A CONVEXITY PROPERTY

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There exist a variety of conditions yielding convexity of a set, dependent upon the nature of the underlying space. It is the purpose here to define a particular restriction involving \(n\)-tuples (the \(n\)-isosceles property) on subsets of a straight line space and study the effect of this restriction in establishing convexity. By a straight line space is meant a finitely compact, convex, externally convex metric space in which the linearity of two triples of a quadruple implies the linearity of the remaining two. The principal theorem states that the \(n\)-isosceles property is a sufficient condition for a closed and arcwise connected subset of a straight line space to be convex if and only if \(n\) is two or three.

In such a space \(S\) we use two of the definitions stated by Marr and Stamey (4).

**DEFINITION 1.** If \(p, q, r\) are distinct points of \(S\) such that at least two of the distances \(pq, pr, qr\) are equal, then the points \(p, q, r\) are said to form an isosceles triple in \(S\).

**DEFINITION 2.** A subset \(M\) of \(S\) is said to have the double-isosceles three-point property if two connecting segments of each of its isosceles triples belong to \(M\).

A proof of (2) together with (4) shows that if \(M\) is a closed connected subset of \(S\) and possesses the double isosceles property, then \(M\) is convex.

**DEFINITION 3.** A subset \(M\) of \(S\) is said to have the \(n\)-isosceles property \((n \geq 2)\) provided for every \((n + 1)\)-tuple \(p_1, p_2, \ldots, p_{n+1}\) of distinct points of \(M\) such that \(p_ip_{i+1} = p_{i+1}p_{i+2}, i = 1, 2, \ldots, n - 1, \) at least \(n\) of the connecting segments lie in \(M\).

A comparison of the double isosceles property and \(n\)-isosceles property shows that in \(S\) the two are equivalent for \(n = 2\). For \(n\) greater than 2, the double isosceles property clearly implies the \(n\)-isosceles property but it is not immediately evident whether the two are equivalent. The question may be raised concerning the conditions under which the \(n\)-isosceles property is sufficient to replace the double-isosceles property in the above-mentioned theorem yielding convexity.
This question is answered in part by the following theorem.

**Theorem 1.** Let $M$ be a closed subset of $S$ such that every pair of points of $M$ can be joined by a rectifiable arc in $M$. If $M$ has the three-isosceles property, then $M$ is convex.

**Proof.** Let $p, q$ be any two points of $M$ and let $A$ denote a rectifiable arc in $M$ with endpoints $p, q$. Then there exists a shortest arc in $M$ joining $p, q$, say $A'$. Let $r, s$ be points of $A'$ such that $pr = rs = sq$. Then since $M$ possesses the three-isosceles property and $A'$ is a geodesic arc in $M$, consideration of the cases reveals $S(p, q) \subset M$ or $A'$ is the union of a finite number of noncollinear metric segments, or all three connecting segments of triples $p, r, s$ or $q, r, s$ are contained in $M$.

We shall suppose $S(p, q) \not\subset M$. If $A'$ is the union of a finite number of noncollinear metric segments, then by the metric transitivities of the space, $r$ or $s$ is noncollinear with $p, q$. Hence for at least one of the points $r, s$ say $r$, that point is the terminal and initial point, respectively, [when traversing $A'$ from $p$ to $q$] of metric segments $S_1 \subset M \cap A'$, $S_2 \subset M \cap A'$, which in turn contain point pairs $u_1, u_2$ and $v_1, v_2$, respectively, such that $u_2u_1 = u_2v_1 = v_1v_2$ while $u_2v_1$ is strictly less than $u_2r + rv_1$. Applying the three-isosceles property to the points $u_1, u_2, v_1, v_2$, it follows that $S(u_i, v_j) \subset M$ for some $i, j = 1, 2$ which violates the shortest arc hypothesis for $A'$.

Now suppose all three connecting segments of a triple (say $p, r, s$) are contained in $M$. If $ps$ is less than $pr + rs$ and $p, r, s$ are met in this or reverse order, a contradiction is encountered. A similar argument holds if the order is $p, s, r$. We may then assume the labeling such that $p, r, s$ are encountered in this order and $ps = pr + rs$. Consider the longest segment containing $S(p, s)$ with one endpoint $p$ and contained within $M$ and denote its remaining terminal point by $s'$. Considering the subarc $A'(s', q)$, it follows as above that it consists of a finite number of metric segments (and hence $A'$, which was discussed previously) or else there exists a metric segment contained in $A' \cap M$ with either $s'$ or $q$ as endpoint.

