POINT-DETERMINING HOMOMORPHISMS ON MULTIPLICATIVE SEMI-GROUPS OF CONTINUOUS FUNCTIONS

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Let $X$ and $Y$ be compact Hausdorff spaces, $C(X)$ and $C(Y)$ the algebras of real valued continuous functions on $X$ and $Y$ respectively with the usual sup norms. If $T$ is an algebra homomorphism from $C(X)$ onto a dense subset of $C(Y)$ then by a theorem of Stone, $T$ induces a homeomorphism $\mu$ from $Y$ to $X$ and it necessarily follows that $Tf(y) = 0$ if and only if $f(\mu(y)) = 0$.

In a more general setting, viewing $C(X)$ and $C(Y)$ as multiplicative semi-groups, let $T$ be a semi-group homomorphism from $C(X)$ onto a dense point-separating set in $C(Y)$. No such map $\mu$ satisfying the above condition need exist. $T$ is called point-determining in case for each $y$ there is an $x$ such that $Tf(y) = 0$ if and only if $f(x) = 0$. It is shown that such a homomorphism $T$ induces a homeomorphism from $Y$ into $X$ in such a way that $Tf(y) = [\text{sgn} f(x)] |f(x)|^{p(x)}$ for some continuous positive function $p$ where $x$ is related to $y$ via the induced homeomorphism, that such a $T$ is an algebra homomorphism followed by a semi-group automorphism, and that $T$ is continuous.

Let $X$ and $Y$ be compact Hausdorff spaces, $C(X)$ and $C(Y)$ the algebras of all continuous real-valued functions on $X$ and $Y$ respectively with the usual sup norm. Let $T$ be an algebra homomorphism of $C(X)$ onto a dense set in $C(Y)$. For each $y \in Y$ consider the mapping $\gamma_y$ of $C(X)$ into the reals defined by

$$\gamma_y(f) = Tf(y).$$

$\gamma_y$ maps $C(X)$ onto the reals for if $Tf(x) = 0$ for all $f \in C(X)$ then the image of $T$ is not dense. The kernel is, by algebra, a maximal ideal in $C(X)$. By a theorem of Stone [3, 80] there is a point $x \in X$ so that the kernel of $\gamma_y$ is the set of all $f \in C(X)$ such that $f(x) = 0$, this point being uniquely determined.

Consider the map $\mu$ of $Y$ into $X$ which assigns to each $y \in Y$ the $x$ as described above. If $e$ and $e_i$ are the unit functions in $C(X)$ and $C(Y)$ respectively it is easy to see that $Te = e_i$ and that $\mu$ is one-to-one. Now for each $f \in C(X)$ consider the function $Tf(y)e - f = g$ in $C(X)$. Then $Tg(y) = 0$ so that $g(\mu(y)) = 0$ and hence $Tf(y) = f(\mu(y))$. We especially note that
As we shall see, under a more general setting for $T$, this condition will imply that $\mu$ is bicontinuous (see Lemma 3.1 below).

In this paper we view $C(X)$ and $C(Y)$ as multiplicative semi-groups and let $T$ be a semi-group homomorphism from $C(X)$ onto a dense set in $C(Y)$; the restriction on $T$ being that for each $y \in Y$ there is an $x \in X$ such that $f(x) = 0$ if and only if $Tf(y) = 0$ (i.e. a condition such as (*) above is satisfied). For such a $T$ we show that $Y$ can be imbedded homeomorphically in $X$ in such a way that $Tf(y) = [\text{sgn } f(x)] \cdot |f(x)|^{p(x)}$ for some continuous positive function $p(x)$ where $y$ is related to $x$ via the induced homeomorphism. It is shown that each such homomorphism $T$ is an algebra homomorphism followed by a semi-group automorphism and that $T$ is continuous.

2. Definitions, Notation and Preliminaries. We first note that in our more general setting no mapping $\mu$ satisfying (*) above need exist. To see this let $X = [0, 1] \cup \{2\}$, $Y = [0, 1]$ with the relative topology of the real line. For $t \in [0, 1]$ set

$$Tf(t) = f(t)f(2).$$

$T$ is a semi-group homomorphism of $C(X)$ onto $C(Y)$ but $Tf(t) = 0$ if and only if either $f(t) = 0$ or $f(2) = 0$.

