

# Pacific Journal of Mathematics

**A NOTE ON THE CLASS GROUP**

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The main result yields some information on the class group of a domain  $R$  in terms of the class group of  $R/xR$ . With slightly stronger hypotheses than are strictly necessary, we state the main result: Let  $R$  be a regular domain,  $x$  a prime element contained in the radical of  $R$ , and suppose that  $R/xR$  is locally a unique factorization domain. Let  $\{I_\alpha\}$  be a set of unmixed height 1 ideals of  $R$  such that the classes of  $\{I_\alpha + xR/xR\}$  generate the class group of  $R/xR$ ; then the classes of  $\{I_\alpha\}$  generate the class group of  $R$ .

The result of Samuel's and Buchsbaum's stating that if  $R$  is a regular U.F.D., then  $R[[X]]$  is a regular U.F.D. [4] has been generalized by P. Salmon and the present author in two different directions. Salmon [2, Prop. 3] showed that if  $R$  is a regular domain,  $x$  is a prime element of  $R$  which is contained in the radical of  $R$ , and  $R/xR$  is a U.F.D., then  $R$  is a U.F.D. It was shown [1, Cor. 4] that the map of the class group of  $R$  into the class group of  $R[[X]]$  is onto if  $R$  is a regular noetherian domain. We have found a theorem which simultaneously generalizes the last two results, and even allows a little weakening of the hypotheses.

To set the notation and terminology, we will say that a domain  $R$  is locally U.F.D. if the quotient ring  $R_M$  is a U.F.D. for all maximal ideals  $M$  of  $R$ . For any Krull domain  $R$ , we will denote the class group (see [3]) of  $R$  by  $C(R)$ . If  $I$  is an unmixed height 1 ideal of a Krull domain  $R$ , we will denote the class of the class group determined by  $I$  by  $\text{cl}(I)$ . Finally, all rings considered will be commutative noetherian domains with identity.

We wish to capitalize on a simple description of the class group valid for domains which are locally U.F.D. We do so and prepare for the main theorem by a sequence of (probably all known) lemmas.

LEMMA 1. *If  $R$  is locally U.F.D., then  $R$  is a Krull domain.*

*Proof.* Since  $R$  is noetherian, it is sufficient to show that  $R$  is integrally closed. Since  $R = \bigcap R_M$  as  $M$  runs over all maximal ideals of  $R$ , it will be enough to see that each  $R_M$  is integrally closed. But each  $R_M$  is a U.F.D., hence integrally closed.

LEMMA 2. *If  $R$  is locally U.F.D. and  $P$  is a height 1 prime of  $R$ , then  $P$  is invertible.*

*Proof.*  $P$  is locally principal, hence locally free (as a module), hence projective, hence invertible.

PROPOSITION 3. If  $R$  is locally a U.F.D., then the unmixed height 1 ideals of  $R$  are precisely all finite products of minimal prime ideals of  $R$ .

Let  $I_1$  and  $I_2$  be two unmixed height 1 ideals of  $R$ , then  $\text{cl}(I_1) = \text{cl}(I_2)$  if and only if there are elements  $a$  and  $b$  in  $R$  such that  $aI_1 = bI_2$ .

*Proof.* From Lemma 2 we know that any product of height 1 prime ideals of  $R$  is invertible. Given an unmixed height 1 ideal  $I$  determined by the valuation data  $I = \{x \mid v_{P_i}(x) \geq n_i\}$  (almost all  $n_i = 0$ ), we form  $J = \prod_{n_i \neq 0} P_i^{n_i}$ . Since  $J$  is invertible, we have  $J = R: (R: J)$ , so  $J$  is also unmixed of height 1. Since  $I$  and  $J$  are determined by same valuation data, this entails  $I = J$ . If now  $I_1$  and  $I_2$  are unmixed height 1 ideals such that  $\text{cl}(I_1) = \text{cl}(I_2)$ , the  $I_1 I_2^{-1}$  is invertible and is determined by the same data as some  $f \cdot R$ , where  $f$  is in the quotient field of  $R$ . We have therefore  $I_1 I_2^{-1} = fR$ , or  $I_1 = fI_2$ , which is equivalent to the final assertion.

