A NOTE ON THE CLASS GROUP

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The main result yields some information on the class group of a domain $R$ in terms of the class group of $R/xR$. With slightly stronger hypotheses than are strictly necessary, we state the main result: Let $R$ be a regular domain, $x$ a prime element contained in the radical of $R$, and suppose that $R/xR$ is locally a unique factorization domain. Let $\{I_a\}$ be a set of unmixed height 1 ideals of $R$ such that the classes of $\{I_a + xR/xR\}$ generate the class group of $R/xR$; then the classes of $\{I_a\}$ generate the class group of $R$.

The result of Samuel's and Buchsbaum's stating that if $R$ is a regular U.F.D., then $R[[X]]$ is a regular U.F.D. [4] has been generalized by P. Salmon and the present author in two different directions. Salmon [2, Prop. 3] showed that if $R$ is a regular domain, $x$ is a prime element of $R$ which is contained in the radical of $R$, and $R/xR$ is a U.F.D., then $R$ is a U.F.D. It was shown [1, Cor. 4] that the map of the class group of $R$ into the class group of $R[[X]]$ is onto if $R$ is a regular noetherian domain. We have found a theorem which simultaneously generalizes the last two results, and even allows a little weakening of the hypotheses.

To set the notation and terminology, we will say that a domain $R$ is locally U.F.D. if the quotient ring $R_M$ is a U.F.D. for all maximal ideals $M$ of $R$. For any Krull domain $R$, we will denote the class group (see [3]) of $R$ by $C(R)$. If $I$ is an unmixed height 1 ideal of a Krull domain $R$, we will denote the class of the class group determined by $I$ by $\text{cl}(I)$. Finally, all rings considered will be commutative noetherian domains with identity.

We wish to capitalize on a simple description of the class group valid for domains which are locally U.F.D. We do so and prepare for the main theorem by a sequence of (probably all known) lemmas.

**Lemma 1.** If $R$ is locally U.F.D., then $R$ is a Krull domain.

*Proof.* Since $R$ is noetherian, it is sufficient to show that $R$ is integrally closed. Since $R = \cap R_M$ as $M$ runs over all maximal ideals of $R$, it will be enough to see that each $R_M$ is integrally closed. But each $R_M$ is a U.F.D., hence integrally closed.

**Lemma 2.** If $R$ is locally U.F.D. and $P$ is a height 1 prime of $R$, then $P$ is invertible.
Proof. P is locally principal, hence locally free (as a module), hence projective, hence invertible.

**PROPOSITION 3.** If \( R \) is locally a U.F.D., then the unmixed height 1 ideals of \( R \) are precisely all finite products of minimal prime ideals of \( R \).

Let \( I_i \) and \( I_j \) be two unmixed height 1 ideals of \( R \), then \( \text{cl}(I_i) = \text{cl}(I_j) \) if and only if there are elements \( a \) and \( b \) in \( R \) such that \( aI_i = bI_j \).

**Proof.** From Lemma 2 we know that any product of height 1 prime ideals of \( R \) is invertible. Given an unmixed height 1 ideal \( I \) determined by the valuation data \( I = \{ x \mid v_{P_i}(x) \geq n_i \} \) (almost all \( n_i = 0 \)), we form \( J = \prod_{n_i \neq 0} P_i^{n_i} \). Since \( J \) is invertible, we have \( J = R: (R: J) \), so \( J \) is also unmixed of height 1. Since \( I \) and \( J \) are determined by same valuation data, this entails \( I = J \). If now \( I_i \) and \( I_j \) are unmixed height 1 ideals such that \( \text{cl}(I_i) = \text{cl}(I_j) \), the \( I_iI_j^{-1} \) is invertible and is determined by the same data as some \( f \cdot R \), where \( f \) is in the quotient field of \( R \). We have therefore \( I_iI_j^{-1} = fR \), or \( I_i = fI_j \), which is equivalent to the final assertion.

**LEMMA 4.** Let \( R \) be locally U.F.D., and suppose that \( R \) is a Macaulay ring. Let \( I \) be an unmixed height 1 ideal of \( R \) and \( x \) an element of the radical of \( R \) such that \( I; xR = I \). Then \( I + xR \) is unmixed of height 2.