DEFINITION 2.1. A semi-group homomorphism $T$ will be called point-determining in case for each $y \in Y$ there is an $x \in X$ such that $f(x) = 0$ if and only if $Tf(y) = 0$.

The following result is immediate.

LEMMA 2.2. If $T$ is a point-determining semi-group homomorphism of $C(X)$ onto a dense set in $C(Y)$, $e$ and $e$, the respective unit functions in $C(X)$ and $C(Y)$ then $Te = e$ and $TO = O$.

DEFINITION 2.3. A subset $A \subseteq C(Y)$ will be called point-separating in case for $y_1 \neq y_2$ in $Y$ there is a $g \in A$ such that $g(y_1) = 0$ and $g(y_2) \neq 0$.

In the development that follows $X$ and $Y$ will be compact Hausdorff spaces, $Y$ having no isolated points; $C(X)$ and $C(Y)$ will be viewed as multiplicative semi-groups and $T$ will be a point-determining semi-group homomorphism of $C(X)$ onto a dense point-separating set in $C(Y)$. The hypothesis $Y$ has no isolated points, however, is not used until Lemma 3.5. Multiplication is defined pointwise in $C(X)$ and $C(Y)$.

$\sim A$ will denote complement of $A$ in any of the spaces considered.
\( \emptyset \) will denote the empty set and the bar notation will denote closure.

We are indebted to a paper of Milgram [2] for suggesting the sequence of ideas and devices employed here.

3. Development of the main results. Notice that for each \( y \in Y \), \( T \) determines a unique point \( x \in X \). Thus \( T \) induces a well-defined single valued mapping \( \mu: Y \rightarrow X \) defined by \( \mu(y) = x \) in case \( f(x) = 0 \) if and only if \( Tf(y) = 0 \). In the material to follow notationally we let \( \mu(Y) = X \). (\( \mu \) turns out to be a special case of the multi-valued mappings studied in [1] although there we assumed \( T \) continuous.)

**Lemma 3.1.** \( \mu \) is a homeomorphism of \( Y \) into \( X \).

**Proof.** \( \mu \) is a one-to-one for say \( \mu(y_1) = \mu(y_2) \). Then \( Tf(y_1) = 0 \) if and only if \( Tf(y_2) = 0 \). If \( y_1 \neq y_2 \), then since the range of \( T \) is point-separating there is an \( h \in C(X) \) such that \( Th(y_1) = 0 \), \( Th(y_2) \neq 0 \) a contradiction.

To see \( \mu \) is continuous we suppose contrarywise that \( \mu \) is not continuous at some point \( t_0 \in Y \). Then there is a net \( \{ t_\beta \} \) in \( Y \), \( t_\beta \to t_0 \) and an open neighborhood \( U \) containing \( \mu(t_\beta) \) such that \( \mu(t_\beta) \in U \) for any \( \beta \). Now there is an \( f \in C(X) \) such that \( f(\mu(t_0)) = 1 \) and \( f(\sim U) = 0 \) so that \( f(\mu(t_\beta)) = 0 \) for all \( \beta \) and hence \( Tf(t_\beta) = 0 \) for all \( \beta \). But \( Tf(t_0) \neq 0 \) since \( f(\mu(t_0)) \neq 0 \) contradicting the fact that \( Tf \in C(Y) \). Thus \( \mu \) is continuous and it follows that \( \mu \) is a homeomorphism.

If \( \sigma \) is a homeomorphism from \( Y \) into \( X \), define

\[
T: C(X) \rightarrow C(Y)
\]

by

\[
Tf(y) = f(\sigma(y)) , \quad f \in C(X) , \quad y \in Y .
\]

\( T \) is onto (and continuous) so that we have the following.

**Theorem 3.2.** There is a point-determining semi-group homomorphism of \( C(X) \) onto a dense point-separating set in \( C(Y) \) if and only if \( Y \) is homeomorphic to a subset of \( X \).

We proceed now, to find the form of the general homomorphism in our theory.