Repeating this latest procedure at most once, it follows that either $A'$ consists of a finite number of metric segments or there exist two noncollinear metric segments contained in $A' \cap M$ with a common endpoint. Applying the three-isosceles condition to the appropriate four points of these two segments results again in a contradiction.

We conclude $M$ is convex.

The following sequence of lemmas will lead to a strengthening of
the above theorem. In each of these lemmas, \( S \) is assumed to be a straight line space and \( M \) to be a closed, arcwise connected subset of \( S \) possessing the three-isosceles property.

**Lemma 1.** Let \( A \) denote an arc in \( M \) with endpoints \( p, q \). If \( p, q \) are not joined by a rectifiable arc, then one of the two points (to be termed 'exceptional') has the property that every arc joining it to other points is nonrectifiable.

**Proof.** Let \( a, b \) be points of \( A \) such that \( pa = ab = bq \). Then since \( M \) possesses the three-isosceles property, the existence of three of the six segments within \( M \) implies that either there exists an arc with endpoints \( p, q \) consisting of one, two, or three segments each contained within \( M \) (and hence there exists a rectifiable arc with endpoints \( p, q \)) or all three connecting segments of some triple (say \( a, b, q \)) of the quadruple are contained within \( M \).

In the latter case, given a positive \( \varepsilon \) less than \( ab/4 \), by the method of proof of Lemma 23.1 (1), there exists a finite sequence \( p_1, p_2, \ldots, p_n \) of distinct points of the arc such that \( p_ip_{i+1} = \varepsilon, p_ip_j \geq \varepsilon \) for \( i \neq j, p = p_1, \) and \( 0 < p_na \leq \varepsilon \) for \( i, j = 1, 2, 3, \ldots, n, \) where \( p_{n+1} \) is defined as follows. If none of the \( p_i, i = 1, 2, \ldots, n \) are elements of \( S(a, b) \) let \( p_{n+1}, p_{n+2} \) be two points of \( S(a, b) \) such that \( p_np_{n+1} = p_{n+1}p_{n+2} = \varepsilon \).

Applying the three-isosceles property to \( p_{n-1}, p_n, p_{n+1}, p_{n+2} \), it follows that at least one other connecting segment of the quadruple must form with \( S(p_{n+1}, p_{n+2}) \) a connected set and be contained within \( M \). Hence there exists a rectifiable arc from \( b \) to \( p_{n-1} \) or \( p_n \) contained within \( M \) and consisting of a finite union of metric segments. Suppose \( p_n[ \) or \( p_{n-1} \) is the endpoint of this arc. Then there exists a point, which may be denoted by \( r_n[ s_{n-1} \) such that \( p_nr_n = \varepsilon \) and \( S(p_{n}, r_n) \subset M \). \( [p_{n-1}s_{n-1} = \varepsilon \) and \( S(p_{n-1}, s_{n-1}) \subset M \). Then applying the three-isosceles condition to the appropriate quadruple, it follows that there exists a rectifiable arc from \( b \) to \( p_{n-1} \) or \( p_{n-2} \). Repeating this process a finite number of times shows the existence of a rectifiable polygonal arc contained in \( M \) with one endpoint \( b \) and the other endpoint \( p_i \) or \( p_0 \) where \( p_0 \) is any point of \( M \) with \( p_0p_1 = \varepsilon \). Hence the lemma is valid, for in the contrary case, if \( p_1 \) and some point \( u \) are the endpoints of a rectifiable arc \( A(p_1, u) \), then, given that all segments of \( a, b, q \) are contained in \( M \), the above method of proof can be followed for a positive \( \delta \) less than \( \min \{ab/4, up\} \) and hence there exists a rectifiable arc \( A(p_1, t) \) where \( t \) is in \( A(p_1, u) \) such that \( p_1t = \delta \). Then by the preceding it is not possible for \( t \) to be \( p_i \) for any \( i = 1, 2, \ldots, n + 2 \) for then there exists a rectifiable arc with endpoints \( b, p_i \), whereas if \( t \) is distinct from these points we may set \( t = p_0 \) and observe that there exists a
rectifiable arc with endpoints $p, b$ which implies the existence of a rectifiable arc contained in $M$ and joining $p, q$, contrary to hypothesis.