**Proof.** Word for word the proof of Lemma 2 of [1].

**LEMMA 5.** Let the hypotheses be as in Lemma 4 and suppose that \( x \) is prime and \( R/xR \) is a Krull domain. Let \( h \) denote the homomorphism of \( R \) onto \( R/xR \). If \( d \) is an element of \( R \) such that \( dI^{-1} \subseteq R \) (for \( I \) an unmixed height 1 ideal of \( R \)), then \( \text{cl}(h(dI^{-1})) = \text{cl}(h(I))^{-1} \).

**Proof.** From \( II^{-1} = R \), we get \( I(dI^{-1}) = dR \). Applying \( h \) to both sides of the last equation, we obtain \( h(I) \cdot h(dI^{-1}) = h(d) \cdot R/xR \), which yields the result.

**THEOREM 6.** Let \( R \) be a Macaulay ring which is locally U.F.D. Let \( x \) be a prime element of the radical of \( R \) such that \( R/xR \) is locally U.F.D. Let \( h \) denote the natural homomorphism of \( R \) onto \( R/xR \). If \( \{ I_\alpha \} \) is a set unmixed height 1 ideals of \( R \) such that \( I_\alpha; xR = I_\alpha \) and \( \{ \text{cl}(h(I_\alpha)) \} \) generates \( C(R/xR) \), then \( \{ \text{cl}(I_\alpha) \} \) generates \( C(R) \).

**Proof.** Let \( P \) be a height 1 prime ideal of \( R \). If \( x \in P \), then
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$P = xR$, and $\text{cl}(P)$ is the identity element of $C(R)$. If $x \notin P$, we must have $P : xR = P$ and Lemma 4 shows that $P + xR$ is unmixed of height 2. Thus $h(P)$ is unmixed of height 1 in $R/xR$, so the hypotheses yield that $h(P) = f h(I_1)^{e_1} \cdots h(I_k)^{e_k}$ for some $f$ in the quotient field of $R/xR$ and integers $e_1, \ldots, e_k$. Write $f = h(a)/h(b)$ for $a, b \in R$. Then $h(b)h(P)h(I_1)^{-e_1} \cdots h(I_k)^{-e_k} = h(a)$. Choose $d_i \in R$ such that $x$ does not divide $d_i$ and $d_iI_i^{-e_i} \subseteq R$ for $i = 1, \ldots, k$. Form the ideal $I = bP(d_1I_1^{-e_1}) \cdots (d_kI_k^{-e_k})$. Lemma 5 shows that $h(I)$ is principal; say $h(I) = h(t)R/xR$. We may assume $t \in I$. From $I \subseteq tR + xR$ and $I : x = I$, we get $I = tR + xI$. Since $x$ is in the radical of $R$, we must have $I = tR$ by Nakayama’s lemma. This implies that $P = t/bd_1 \cdots d_k \cdot I_1 \cdots I_k^{e_k}$, so $\text{cl}(P)$ is in the subgroup of $C(R)$ generated by $\{\text{cl}(I_a)\}$. Since $P$ is an arbitrary height 1 prime ideal, the theorem is established.

REMARKS. (1) Salmon’s result cited in the introduction is obtained by choosing the set $\{I_a\}$ to consist of all principal ideals of $R$ generated by elements of $R$ which are not divisible by $x$.

(2) If $R$ is a regular domain, then $R[[X]]$ is also, and Theorem 6 may be applied with $x = X$ and the set of ideals $\{P_aR[[X]]\}$ where $P_a$ ranges over the height 1 prime ideals of $R$. We get that $\{\text{cl}(P_aR[[X]])\}$ generate $C(R[[X]])$ which shows that the natural homomorphism $C(R) \to C(R[[X]])$ is onto (it is easily seen that it is one to one).

(3) Should Samuel’s question “Does U.F.D. imply Macaulay?” [4] have an affirmative answer, then the hypotheses of Theorem 6 could be further weakened in the obvious fashion.

REFERENCES

1. L. Claborn, Note generalizing a result of Samuel’s, Pacific J. Math. 15 (1965), 805-808.

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