**Lemma 3.3.** Let \( U \) be open in \( X \). If \( f \equiv 1 \) on \( U \) then \( Tf \equiv 1 \) on \( \mu^{-1}(U \cap X_0) \).
Proof. \( f \equiv 1 \) on \( U \) implies that \( fg = g \) for all \( g \in C(X) \) such that 
\( g(x) = 0 \) on \( \sim U \) and hence \( TfTg = Tg \) for all such \( g \). Let \( y_0 \in \mu^{-1}(U \cap X_0) \)
so that \( \mu(y_0) = x_0 \in U \cap X_0 \subseteq U \) and note that \( Tf(y_0) \neq 0 \) since \( f(x_0) = 1 \).
Now there is an \( h \in C(X) \) such that \( h(x_0) = 1 \) and \( h(\sim U) = 0 \) so from the above \( Tf(y_0)Th(y_0) = Th(y_0) \). But \( h(x_0) = 1 \) implies that \( Th(y_0) \neq 0 \) and therefore \( Tf(y_0) = 1 \) and since \( y_0 \) was arbitrary the result follows.

**Lemma 3.4.** Let \( U \) be open in \( X \). If \( f \equiv g \) on \( U \) then \( Tf \equiv Tg \)
on \( \mu^{-1}(U \cap X_0) \).

Proof. Let \( y_0 \in \mu^{-1}(U \cap X_0) \) so that \( \mu(y_0) = x_0 \in U \cap X_0 \subseteq U \). If 
\( f(x_0) = 0 = g(x_0) \) then \( Tf(y_0) = 0 = Tg(y_0) \). If \( f(x_0) = g(x_0) \neq 0 \) we may assume without loss of generality that \( f(x_0) = g(x_0) = c > 0 \). Then 
\( W' = \{ x | f(x) > c/2 \} \) is open in \( X \) and \( x_0 \in W' \). For \( x \in X \) set \( h'(x) = \max[f(x), c/2] \). Then \( h' \) and \( h = 1/h' \), are in \( C(X) \). Now \( fh \equiv 1 \) on 
\( W' \) and hence \( fh \equiv 1 \equiv gh \) on \( W = W' \cap U \). Thus by Lemma 3.3 
\( Tfh = 1 = Tgh \) on \( \mu^{-1}(W \cap X_2) \) and so in particular \( Tf(y_0)Th(y_0) = 1 = Tg(y_0)Th(y_0) \). Now \( h(x_0) \neq 0 \) so \( Th(y_0) \neq 0 \). Thus \( Tf(y_0) = Tg(y_0) \) and the result follows.

**Lemma 3.5.** Let \( x_0 = \mu(y_0) \). If \( f(x_0) = 1 \) then \( Tf(y_0) = 1 \).

Proof. Suppose first that \( f(x_0) = 1 \) but that \( Tf(y_0) > 1 \). Then there is an open neighborhood \( W \) containing \( y_0 \) such that \( Tf(y_0) \geq c > 1 \) for all \( y \in W \). Now \( \mu(W) = U \cap X_0 \) for some open set \( U \) in \( X \) such that \( x_0 \in U \). Let \( V_n = \{ x \in X \mid |f^n(x) - 1| < 1/n \} \) \( n = 1, 2, 3, \ldots \) and set 
\( U_n = V_n \cap U \). Note that \( x_0 \in U_n \) an open set in \( X \) for each \( n \) and that there are points of \( X_0 - \{ x_0 \} \) in \( U_n \) for every \( n \) since \( Y \) has no isolated points (and hence \( X_0 \) has none). We construct a sequence \( \{ x_n \} \) of distinct points such that \( x_n \in U_n \cap X_0 \) as follows:

Select \( x_1 \in U_1 \cap X_0 \) such that \( x_1 \neq x_0 \) and set \( U_1 = W_0^{(n)} \). Select disjoint neighborhoods \( W_0^{(1)} \) containing \( x_0 \) and \( W_1^{(1)} \) containing \( x_1 \) such that 
\( W_0^{(1)} \subset W_0^{(0)} \) and \( W_1^{(1)} \subset W_0^{(0)} \).

