $x \in X$  and each open  $U$  containing  $x$ , there exists an  $S \in \gamma$  with  $x \in S \subseteq U$  (see [3]). We write  $\text{card } A$  for the cardinality of  $A$ , and  $\mathfrak{c}$  for the cardinality of the continuum. If  $\gamma$  is a family of subsets of a space  $X$ , and if  $x \in X$ , then  $\gamma x$  denotes the union of all elements of  $\gamma$  containing  $x$ . As usual, we call a space *countable-dimensional* if it is a countable union of subspaces with  $\dim = 0$ ;

otherwise, we call the space *uncountable-dimensional*. We write  $\beta X$  for the Stone-Čech bicomactification of  $X$ , and we call  $X$  a  $G_\delta$ -space if it is a  $G_\delta$  in  $\beta X$ . We call  $X$  *point-paracompact* if every (open) covering of  $X$  has an (open) point-finite refinement. Finally,  $X$  is called a  $k$ -space if a subset  $U \subseteq X$  is open whenever its intersection with every compact  $K \subseteq X$  is open in  $K$ .

### 1. Closed mappings of point-paracompact $G_\delta$ -spaces.

**THEOREM (1.1).** *Let  $X$  be a point-paracompact  $G_\delta$ -space, and let  $f: X \rightarrow Y$  be a closed map. Then*

$$Y = Y_0 \cup (\bigcup_{n=1}^{\infty} Y_n),$$

where  $Y_n$  is discrete in  $Y$  for all  $n$ , and  $f^{-1}y$  is bicomact for all  $y \in Y_0$ .

For the proof of this theorem, we need

**LEMMA (1.2).** *Let  $X$  be a  $k$ -space, let  $\gamma$  be a point-finite covering of  $X$ , and let  $f: X \rightarrow Y$  be a closed mapping of  $X$  onto some  $Y$ . Then*

$$N = \{y \in Y \mid \text{no finite } \gamma' \subseteq \gamma \text{ covers } f^{-1}y\}$$

is discrete in  $Y$ .

*Proof.* Suppose that some  $y \in Y$  is an accumulation point for  $N$ . Then the set  $N_1 = N \setminus \{y\}$  is not closed. Since  $X$  is a  $k$ -space and  $f$  a closed mapping,  $Y$  is a  $k$ -space [8]. Therefore there exists a bicomact  $F \subseteq Y$  such that  $F \cap N_1$  is not closed, and hence is infinite. Let  $\{y_n\}$  be a sequence of distinct points from  $F \cap N_1$ . Since  $F$  is bicomact, there exists an accumulation point  $y'$  for this sequence, which we may suppose different from all  $y_n$ . We let  $A_n = f^{-1}y_n$  for  $n = 1, 2, \dots$ . Next we shall define a sequence  $\{x_n\}$ , with  $x_n \in A_n$ , where for  $x_1$  we take any point from  $A_1$ . Suppose the points  $x_k$  are defined for all  $k < m$ . We take for  $x_m$  any point of the set  $A_m \setminus \bigcap_{i=1}^{m-1} \gamma x_i$ ; this set is not empty by the very definition of  $N$ .

Now we prove that the sequence  $\{x_n\}$  is discrete. Consider any point  $x \in X$ . We only have to consider the case where  $\gamma x \cap \{x_n\} \neq \emptyset$ . Let  $x_m \in \gamma x$ ; then  $x \in \gamma x_m$  and  $U = \gamma x_m$  is a neighborhood of  $x$ ; by the definition of  $x_n$ , this  $U$  can contain only points  $x_n$  with  $n \leq m$ . Thus the discreteness of  $\{x_n\}$  is proved.

It follows that  $P = \bigcup_{n=1}^{\infty} x_n$  is closed, while the set  $Q = fP = \bigcup_{n=1}^{\infty} y_n$  is not (since  $y' \in \bar{Q} \setminus Q$ ). This contradiction completes the proof

of the lemma.