If $S(a, b) \cap \{p_i\}$ is not null, let $p_j$ denote the point with minimum index and delete the members of the sequence with higher index. Then relabel as $p_{j+1}$ a point of $S(a, b)$ such that $p_j p_{j+1} + \varepsilon$, and in the above proof replace $n$ by $j - 1$. 

**Lemma 2.** There exists at most one 'exceptional' point.

**Proof.** Suppose the contrary, and let $x, y$ denote two such points. The method of proof of the preceding lemma involving $p, q$ and now applied to $x, y$ shows that there exists a rectifiable arc $A(x, y_0) \subset M$ where $x \neq y_0$ or a rectifiable arc $A(y, x_0) \subset M$, where $y \neq x_0$, which violates our supposition.

**Lemma 3.** The set of points of $M$ that is not 'exceptional' is convex.

**Proof.** Denote by $x$ the 'exceptional' point of $M$ if such exists. Given any two points $p, q$ of $M - \{x\}$, $p \neq q$, it follows from Lemma 2 that neither $p$ nor $q$ is 'exceptional' and hence by Lemma 1 they are the endpoints of a rectifiable arc in $M$. As in Theorem 1, considering $M$ as a finitely compact metric space it follows that there exists in $M$ a geodesic arc $A$ joining $p, q$. Since $x$ is an 'exceptional' point, $x$ is not in $A$. Again as in Theorem 1, there exist two points of $A$ which, with $p, q$, form a quadruple to which the three-isoceles condition can be applied. Again $x$ is not a point of any of the connecting segments in $M$ whose existence is determined since it is 'exceptional'. Hence the proof proceeds as in Theorem 1, yielding a contradiction unless the segment joining $p, q$ is contained in $M - \{x\}$.

**Lemma 4.** The set $M$ is convex.

**Proof.** In view of Lemma 3, it suffices to show that if $x$ denotes the 'exceptional' point and $p$ is a point of $M - \{x\}$, there exists a point of $M$ between $p$ and $x$.

Since $M$ is connected, let $\{x_n\}$ denote a sequence of points of $M - \{x\}$ such that $\lim x_n = x$. Denote by $m_n$ the midpoint of $y, x_n$ for $n = 1, 2, \cdots$. Since $M$ is finitely compact, there exists a point $m$ of $M$ such that $m$ is the limit of a subsequence $\{m_{i_n}\}$ of $\{m_n\}$. Hence $\lim x_{i_n} = x$ and $p m_{i_n} + m_{i_n} x_{i_n} = p x_{i_n}$ for all $n$ implies $p m + m x = p x$.

From these lemmas, it follows that the theorem below is valid.
THEOREM 2. Let $M$ be a closed arcwise connected subset of a straight line space $S$. If $M$ has the three-isosceles property, then $M$ is convex.

The above theorem is not valid when the condition that $M$ possess the three-isosceles property is replaced by the demand that $M$ possess the $n$-isosceles property with $n \geq 4$. This may be observed by considering any nonlinear isosceles triple $q, r, s$ of the euclidean plane. Let $M_0$ be the union of the equal segments $S(q, r), S(r, s)$. Since $M_0$ clearly is not convex, it suffices to show that $M_0$ possesses the $n$-isosceles property for all $n$ greater than three.

Let $p_1, p_2, \ldots, p_{n+1}$ be any $n + 1$ distinct points of $M_0$ such that $p_ip_{i+1} = p_{i+1}p_{i+2}, i = 1, 2, \ldots, n - 1$. If $n$ is even the minimum number of segments lying entirely within $M_0$ will occur when $n/2$ points lie on one of the two segments comprising $M_0$ and $(n + 2)/2$ points on the other segment. Hence there always exist at least $n(n - 2)/8 + n(n + 2)/8$ connecting segments contained within $M_0$ which is greater than or equal to $n$ for $n \geq 4$. If $n$ is odd, the minimum number of segments lying entirely within $M_0$ will occur when $(n + 1)/2$ points lie on each segment. Hence since $(n^2 - 1)/8 + (n^2 - 1)/8 \geq n$ for $n \geq 5$, it follows that $M_0$ has the $n$-isosceles property for $n \geq 4$.

REFERENCES


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