In general select \( x_n \in W_0^{(n-1)} \cap X_0 \cap U_n \) such that \( x_n \neq x_0 \) and disjoint neighborhoods \( W_0^{(n)} \) containing \( x_n \) and \( W_1^{(n)} \) containing \( x_n \) such that 
\( W_0^{(n)} \subset W_0^{(n-1)} \) and \( W_1^{(n)} \subset W_0^{(n-1)} \). Note that \( \{ W_0^{(n)} \} \) is a decreasing sequence of neighborhoods containing \( x_0 \) where \( x_1 \in W_0^{(i-1)} \); \( \{ W_1^{(n)} \} \) is a collection of neighborhoods where \( x_i \in W_1^{(i)} \) and \( W_1^{(n)} \cap W_0^{(n)} = \emptyset \) \( n = 1, 2, 3, \ldots \).

For the sequence \( \{ x_n \} \) we have \( \{ x_{n+1}, x_{n+2}, \ldots \} \subset W_0^{(n)} \) and \( W_1^{(n)} \) is a neighborhood containing \( x_n \) such that \( \{ x_{n+1}, x_{n+2}, \ldots \} \cap W_1^{(n)} = \emptyset \). Therefore \( x_n \in \{ x_{n+1}, x_{n+2}, \ldots \} \). Hence we can select a collection \( \{ O_n \} \) of open sets as follows:
Let $O_i$ be an open subset of $U_i$ such that $x_i \in O_i \subset \bar{O}_i \subset U_i$ and $\bar{O}_i$ does not contain $x_2, x_3, \ldots$. In general let $O_n$ be an open subset of $U_n$ such that $x_n \in O_n \subset \bar{O}_n \subset U_n$ and $\bar{O}_n$ does not contain $x_{n+1}, x_{n+2}, \ldots$ and such that $\bar{O}_i \cap \bar{O}_n = \emptyset$ for $i = 1, 2, \ldots, n - 1$.

Now define a function $g'$ on $\bigcup_{n=1}^{\infty} \bar{O}_n$ by

$$g' = \begin{cases} f^n & \text{on } \bar{O}_n \\ 1 & \text{elsewhere on } \bigcup_{n=1}^{\infty} \bar{O}_n. \end{cases}$$

Then $g'$ is continuous on $\bigcup_{n=1}^{\infty} \bar{O}_n$. To see this we need only examine $t \in \bigcup_{n=1}^{\infty} \bar{O}_n - \bigcup_{n=1}^{\infty} \bar{O}_n$. At such a $t$, $g'(t) = 1$. Let $Q$ be any open set in the reals containing 1 and choose $N > 0$ such that $(1 - 1/N, 1 + 1/N) \subset Q$. Now since $t \in \bigcup_{k=1}^{\infty} \bar{O}_k$, a closed set, there is a neighborhood $V$ containing $t$ such that $V \cap \bigcup_{k=N+1}^{\infty} \bar{O}_k = \emptyset$. For $s \in V \cap \bigcup_{k=N+1}^{\infty} \bar{O}_k$, either $g'(s) = 1$ or $s \in \bar{O}_k$ for some $k \in \{N + 1, N + 2, \ldots\}$ in which case $|g'(s) - 1| = |f^k(s) - 1| < 1/k < 1/N$. So in any case $|g'(s) - 1| < 1/N$ i.e. $g'(s) \in Q$ and hence $g'$ is continuous. By Tietze’s extension theorem $g'$ can be extended to a function $g \in C(X)$.

Now for $y \in \mu^{-1}(O_n \cap X_0) \subset W$ we have by Lemma 3.4

$$Tg(y) = Tf^n(y) = [Tf(y)]^n \geq c^n$$

so $Tg$ is not bounded and hence $Tg \in C(Y)$ a contradiction. Thus if $f(x_0) = 1$ then $Tf(y_0) \leq 1$.

Now suppose that $f(x_0) = 1$ but that $Tf(y_0) < 1$. Set $f_1 = f^2$. Then $f_1(x_0) = 1$ so $Tf_1(y_0) \neq 0$ and $Tf_1(y_0) = [Tf(y_0)]^2 > 0$. By the first part of the proof $Tf_1(y_0) \neq 1$. We rule out $Tf_1(y_0) < 1$ as follows.