We now proceed to the

*Proof of (1.1).* Let  $\{G_n\}$  be a countable family of open subsets of  $\beta X$  such that  $X = \bigcap_{n=1}^{\infty} G_n$ . We write  $\gamma_n$  for some covering of  $X$  (by open sets in  $X$ ) such that the closure in  $\beta X$  of each element of  $\gamma$  is contained in  $G_n$ . We take a point-finite refinement  $\lambda_n$  of  $\gamma_n$ . For  $n = 1, 2, \dots$ , let

$$Y_n = \{y \in Y \mid \text{no finite } \lambda'_n \subseteq \lambda_n \text{ covers } f^{-1}y\}.$$

It is known [5] that every  $G_\delta$ -space is a  $k$ -space. Thus, by Lemma (1.2),  $Y_n$  is discrete in  $Y$ . We write  $Y_0 = Y \setminus \bigcup_{n=1}^{\infty} Y_n$ .

We now prove that  $f^{-1}y$  is bicomact for every  $y \in Y_0$ . For each  $n$  there exists a finite  $\lambda'_n \subseteq \lambda_n$ , say  $\lambda'_n = \{V_i^n \mid i = 1, \dots, k(n)\}$ , such that  $f^{-1}y \subseteq \bigcup_{i=1}^{k(n)} V_i^n$ . Then  $F_n$ , the closure of  $\bigcup_{i=1}^{k(n)} V_i^n$  in  $\beta X$ , is bicomact, and  $f^{-1}y \subseteq F_n \subseteq G_n$ . Therefore

$$f^{-1}y \subseteq F = \bigcap_{n=1}^{\infty} F_n \subseteq \bigcap_{n=1}^{\infty} G_n = X,$$

where  $F$  is bicomact. As  $f^{-1}y$  is closed in  $X$ , it follows that  $f^{-1}y$  is bicomact too. This completes the proof of (1.1).

REMARK. As the proof shows, a result analogous to (1.1) could be obtained for  $k$ -spaces  $X$  which, for some cardinal  $\tau$ , are  $G_{\delta(\tau)}$ -spaces (i.e. an intersection of  $\tau$  open subsets of  $\beta X$ ). In particular, taking  $\tau = 1$ , we conclude: *If  $f: X \rightarrow Y$  is a closed mapping, and if  $X$  is point-paracompact and locally bicomact, then the set of all  $y \in Y$  such that  $f^{-1}y$  is not bicomact is discrete in  $Y$ .* This is a slight generalization of a theorem of K. Morita [11], whose proof needs the assumption that  $X$  is paracompact and locally bicomact.

In case  $X$  is a Lindelöf space, the conclusion in (1.1) can be simplified.

COROLLARY (1.3). *If  $X$  is a Lindelöf  $G_\delta$ -space, and if  $f: X \rightarrow Y$  is a closed map, then  $f^{-1}y$  is bicomact for all but countably many  $y \in Y$ .*

*Proof.* Since a (regular) Lindelöf space is paracompact, and hence surely point-paracompact, Theorem (1.1) is applicable. Now  $Y$ , as the continuous image of the Lindelöf space  $X$ , is itself Lindelöf, and hence all its discrete subsets are countable. Hence the set  $\bigcup_{n=1}^{\infty} Y_n$  in (1.1) is countable, and that proves the corollary.

2. Closed mappings of spaces with countable net. The class

of spaces with countable net (see the introduction for definition of net) contains all separable metric spaces and all their continuous images. Spaces with countable net are Lindelöf, and hence paracompact.

The main result of this section (Theorem (2.1)) is similar to Corollary (1.3), but the hypotheses are different. Note that the hypotheses of (1.3) are satisfied by complete separable metric spaces, while the hypotheses of (2.1) are satisfied by *all* separable metric spaces.<sup>1)</sup>

**THEOREM (2.1).** *If  $X$  is a space with countable net, and  $f: X \rightarrow Y$  is a closed mapping, then  $f^{-1}y$  is bicomact for all but countably many  $y \in Y$ .*

The proof is based on Lemma (2.2) below, which will also be used in the proof of Theorem (2.3). We will use the following terminology: If  $X$  is a space with a net  $\gamma$ , and if  $x \in X$ , then an  $x$ -sequence is a sequence  $\{S_n(x)\}$ , with  $x \in S_n(x) \in \gamma$  for all  $n$ , such that any sequence  $\{x_n\}$  with  $x_n \in S_n(x)$  for all  $n$  has an accumulation point in  $X$ .

