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OSCILLATION CRITERIA FOR THIRD ORDER NONLINEAR DIFFERENTIAL EQUATIONS

PAUL ELVIS WALTMAN

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Two oscillation theorems are provided for certain nonlinear nonautonomous third order differential equations. Both results involve integral conditions and are of the form that any solution which has one zero is oscillatory. Theorem. Let $p(t)$, $q(t)$ be continuous and let $p'(t)$ be nonpositive. Define $Q(t)=\int_0^t q(s)ds$. If $A+Bt-\int_{t_0}^t Q(s)ds<0$ for t sufficiently large, any A and B, if γ is positive and the quotient of two odd integers, then any continuable solution of $y'''+p(t)y'+q(t)y^\gamma=0$ which has a zero is oscillatory.

THEOREM. Let $p(t)$ and $q(t)$ be continuous and nonnegative and let $f(y)/y \geqq \alpha > 0$ for some α. If $\alpha q(t) - p'(t)$ is positive and if $\int^\infty t(\alpha q(t) - p'(t))dt = \infty$ then any continuable solution of $y''' + p(t)y' + q(t)f(y) = 0$ which has a zero is oscillatory.

A nontrivial solution of a differential equation is said to be oscillatory if it has zeros for arbitrarily large values of the independent variable. Finding oscillation criteria is a problem of general interest in the theory of ordinary differential equations. The oscillation of solutions of the linear nonautonomous second order equation

$$y'' + a(t)y = 0 \tag{1}$$

and third order equation

$$y''' + p(t)y' + q(t)\,y = 0 \tag{2}$$

has been widely studied and has a vast literature. Oscillation theorems for nonlinear generalizations of (1), in particular in the form

$$y'' + a(t)y^{2n+1} = 0\ ,$$

are known [1], [3], [4], [5], [6], but oscillation criteria for nonlinear counterparts of (2) seem to have been ignored.

Here, the linear term y, in equation (2), will be replaced by a nonlinear function and two oscillation criteria established. Both of these yield the conclusion that any solution which has a zero is oscillatory. Before stating the theorems, the author wishes to cite a very interesting paper of Hannan [2] on the third order linear equation which was the starting point for this investigation.

THEOREM 1. *Let $p(t)$, $q(t)$ be continuous and nonnegative and*

let $p'(t)$ be nonpositive. Define $Q(t) = \int_0^t q(s)ds$. If $A+Bt-\int_0^t Q(s)ds < 0$ for t sufficiently large, any A and B, if γ is positive and is the quotient of two odd integers, then any continuable solution of

$$y''' + p(t)y' + q(t)y^\gamma = 0 , \tag{3}$$

which has a zero is oscillatory.

REMARK. $q(t) > 1/t$ is sufficient to satisfy the above integral condition.

Proof. Suppose $y(t)$ is a solution of (3) which has a zero but does not oscillate. Then it has a last zero which we will denote by t_0. Since $-y(t)$ is also a solution of (3), we may assume that for $t > t_0$, $y(t) > 0$, and thus that $y'(t) > 0$ in some interval (t_0, α). Let $t_0 < t_1 < \alpha$. For $t \geqq t_1$ equation (3) may be rewritten as

$$\frac{y'''(t)}{y^\gamma(t)} + \frac{p(t)y'(t)}{y^\gamma(t)} = -q(t) .$$

Integration from t_1 to t yields

$$\frac{y''(t)}{y^\gamma(t)} - \frac{y''(t_1)}{y^\gamma(t_1)} + (\gamma)\int_{t_1}^t \frac{y''(s)}{y^{\gamma+1}(s)}y'(s)ds + \int_{t_1}^t \frac{p(s)y'(s)}{y^\gamma(s)}ds = Q(t_1) - Q(t) .$$

Expanding the first integral gives

$$\frac{y''(t)}{y^\gamma(t)} + \frac{(\gamma)[y'(t)]^2}{2y^{\gamma+1}(t)} + \frac{\gamma(\gamma+1)}{2}\int_{t_1}^t \frac{[y'(s)]^3}{y^{\gamma+2}(s)}ds + \int_{t_1}^t \frac{p(s)y'(s)}{y^\gamma(s)}ds = K - Q(t)$$

where all of the constants have been combined. Integrating again produces

$$\frac{y'(t)}{y^\gamma(t)} + \frac{3\gamma}{2}\int_{t_1}^t \frac{[y'(s)]^2}{y^{\gamma+1}(s)}ds + \frac{\gamma(\gamma+1)}{2}\int_{t_1}^t \frac{(t-s)[y'(s)]^3}{y^{\gamma+2}(s)}ds + \int_{t_1}^t (t-s)\frac{p(s)y'(s)}{y^\gamma(s)}ds = M + Kt - \int_0^t Q(s)ds$$

where the constants of integration have been combined with $\int_0^{t_1} Q(s)ds$

Suppose that $y'(t) > 0$ for all $t > t_1$. Then

$$\frac{y'(t)}{y^\gamma(t)} < M + Kt - \int_0^t Q(s)ds .$$

Since, by hypothesis, $M + Kt - \int_0^t Q(s)ds < 0$ for t sufficiently large,

and since $y(t) > 0$ for all $t > t_0$, it follows that $y'(t) < 0$ for t sufficiently large, contradicting the assumption that $y'(t) > 0$ for $t > t_1$. Thus there exists a point $t_2 > t_0$ such that $y'(t_2)=0$.