Set $W = \{x \mid f_1(x) > 1/2\}$. $W$ is open, $x_0 \in W$ and if we set $f_2(x) = \max\{1/2, f_1(x)\}$ then $f_2$ is nowhere zero, $f_2 \in C(X)$ and $f_2$ agrees with $f_1$ on $W$. Hence by Lemma 3.4 $Tf_1 \equiv Tf_2$ on $\mu^{-1}(W \cap X_0)$ and so $0 < Tf_2(y_0) < 1$. Now $f_3 = 1/f_2 \in C(X)$, $f_3(x_0) = 1$ and $Tf_3(y_0) = 1/Tf_2(y_0) > 1$. But $f_3(y_0) = \pm 1$. But by assumption $Tf_2(y_0) < 1$ so $Tf_2(y_0) = -1$.

As done above let $g$ be a strictly positive function in $C(X)$ agreeing with $f$ on some neighborhood $U$ containing $x_0$. Then $Tf$ and $Tg$ agree on $\mu^{-1}(U \cap X_0)$. But $g > 0$ everywhere on $X$ implies that $Tg \geq 0$ everywhere on $Y$ and hence $Tg(y_0) = -1$ so $Tf(y_0) \neq -1$, a contradiction. Thus $Tf(y_0) = +1$ and the proof is finished.

**Lemma 3.6.** If $x_0 = \mu(y_0)$ and if $f(x_0) = g(x_0)$ then $Tf(y_0) = Tg(y_0)$.

**Proof.** We need only consider $f(x_0) = c = g(x_0) \neq 0$. Let $h(x) = 1/c$ for all $x \in X$ so that $h \in C(X)$ and $hf(x_0) = hg(x_0) = 1$. By Lemma 3.5 $Thf(y_0) = 1 = Thg(y_0)$ i.e. $Th(y_0)Tf(y_0) = Th(y_0)Tg(y_0) = 1$. But $Th(y_0) \neq 0$.
so the result follows.

Notice that Lemma 3.6 implies us that functions in $C(X)$ which agree on $X_0 = \mu(Y)$ have the same images in $C(Y)$. We will show that $T$ is actually restriction to $X_0$ followed by a semi-group automorphism.

Suppose we regard the real numbers, $R$, as a multiplicative semi-group. We have the following.

**Lemma 3.7.** Let $\alpha$ be a semi-group homomorphism from $R$ onto a dense subset of $R$. Then $\alpha$ is either unbounded in every neighborhood of zero or $\alpha$ is order preserving.

**Proof.** Since the range of $\alpha$ is dense in $R$ it follows that $\alpha(0) = 0$ and $\alpha(1) = 1$. If we show that $\alpha(-t) = -\alpha(t)$ for all $t$ then only positive numbers need be considered in verifying the lemma. To this end note $\alpha(1) = [\alpha(-1)]^2$ so $\alpha(-1) = \pm 1$. We rule out $\alpha(-1) = +1$ for suppose $\alpha(-1) = +1$. Then $\alpha(\pm t) = \alpha(t)$ for all $t$. Let $\{t_n\}$ be a sequence in $R$ such that $\alpha(t_n) \rightarrow -1$. Then $\alpha(-t_n) = \alpha(t_n) \rightarrow -1$ so that $\alpha(\lvert t_n \rvert) \rightarrow -1$. But $|t_n| = s_n^2$ for some $s_n \in R$ and $\alpha(s_n^2) = [\alpha(s_n)]^2 \rightarrow -1$ a contradiction. Hence $\alpha(-t) = -\alpha(t)$.