**LEMMA (2.2).** *Let  $f: X \rightarrow Y$  be a closed mapping of a normal space  $X$  with net  $\gamma$ , such that for each  $y \in Y$  there exists an  $x \in f^{-1}y$  possessing an  $x$ -sequence  $\{S_n(x)\}$ . Let  $Y_1$  be the set of all  $y \in Y$  such that  $f^{-1}y$  is not countably compact. Then  $\text{card } Y_1 \leq \text{card } \gamma$ .*

*Proof.* Without loss of generality, we may suppose that  $S_1 \cap S_2 \in \gamma$  for every pair  $S_1, S_2 \in \gamma$ . We define  $Y_0$  as the set of all  $y \in Y$  such that  $y = \bigcap \{f(S) \mid S \in \gamma'\}$  for some finite subcollection  $\gamma' \subseteq \gamma$ . Clearly  $\text{card } Y_0 \leq \text{card } \gamma$  (if  $\gamma$  is infinite). Consider any point  $y \in Y \setminus Y_0$ . Our purpose is to show that  $f^{-1}y$  is countably compact. Then the conclusion of the theorem will follow.

Let  $x \in f^{-1}y$  be a point with an  $x$ -sequence  $\{S_n(x)\}$ . Suppose that  $F = f^{-1}y$  is not countably compact, and pick  $x_1, x_2, \dots$  in  $F$  such that  $x_i$  is discrete. By an obvious induction (using the normality of  $X$ ), we can construct a discrete sequence  $\{U_i\}$  of open sets such that  $x_i \in U_i$ . Now, again by induction, we define three sequences  $\{x_n\}$ ,  $\{x'_n\}$ , and  $\{0_n x\}$ , where  $x_n, x'_n \in X$  and  $0_n x$  is a neighborhood of  $x$ , such that

- (a) for  $n > k$ ,  $0_k x$  contains  $x_n$ ;
- (b) for  $n \leq k$ ,  $X \setminus (0_k x)^- \supseteq f^{-1}f x_n$ ;
- (c) for all  $n$ ,  $x_n \in S_n(x)$ ;
- (d) for all  $n > 1$ ,  $x'_n \in U_n$ ;
- (e) for all  $n$ ,  $f x_n = f x'_n$ .

We take  $x_1$  to be any point of  $X \setminus F$ , and for  $0_1 x$  any neighborhood of

<sup>1</sup> For separable metric  $X$ , (2.1) also follows from a recent result of N. Lashnev.

$x$  which satisfies (b) (with  $n = k = 1$ ). Suppose that  $x_n$ ,  $x'_n$ , and  $0_n x$  are already defined for  $n \leq k$  so as to satisfy (a)-(e). Since  $\gamma$  is a net, there exist  $S, S' \in \gamma$  such that  $x_{k+1} \in S \subseteq 0_k x$  and  $x_{k+1} \in S' \subseteq U_{k+1}$ . Let  $S^* = S \cap S_{k+1}(x)$ . Then  $S^* \in \gamma$  and  $(fS^* \cap fS') \setminus y \neq \emptyset$ .

Let  $x_{k+1} \in S^*$ ,  $x'_{k+1} \in S'$  be points such that  $fx_{k+1} = fx'_{k+1} \neq y$ . Finally, we take for  $0_{k+1}x$  some neighborhood of  $x$  such that

$$x \in 0_{k+1}x \subseteq (0_{k+1}x)^- \subseteq X \setminus \bigcup_{i=1}^{k+1} f^{-1}fx_i.$$

Clearly conditions (a)-(e) are satisfied, and we can go further in our induction.

