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The main result yields some information on the class group of a domain R in terms of the class group of R/xR. With slightly stronger hypotheses than are strictly necessary, we state the main result: Let R be a regular domain, x a prime element contained in the radical of R, and suppose that R/xR is locally a unique factorization domain. Let $\{I_\alpha\}$ be a set of unmixed height 1 ideals of R such that the classes of $\{I_\alpha + xR/xR\}$ generate the class group of R/xR; then the classes of $\{I_\alpha\}$ generate the class group of R.

The result of Samuel's and Buchsbaum's stating that if R is a regular U.F.D., then $R[[X]]$ is a regular U.F.D. [4] has been generalized by P. Salmon and the present author in two different directions. Salmon [2, Prop. 3] showed that if R is a regular domain, x is a prime element of R which is contained in the radical of R, and R/xR is a U.F.D., then R is a U.F.D. It was shown [1, Cor. 4] that the map of the class group of R into the class group of $R[[X]]$ is onto if R is a regular noetherian domain. We have found a theorem which simultaneously generalizes the last two results, and even allows a little weakening of the hypotheses.

To set the notation and terminology, we will say that a domain R is locally U.F.D. if the quotient ring R_M is a U.F.D. for all maximal ideals M of R. For any Krull domain R, we will denote the class group (see [3]) of R by $C(R)$. If I is an unmixed height 1 ideal of a Krull domain R, we will denote the class of the class group determined by I by $\mathrm{cl}(I)$. Finally, all rings considered will be commutative noetherian domains with identity.

We wish to capitalize on a simple description of the class group valid for domains which are locally U.F.D. We do so and prepare for the main theorem by a sequence of (probably all known) lemmas.

LEMMA 1. *If R is locally U.F.D., then R is a Krull domain.*

Proof. Since R is noetherian, it is sufficient to show that R is integrally closed. Since $R = \cap R_M$ as M runs over all maximal ideals of R, it will be enough to see that each R_M is integrally closed. But each R_M is a U.F.D., hence integrally closed.

LEMMA 2. *If R is locally U.F.D. and P is a height* 1 *prime of R, then P is invertible.*

Proof. P is locally principal, hence locally free (as a module), hence projective, hence invertible.

PROPOSITION 3. If R is locally a U.F.D., then the unmixed height 1 ideals of R are precisely all finite products of minimal prime ideals of R.

Let I_1 and I_2 be two unmixed height 1 ideals of R, then $\mathrm{cl}(I_1) = \mathrm{cl}(I_2)$ if and only if there are elements a and b in R such that $aI_1 = bI_2$.

Proof. From Lemma 2 we know that any product of height 1 prime ideals of R is invertible. Given an unmixed height 1 ideal I determined by the valuation data $I = \{x \mid v_{P_i}(x) \geqq n_i\}$ (almost all $n_i = 0$), we form $J = \prod_{n_i \neq 0} P_i^{n_i}$. Since J is invertible, we have $J = R\colon (R\colon J)$, so J is also unmixed of height 1. Since I and J are determined by same valuation data, this entails $I = J$. If now I_1 and I_2 are unmixed height 1 ideals such that $\mathrm{cl}(I_1) = \mathrm{cl}(I_2)$, the $I_1 I_2^{-1}$ is invertible and is determined by the same data as some $f \cdot R$, where f is in the quotient field of R. We have therefore $I_1 I_2^{-1} = fR$, or $I_1 = fI_2$, which is equivalent to the final assertion.

LEMMA. 4. *Let R be locally U.F.D., and suppose that R is a Macaulay ring. Let I be an unmixed height 1 ideal of R and x an element of the radical of R such that $I\colon xR = I$. Then $I + xR$ is unmixed of height 2.*

Proof. Word for word the proof of Lemma 2 of [1].

LEMMA 5. *Let the hypotheses be as in Lemma 4 and suppose that x is prime and R/xR is a Krull domain. Let h denote the homomorphism of R onto R/xR. If d is an element of R such that $dI^{-1} \subseteq R$ (for I an unmixed height 1 ideal of R), then $\mathrm{cl}\,(h(dI^{-1})) = \mathrm{cl}(h(I))^{-1}$.*

Proof. From $II^{-1} = R$, we get $I(dI^{-1}) = dR$. Applying h to both sides of the last equation, we obtain $h(I) \cdot h(dI^{-1}) = h(d) \cdot R/xR$, which yields the result.

THEOREM 6. *Let R be a Macaulay ring which is locally U.F.D. Let x be a prime element of the radical of R such that R/xR is locally U.F.D. Let h denote the natural homomorphism of R onto R/xR. If $\{I_\alpha\}$ is a set unmixed height 1 ideals of R such that $I_\alpha\colon xR = I_\alpha$ and $\{\mathrm{cl}\, h(I_\alpha)\}$ generates $C(R/xR)$, then $\{\mathrm{cl}\,(I_\alpha)\}$ generates $C(R)$.*

Proof. Let P be a height 1 prime ideal of R. *If* $x \varepsilon P$, then

$P = xR$, and $\mathrm{cl}(P)$ is the identity element of $C(R)$. If $x \notin P$, we must have $P: xR = P$ and Lemma 4 shows that $P + xR$ is unmixed of height 2. Thus $h(P)$ is unmixed of height 1 in R/xR, so the hypotheses yield that $h(P) = fh(I_1)^{e_1} \cdots h(I_k)^{e_k}$ for some f in the quotient field of R/xR and integers $e_1, \cdots, e_k$. Write $f = h(a)/h(b)$ for $a, b \varepsilon R$. Then $h(b)h(P)h(I_1)^{-e_1} \cdots h(I_k)^{-e_k} = h(a)$. Choose $d_i \varepsilon R$ such that x does not divide d_i and $d_i I_i^{-e_i} \subseteq R$ for $i = 1, \cdots, k$. Form the ideal $I = bP(d_1 I_1^{-e_1}) \cdots (d_k I_k^{-e_k})$. Lemma 5 shows that $h(I)$ is principal; say $h(I) = h(t)R/xR$. We may assume $t \varepsilon I$. From $I \subseteq tR + xR$ and $I: x = I$, we get $I = tR + xI$. Since x is in the radical of R, we must have $I = tR$ by Nakayama's lemma. This implies that $P = t/bd_1 \cdots d_k \cdot I_1^{e_1} \cdots I_k^{e_k}$, so $\mathrm{cl}\,(P)$ is in the subgroup of $C(R)$ generated by $\{\mathrm{cl}\,(I_\alpha)\}$. Since P is an arbitrary height 1 prime ideal, the theorem is established.

REMARKS. (1) Salmon's result cited in the introduction is obtained by choosing the set $\{I_\alpha\}$ to consist of all principal ideals of R generated by elements of R which are not divisible by x.

(2) If R is a regular domain, then $R[[X]]$ is also, and Theorem 6 may be applied with $x = X$ and the set of ideals $\{P_\alpha R[[X]]\}$ where P_α ranges over the height 1 prime ideals of R. We get that $\{\mathrm{cl}(P_\alpha R[[X]])\}$ generate $C(R[[X]])$ which shows that the natural homomorphism $C(R) \rightarrow C(R[[X]])$ is onto (it is easily seen that it is one to one).

(3) Should Samuel's question "Does U.F.D. imply Macaulay?" [4] have an affirmative answer, then the hypotheses of Theorem 6 could be further weakened in the obvious fashion.

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