Now let $a, b \in R$ such that $0 < a < b$. Suppose $\alpha(a) > \alpha(b)$. Then since $\alpha(a/b) = \alpha(a)/\alpha(b)$ we have $\alpha[(a/b)^n] = [\alpha(a)/\alpha(b)]^n \rightarrow \infty$ while $(a/b) \rightarrow 0$ i.e. $\alpha$ is unbounded in every neighborhood of zero. Now suppose $\alpha(a) = \alpha(b)$ and that $\alpha$ is bounded in some neighborhood of zero. Then $r \in [a, b]$ implies that $\alpha(r) = \alpha(a)$ since otherwise either $\alpha(r) < \alpha(a)$ or $\alpha(r) > \alpha(b)$ and in both cases by the above $\alpha$ would be unbounded in every neighborhood of zero contradicting our assumption. Hence for $r, r' \in [a, b]$ $\alpha(r/r') = \alpha(r)/\alpha(r') = 1$. Now let $z$ be any positive real number. There is an $n$ such that $a/b \leq z^{1/n} \leq b/a$ i.e. there is an $n$ such that $z^{1/n} = r/r'$, where $r, r' \in [a, b]$. Then $1 = \alpha(r/r')$ and so $\alpha(z) = [\alpha(z^{1/n})]^n = 1$. So $z > 0$ implies that $\alpha(z) = 1$, $z < 0$ implies that $\alpha(z) = -1$ and $\alpha(0) = 0$, a contradiction since the image of $\alpha$ is dense in $R$. Thus $\alpha(a) = \alpha(b)$ implies that $\alpha$ is unbounded in every neighborhood of zero.

**Lemma 3.8.** If $\alpha$ is order preserving then $\alpha$ is actually onto and in this case $\alpha(t) = (\text{sgn } t) \lvert t \rvert^p$ for some positive number $p$.

**Proof.** Let $r_0 \in R$ and $\{r_n\}$ a sequence in $R$ such that $\{r_n\} \downarrow r_0$. Then $\alpha(r_n) \rightarrow \alpha(r_0)$ since $\alpha(r_n) > \alpha(r_0)$ and if $\alpha(r_n) \geq m > \alpha(r_0)$ there is an $s \in R$, $m > s > \alpha(r_0)$ and a $q \in R$ such that $\alpha(q) = s$. But $\alpha(r_n) < \alpha(q) < \alpha(r_n)$ for all $n$ so $r_0 < q < r_n$, a contradiction since $r_n \rightarrow r_0$. 


To see $\alpha$ is onto say $r_0$ is such that $\alpha(r) \neq r_0$ for any $r \in R$. We can choose a sequence of distinct points $\{\alpha(r_n)\}$ such that $\alpha(r_n) \downarrow r_0$. This implies $\{r_n\}$ is a bounded decreasing sequence so there is an $r'$ such that $r_n \downarrow r'$ and hence from the above $\alpha(r_n) \to \alpha(r')$, a contradiction since $\alpha(r') \neq r_n$. Thus $\alpha$ is onto. Milgram [2, 4.3] has shown that in this case there is a $p > 0$ such that $\alpha(t) = (\text{sgn } t) |t|^p$ which completes the proof.

In view of Lemma 3.6 for each $y \in Y$, $\alpha_{\mu(y)} : R \to R$ defined for arbitrary $f \in C(X)$ by $\alpha_{\mu(y)}(f(\mu(y))) = Tf(y)$ is well-defined. The image of $\alpha_{\mu(y)}$ is a dense subset in $R$, for fix $y \in Y$ and let $r \in R$. There is a function $g \in C(Y)$ such that $g(y) = r$ and a sequence $\{f_n\} \subset C(X)$ such that $Tf_n(y) \to g(y) = r$ i.e. $\alpha_{\mu(y)}(f_n(\mu(y))) \to r$.

Note that from Lemmas 3.7 and 3.8 we can say that $\alpha_{\mu(y)}$ is unbounded in every neighborhood of zero or $\alpha_{\mu(y)}$ is continuous.

**Lemma 3.9.** The mappings $\{\alpha_{\mu(y)}\}$ are discontinuous for at most a finite number of points.

**Proof.** Suppose otherwise at $\{\mu(y'_n)\}$ where the $y'_n$ are all distinct $n = 1, 2, 3, \cdots$. We can choose a subsequence $\{\mu(y_n)\}$ of distinct points such that no $\mu(y_n)$ is a limit point of the others as follows:

If no point in $\{\mu(y'_n)\}$ is a limit point of the other we are finished. If $y'_{n_0}$ is a limit point of a subset of $\{\mu(y'_n)\}$ where $y'_{n_0} \in \{\mu(y'_n)\}$, by a process similar to that used in selecting the sequence $\{x_n\}$ in the proof of Lemma 3.5 with $y'_{n_0}$ in the role of $x_0$ we obtain a sequence $\{\mu(y'_n)\}$ such that $\mu(y'_n) \in \{\mu(y_{n+1}'), \mu(y_{n+2}'), \cdots\}$, $n = 1, 2, 3, \cdots$. Hence for any $\mu(y_n)$ there is an open set $V$ containing $\mu(y_n)$ such that $V \cap \{\mu(y_n)\} - \mu(y_n) = \emptyset$ so that $\{\mu(y_n)\}$ is the desired collection.