Since  $\{S_n(x)\}$  is an  $x$ -sequence, the set  $P = \{x_n\}$  has in  $X$  an accumulation point, say  $x^*$ . The conditions imply that  $x^* \notin f^{-1}fP$ . Then  $fx^* \in (fP)^- \setminus fP$ , and hence  $fP$  is not closed. On the other hand,  $fP = fQ$ , where  $Q = \{x'_n\}$ . Since  $x'_n \in U_n$ , and the family  $U_n$  is discrete,  $Q$  is closed in  $X$ . Thus  $fQ$  is closed in  $Y$ , and we have a contradiction which completes the proof of (2.2).

*Proof of (2.1).* In a space  $X$  with a countable net  $\gamma$ , every  $x \in X$  has an  $x$ -sequence, namely all elements of  $\gamma$  containing  $x$ . Also, as observed earlier, such an  $X$  is paracompact, and hence it is normal and all closed, countably compact subsets are bicomact. We therefore see that (2.1) follows from (2.2).

In the following theorem, a space  $X$  is of *point-countable type* (see [5]) if it is the union of bicomact subsets  $K$  having a countable base of neighborhoods  $\{U_n\}$  (i.e., if  $V \supseteq K$  is open, then  $K \subseteq U_n \subseteq V$  for some  $n$ ).<sup>2</sup> All first-countable spaces, and all  $p$ -spaces (in the sense of [5]) are spaces of point-countable type.

**THEOREM (2.3).** *Let  $X$  be a normal point-paracompact space of point-countable type, with a net  $\gamma$  of cardinality  $\leq \tau$ . If  $f: X \rightarrow Y$  is a closed mapping, and*

$$Y_1 = \{y \in Y \mid f^{-1}y \text{ is not bicomact}\},$$

*then  $\text{card } Y_1 \leq \tau$ .*

*Proof.* Let us show that every  $x \in X$  has a  $\gamma$ -sequence  $\{S_n(x)\}$ , so that (2.2) applies. Pick a compact  $K \subseteq X$  such that  $x \in K$  and  $K$  has a countable base of neighborhoods  $\{U_n\}$ . Clearly, if we pick  $S_n(x)$  so that  $x \in S_n(x) \subseteq U_n$ , then  $\{S_n(x)\}$  is an  $x$ -sequence. Applying (2.2), we have the conclusion of the theorem since, in a point-paracompact space, closed countably compact subsets are bicomact.

<sup>2</sup> For point-paracompact spaces, the spaces of point-countable type are the same as the  $q$ -spaces in the sense of E. Michael [10].

### 3. An application.

**THEOREM (3.1).** *If  $X$  is a countable-dimensional space with countable net  $\gamma$ , and  $f: X \rightarrow Y$  is a closed mapping onto an uncountable-dimensional space  $Y$ , then*

$$Y_1 = \{y \in Y \mid \text{card } (f^{-1}y) \geq c\}$$

*is uncountable dimensional.*

The proof is based on two lemmas.

**LEMMA (3.2).** *Let  $f: X \rightarrow Y$  be a closed mapping of a space  $X$  with a countable net  $\gamma = \{S_i\}$  such that, for each  $y \in Y$ , the set  $f^{-1}y$  contains a point which is isolated in  $f^{-1}y$ . Then  $Y$  is a countable sum of subspaces, each of which is homeomorphic to a subspace of  $X$ .*

*Proof.* Without loss of generality, we may suppose that all  $S_i \in \gamma$  are closed in  $X$ . For each  $i$ , let  $f_i^* = f|_{S_i}$ , and let

$$X_i = \{x \in S_i \mid f_i^{*-1}f_i^*x = x\}.$$

Let  $Y_i = fX_i$ , and let us show that  $X_i$  is homeomorphic to  $Y_i$  and  $Y = \bigcup_{i=1}^{\infty} Y_i$ .

Since  $f$  is closed, so is its restriction  $f_i^*$  to the closed set  $S_i$ . Now  $X_i = f_i^{*-1}Y_i$ , so if  $f_i: X_i \rightarrow Y_i$  is defined by  $f_i = f_i^*|_{X_i}$ , then  $f_i$  is also closed. Since  $f_i$  is clearly continuous and one-to-one, it is a homeomorphism.