LEMMA 4. Let  $R$  be locally U.F.D., and suppose that  $R$  is a Macaulay ring. Let  $I$  be an unmixed height 1 ideal of  $R$  and  $x$  an element of the radical of  $R$  such that  $I: xR = I$ . Then  $I + xR$  is unmixed of height 2.

*Proof.* Word for word the proof of Lemma 2 of [1].

LEMMA 5. Let the hypotheses be as in Lemma 4 and suppose that  $x$  is prime and  $R/xR$  is a Krull domain. Let  $h$  denote the homomorphism of  $R$  onto  $R/xR$ . If  $d$  is an element of  $R$  such that  $dI^{-1} \subseteq R$  (for  $I$  an unmixed height 1 ideal of  $R$ ), then  $\text{cl}(h(dI^{-1})) = \text{cl}(h(I))^{-1}$ .

*Proof.* From  $II^{-1} = R$ , we get  $I(dI^{-1}) = dR$ . Applying  $h$  to both sides of the last equation, we obtain  $h(I) \cdot h(dI^{-1}) = h(d) \cdot R/xR$ , which yields the result.

THEOREM 6. Let  $R$  be a Macaulay ring which is locally U.F.D. Let  $x$  be a prime element of the radical of  $R$  such that  $R/xR$  is locally U.F.D. Let  $h$  denote the natural homomorphism of  $R$  onto  $R/xR$ . If  $\{I_\alpha\}$  is a set unmixed height 1 ideals of  $R$  such that  $I_\alpha: xR = I_\alpha$  and  $\{\text{cl } h(I_\alpha)\}$  generates  $C(R/xR)$ , then  $\{\text{cl } (I_\alpha)\}$  generates  $C(R)$ .

*Proof.* Let  $P$  be a height 1 prime ideal of  $R$ . If  $x \in P$ , then

$P = xR$ , and  $\text{cl}(P)$  is the identity element of  $C(R)$ . If  $x \notin P$ , we must have  $P: xR = P$  and Lemma 4 shows that  $P + xR$  is unmixed of height 2. Thus  $h(P)$  is unmixed of height 1 in  $R/xR$ , so the hypotheses yield that  $h(P) = fh(I_1)^{e_1} \cdots h(I_k)^{e_k}$  for some  $f$  in the quotient field of  $R/xR$  and integers  $e_1, \dots, e_k$ . Write  $f = h(a)/h(b)$  for  $a, b \in R$ . Then  $h(b)h(P)h(I_1)^{-e_1} \cdots h(I_k)^{-e_k} = h(a)$ . Choose  $d_i \in R$  such that  $x$  does not divide  $d_i$  and  $d_i I_i^{-e_i} \subseteq R$  for  $i = 1, \dots, k$ . Form the ideal  $I = bP(d_1 I_1^{-e_1}) \cdots (d_k I_k^{-e_k})$ . Lemma 5 shows that  $h(I)$  is principal; say  $h(I) = h(t)R/xR$ . We may assume  $t \in I$ . From  $I \subseteq tR + xR$  and  $I: x = I$ , we get  $I = tR + xI$ . Since  $x$  is in the radical of  $R$ , we must have  $I = tR$  by Nakayama's lemma. This implies that  $P = t/bd_1 \cdots d_k \cdot I_1^{e_1} \cdots I_k^{e_k}$ , so  $\text{cl}(P)$  is in the subgroup of  $C(R)$  generated by  $\{\text{cl}(I_\alpha)\}$ . Since  $P$  is an arbitrary height 1 prime ideal, the theorem is established.

REMARKS. (1) Salmon's result cited in the introduction is obtained by choosing the set  $\{I_\alpha\}$  to consist of all principal ideals of  $R$  generated by elements of  $R$  which are not divisible by  $x$ .

(2) If  $R$  is a regular domain, then  $R[[X]]$  is also, and Theorem 6 may be applied with  $x = X$  and the set of ideals  $\{P_\alpha R[[X]]\}$  where  $P_\alpha$  ranges over the height 1 prime ideals of  $R$ . We get that  $\{\text{cl}(P_\alpha R[[X]])\}$  generate  $C(R[[X]])$  which shows that the natural homomorphism  $C(R) \rightarrow C(R[[X]])$  is onto (it is easily seen that it is one to one).

(3) Should Samuel's question "Does U.F.D. imply Macaulay?" [4] have an affirmative answer, then the hypotheses of Theorem 6 could be further weakened in the obvious fashion.

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