Now the $\alpha_{\mu(y'_n)}$ are unbounded in each neighborhood of the origin so that if $\{t'_m\}$ is a sequence of distinct points decreasing to zero we have $\alpha_{\mu(y'_n)}(t'_m) \to \infty$ for all $n$ as $m \to \infty$. We select a subsequence $\{t_n\}$ such that $\alpha_{\mu(y'_n)}(t_n) \to \infty$ as follows:

There is a $t \in \{t'_m\}$ such that $\alpha_{\mu(y'_n)}(t) > 1$. Set $t = t_1$. In general there is a $t < t_{n-1} < \cdots < t_n, t \in \{t'_m\}$ such that $\alpha_{\mu(y'_n)}(t) > n$. Set $t = t_n$ to yield the desired sequence.

Define a function $f'$ on $\{\mu(y_n)\}$ by $f'(\mu(y_n)) = t_n$ and $f' = 0$ on $\{\mu(y'_n)\} - \{\mu(y_n)\}$. $f'$ is continuous on $\{\mu(y'_n)\}$ since for $y_n \in \{\mu(y'_n)\} - \{\mu(y_n)\}$ we have $f'(y_n) = 0$ and letting $\{\mu(y'_n)\}$ be any subsequence converging to $y_0$, $f'(\mu(y'_n)) = t_m \to 0 = f'(y_0)$.

Now we can extend $f'$ to a continuous function $f$ on all of $X$. But then $Tf(y_n) = \alpha_{\mu(y'_n)}(f(\mu(y'_n))) = \alpha_{\mu(y'_n)}(t_n) \to \infty$ contradicting the fact that $Tf \in C(Y)$ and the lemma is proved.
We have via Lemma 3.8, that except for at most a finite number of points \( y \),

\[
\alpha_{\mu(y)} f(\mu(y)) = [\text{sgn } f(\mu(y))] |f(\mu(y))| p(\mu(y))
\]

where \( p(\mu(y)) \) is a positive function. We note that \( p \) is continuous where it is defined, i.e. on the set \( \{ \mu(y) \mid \alpha_{\mu(y)} \) is continuous\}, since for the constant function \( 2 \) we have \( T_2(y) = \alpha_{\mu(y)}(2) = [\text{sgn } 2] 2 |p(\mu(y))| = 2 p(\mu(y)) \) and since \( T_2 \) is continuous the result follows.

Using the fact that \( Y \) has no isolated points we show a stronger result.

**Lemma 3.10.** There is a positive continuous function \( p \) on \( X_o \) such that

\[
\alpha_{\mu(y)} f(\mu(y)) = [\text{sgn } f(\mu(y))] |f(\mu(y))| p(\mu(y)).
\]

**Proof.** In view of the previous remarks we need only show that \( \alpha_{\mu(y)} \) is continuous for all \( y \). To this end suppose that \( \alpha_{\mu(y)} \) is discontinuous at \( y_0 \). Set \( A = \{ \mu(y) \mid \alpha_{\mu(y)} \) is continuous\}. By Lemma 3.9 all but a finite number of the \( \mu(y) \) are in \( A \) and hence since \( Y \) has no isolated points every open neighborhood about \( \mu(y_0) \) contains points of \( A \).