The theorem will be established by showing that $y(t_0) = y'(t_2) = 0$ contradicts $y(t) > 0$, $t > t_0$. Multiplying equation (3) by $y(t)$ and integrating from t_0 to t yields

$$y''(t)y(t) + p(t)\frac{[y(t)]^2}{2} + \frac{[y'(t_0)]^2}{2} - \frac{[y'(t)]^2}{2} + \int_{t_0}^{t} q(s)y^{\gamma+1}(s)ds - \int_{t_0}^{t} \frac{y^2(s)}{2}p'(s)ds=0 .$$

At any zero of $y'(t)$ it follows that $y''(t)y(t) < 0$. Hence, since $y(t)>0$, t_2 is the only zero of $y'(t)$ $t\geqq t_2$, so that $y'(t) < 0$ for $t > t_2$, and hence $\lim_{t\to\infty}y(t)$ exists. If $y''(t)$ eventually remains negative, if follows immediately that $y(t)$ has a zero. If $y''(t)$ eventually remains positive, then $\lim_{t\to\infty}y'(t)$ exists. Writing the last equation as

$$(4)\qquad \frac{[y'(t)]^2}{2} = \frac{[y'(t_0)]^2}{2} + y''(t)y(t) + \frac{[y(t)]^2}{2}p(t) + \int_{t_0}^{t} y^2(s)\left[q(s)y^{\gamma-1}(s) - \frac{p'(s)}{2}\right]ds$$

and noting that all right hand terms are nonnegative and that the last term is increasing, it follows that $\lim_{t\to\infty} y'(t) = a < 0$, and from this, that $y(t)$ has a zero in (t_2, ∞). If $y''(t)$ changes sign for arbitrarily large t, $y'(t)$ has maxima for arbitrarily large t. Since $\lim_{t\to\infty}y(t)$ exists and $y'(t) < 0$, $\limsup_{t\to\infty}y'(t_n) = 0$. The set of maxima must then contain a subsequence s_n such that $\lim y'(s_n) = 0$. Putting s_n into equation (4), it follows that $\lim_{n\to\infty}[y'(s_n)]^2 > 0$, which is the desired contradiction. This exhausts the possibilities for $y''(t)$ and completes the proof of the theorem.

The concluding argument above is essentially Lemma 5.2 of [2]. It is included here because of some ambiguity in the proof in the reference for the case $y''(t)$ oscillates. If equation (3) is linear ($\gamma = 1$), then (4) shows that the condition, $p'(t)$ is nonpositive, can be replaced by $2q(t) - p'(t) > 0$.

THEOREM 2. *Let $p(t)$ and $q(t)$ be continuous and nonnegative and let $f(y)/y > \alpha > 0$ for some α. If $\alpha q(t) - p'(t)$ is positive and if*

$$\int_{t_0}^{\infty} t(\alpha q(t) - p'(t))dt = \infty$$

then any continuable solution of

$$(5)\qquad y''' + p(t)y' + q(t)f(y) = 0$$

which has a zero is oscillatory.

Proof. Suppose that there exists a nonoscillatory solution $y(t)$ of (5) which has a zero. Then the solution has a last zero which we label t_0. Suppose that $y(t) > 0$, $t > t_0$ (a similar proof follows if $y(t) < 0$, $t > t_0$). If there exists a $t_1 > t_0$ such that $y'(t_1) = 0$, then it can be shown that there exists a $t_2 > t_1$ such that $y(t_2) = 0$, contradicting the fact that t_0 was the last zero of $y(t)$. The proof of this is similar to the proof of the same fact in the concluding part of Theorem 1 and will be omitted. Since $y'(t) > 0$ for t immediately to the right of t_0, it follows that $y'(t) > 0$ for all $t > t_0$. Then since $p(t)$ and $q(t)$ cannot both be identically zero on (t_0, ∞), and since

$$y'''(t) = -p(t)y'(t) - q(t)f(y(t)),$$

$y'''(t) \leqq 0$ and $y''' \not\equiv 0$. Thus $y''(t)$ is a decreasing function for $t > t_0$.

Suppose there is a $t_1 > t_0$ such that $y''(t) \leqq 0$. Then there is a $t_2 > t_1$ such that $y''(t) < 0$, $t > t_2$. Hence $y'(t)$ is positive and monotone decreasing and must therefore tend to a nonnegative limit. But

$$y'(t) - y'(t_2) = \int_{t_2}^{t} y''(s)ds < y''(t_2)(t - t_2) .$$

Since $y''(t_2) < 0$, it follows that $y'(t) \to -\infty$ as $t \to \infty$, which is a contradiction since $y(t) > 0$.

Thus it may be assumed that $y''(t)$ is nonnegative for $t > t_0$. For any $t_1 > t_0$, $y(t) > y'(t_1)(t - t_1)$. Integrating the equation and eliminating nonnegative terms

$$\begin{aligned} y''(t_0) + p(t_0)y(t_0) &\geqq \int_{t_1}^{t} (q(s)f(y) - p'(s)y(s))ds \\ &\geqq \int_{t_1}^{t} y(s)[\alpha q(s) - p'(s)]ds \\ &\geqq y'(t_1) \int_{t_1}^{t} (s - t_1)[\alpha q(s) - p'(s)]ds . \end{aligned}$$

The left side is independent of t and thus the integral on the right must converge as $t \to \infty$. The convergence of this integral is equivalent to the convergence of $\int_{t_1}^{\infty} t[\alpha q(t) - p'(t)]dt$ which proves the theorem.

The concluding argument above follows the concluding argument of Theorem 5.12 of [2]. Theorem 2 generalizes this theorem of [2] not only by introducing a nonlinear term but by eliminating a hypothesis concerning a related second order equation.

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