It remains to show that  $Y = \bigcup_{i=1}^{\infty} Y_i$ . Let  $y \in Y$ , and let  $x$  be an isolated point of  $f^{-1}y$ . Then  $0x = X \setminus (f^{-1}y \setminus x)$  is a neighborhood of  $x$ . Since  $\gamma$  is a net in  $X$ , there exists an  $S_k \in \gamma$  such that  $x \in S_k \subseteq 0x$ . Then  $x = S_k \cap f^{-1}y$ , and thus  $x \in X_k$ . It follows that

$$y = fx \in f_k X_k = Y_k \subseteq \bigcup_{i=1}^{\infty} Y_i.$$

This completes the proof of (3.2).

**LEMMA (3.3).** *If  $f: X \rightarrow Y$  is a closed mapping of a space  $X$  with a countable net  $\gamma$ , such that  $\text{card } (f^{-1}y) < c$  for all  $y \in Y$ , then  $Y$  is a countable sum of subspaces each of which is homeomorphic to a subspace of  $X$ .*

*Proof.* By (2.1),  $Y = Y_0 \cup Y_1$ , where  $Y_0$  is countable and  $f^{-1}y$  is bicomact for each  $y \in Y_1$ . Consider  $f_1 = f|_{X_1}$ , where  $X_1 = f^{-1}Y_1$ . If  $y \in Y_1$ , then  $\text{card } f_1^{-1}y < c$ , so  $f_1^{-1}y$  has an isolated point. Thus  $X_1$ ,

$f_1$  satisfy the condition of (3.2), and hence  $Y_1$  is a countable sum of subspaces which are homeomorphic to subspaces of  $X_1 \subseteq X$ . But  $Y_0$  is a countable sum of points, and hence the conclusion follows.

*Proof of (3.1).* Let  $Y_0 = \{y \in Y \mid \text{card } f^{-1}y < \aleph_1\}$ . Consider  $f_0 = f|X_0$ , where  $X_0 = f^{-1}Y_0$ . The map  $f_0: X_0 \rightarrow Y_0$  satisfies all the conditions of (3.3). Since  $X$  is a space with a countable net,  $X$  is hereditarily Lindelöf. Hence every subspace of  $X$  is countable-dimensional [17]. By (3.3),  $Y_0$  is thus also countable-dimensional. Therefore  $Y \setminus Y_0$  is uncountable-dimensional, and the proof of (3.1) is complete.

Theorem (3.1) is new even for compact metric  $X$ , where it can be rephrased as follows:

**COROLLARY (3.4).** *If  $f: X \rightarrow Y$  is a mapping of a countable-dimensional compact metric space  $X$  onto an uncountable-dimensional space  $Y$ , then  $\{y \in Y \mid \text{card } f^{-1}y = \aleph_1\}$  is uncountable-dimensional.*

**REMARK 1.** Another application of (2.2): A space is called *dyadic* [14] if there exists a dyadic bicomact extension of this space. We call a dyadic space *nowhere countable* if there are no nonempty countable open sets in it.

**THEOREM (3.5).** *If a nowhere countable dyadic space  $Y$  is a closed image of a separable metric space, then there exists a countable base in  $Y$ .*

*Proof.* By (3.1), there exists a countable set  $Y_0 \subseteq Y$  such that, if  $Y_1 = Y \setminus Y_0$ ,  $X_1 = f^{-1}Y_1$ , and  $f_1: X_1 \rightarrow Y_1$  is defined by  $f_1 = f|X_1$ , then  $f_1$  is a perfect map. Hence  $Y_1$  is a space with countable base. Since  $Y$  is nowhere countable,  $Y_1$  is dense in  $Y$ . Hence  $Y$  is metrizable, by a theorem of B. Efimov [7]. This completes the proof.<sup>3)</sup>

**REMARK 2.** The following generalization of a theorem of J. Nagata [13] could also be proved.

**THEOREM (3.6).** *Let  $X$  be a compact metric space, and  $f: X \rightarrow Y$  a map such that  $f^{-1}y$  is finite for all  $y \in Y$ . Then, for each countable-dimensional  $X' \subseteq X$ , the space  $Y' = fX'$  is countable-dimensional.*

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<sup>3</sup> This theorem shows a simple way for constructing nondyadic spaces.



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