Now for \( 0 < s < 1 \) define \( S \in C(X) \) by \( S(x) = s \). Since \( \alpha_{\mu(y)} \) is unbounded in every neighborhood of zero we can find an \( s_0 \in (0, 1) \) such that \( \alpha_{\mu(y_0)}(s_0) > 2 \). Let \( U \) be any neighborhood containing \( \mu(y_0) \) and take \( \mu(y) \in U \cap A \). Then \( TS_0(y) = \alpha_{\mu(y)}(s_0) = [\text{sgn } s_0] s_0 |p(\mu(y))| < 1 \) but \( TS_0(y_0) = \alpha_{\mu(y_0)}(s_0) > 2 \) which contradicts the continuity of \( TS_0 \).

**Lemma 3.11.** The semi-group homomorphism \( T \) is an algebra homomorphism followed by a semi-group automorphism. Moreover \( T \) is continuous.

**Proof.** From Lemma 3.10 we have

\[
Tf(y) = [\text{sgn } f(\mu(y))] |f(\mu(y))| p(\mu(y)).
\]

Identify \( Y \) as the subset \( X_o \) of \( X \) and define \( T_1 : C(X) \to C(Y) \) by \( T_1 f = f \upharpoonright Y \) (i.e. \( f \) restricted to \( Y \)) and note that \( T_1 \) is an onto algebra homomorphism. Define \( T_2 : C(Y) \to C(Y) \) by \( T_2 g(y) = [\text{sgn } g(y)] |g(y)| p(\mu(y)) \) where \( p(\mu(y)) \) is the continuous positive function arising in the previous lemma. \( T_2 \) is a semi-group automorphism. To see that \( T_2 \) is one-to-one suppose \( f_1, f_2 \in C(Y) \) where \( f_1 \neq f_2 \). Then there is a \( y \in Y \) such that \( f_1(y) \neq f_2(y) \). Now if \( |f_1(y)| \neq |f_2(y)| \) then \( T_2 f_1(y) \neq T_2 f_2(y) \) and if \( |f_1(y)| = |f_2(y)| \) then \( \text{sgn } f_1(y) \neq \text{sgn } f_2(y) \) so that \( T_2 f_1(y) \neq T_2 f_2(y) \). Thus \( T_2 \) is one-to-one. Clearly \( T = T_2 T_1 \).
To see that $T$ is continuous it suffices to show that $T^2$ is continuous ($T^2$ is clearly continuous). A standard argument shows this to be the case.

Combining some of the previous results we have the following.

**Theorem 3.12.** Let $X$ and $Y$ be compact Hausdorff spaces, $Y$ having no isolated points. Let $C(X)$ and $C(Y)$ be the multiplicative semi-groups of all continuous real valued function on $X$ and $Y$ respectively. If $T$ is a point-determining semi-group homomorphism of $C(X)$ onto a dense point-separating set in $C(Y)$ then $Y$ can be imbedded homeomorphically in $X$ in such a way that

$$Tf(y) = [\text{sgn } f(x)] |f(x)|^{p(x)}$$

for some continuous positive function $p$ where $x$ is the unique point related to $y$ by the induced homeomorphism. Such a homomorphism is continuous and is an algebra homomorphism followed by a semi-group automorphism.

**Corollary 3.13.** Let $X$ and $Y$ be compact Hausdorff spaces, $Y$ having no isolated points. Let $T$ be a semi-group homomorphism of $C(X)$ onto a dense point-separating set of $C(Y)$. Then

(i) $T$ is an algebra homomorphism of $C(X)$ into $C(Y)$ if and only if $T$ is point-determining and $Tc = c$ for each constant function $c$.

(ii) If $T$ is point-determining then $T(-f) = -Tf$.

**Proof.** (i) If $T$ is an algebra homomorphism of $C(X)$ we have already seen that $T$ is point-determining and in fact that $Tf(y) = f(\mu(y))$ where $\mu$ is the induced homeomorphism. Hence $Tc = c$ for all constant functions $c$.

If $T$ is point-determining and $Tc = c$ for all constant functions $c$ then by the above theorem, for all $y$,

$$2 = T_2(y) = [\text{sgn } 2]|2^{p(\mu(y))}| = 2^{p(\mu(y))}$$

and hence $p(\mu(y)) = 1$ for all $y$. Thus for $f \in C(X)$, $Tf(y) = f(\mu(y))$ so $T$ is an algebra homomorphism.

The proof of (ii) is obvious by the form of the homomorphism shown in the above theorem.

**Bibliography**


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