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# ON SETS REPRESENTED BY THE SAME FORMULA IN DISTINCT CONSISTENT AXIOMATIZABLE ROSSER THEORIES

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**In this note a theorem is proved which includes the following: if $T$ is a consistent, axiomatizable Rosser theory in which all recursive functions of one argument are definable and $S$ is any sentence undecidable in $T$, then given any pair $(d_1, d_2)$ of *re* (recursively enumerable) degrees, there is a formula $F$ which represents a set of degree $d_1$ in $T$ and of degree $d_2$ in $T' = T(S)$, the theory obtained from $T$ by adjoining $S$ as a new axiom.**

For the theory of recursive functions, we follow [1]. If $T$ is a theory and $S$ a sentence undecidable in $T$, we write $T(S)$ for the theory obtained by adding $S$ to $T$ as a new axiom.

THEOREM. *If $T$ is a consistent, axiomatizable theory in which all recursive functions of one argument are definable, and in which some EI (effectively inseparable) pair of re sets is separable, and $S$ is any sentence undecidable in $T$, then if $(A, B)$ is any pair of re sets with $A \subset R \subset B$, where $R$ is recursive, there is a formula which represents $A$ in $T$ and $B$ in $T(S)$.*

*Proof.* The quite simple proof proceeds by way of two lemmas.

LEMMA 1. *If $T$ and $S$ are as in the theorem, $A$ is an re set and $R$ is a recursive subset of $A$, there is a formula which represents $R$ in $T$ and $A$ in $T(S)$.*

*Proof.* We take formulas $F(x)$ and $G(x)$ such that $F(x)$ represents $A$ in $T(S)$ and $G$ *defines* $R$ in $T$ and hence in $T(S)$. The formula $H(x) = (F(x) \wedge S) \vee G(x)$ represents $R$ in $T$ and $A$ in $T(S)$.

LEMMA 2. *If $T$ and $S$ are as above and $A$ is any re set, there is a formula which represents $A$ in $T$ and the set $I$ of nonnegative integers in $T(S)$.*

*Proof.* Consider an *re EI* pair $(U_1, U_2)$ and a formula $F(x)$ which separates $(U_1, U_2)$ in $T$. The formula $F(x) \vee S$ represents $I$ in $T(S)$; it represents in $T$ a superset of $U_1$ disjoint from $U_2$, and consequently represents a creative set $C$ in $T$. Using a well-known theorem of Myhill, we take a recursive function $f$ such that $A = f^{-1}(C)$. Using an argument similar to that of Lemma 1 of [2], we can find a formula $G(X)$ which represents $A = f^{-1}(C)$ in $T$ and $I = f^{-1}(I)$ in $T(S)$. The lemma is proved.

To complete the proof of the theorem, we take $F(x)$ representing $A$ in $T$ and the set $I$ in $T(S)$, by Lemma 2, and $G(x)$ representing $R$ in $T$ and $B$ in $T(S)$. The formula $H(x) = F(x) \wedge G(x)$ represents $A$ in $T$ and $B$ in $T(S)$.

If $(d_1, d_2)$ is any pair of *re* degrees, we can find *re* sets $A$ and $B$, with $A \subset R \subset B$, where $R$ is recursive, such that $A$ is of degree $d_1$ and $B$ of degree $d_2$. We consequently have:

COROLLARY. *If $T$ and $S$ are as in the theorem and $(d_1, d_2)$ is any pair of re degrees, there is a formula $F$ which represents a set of degree $d_1$ in $T$ and of degree $d_2$ in $T(S)$.*

Thus, with regard to the consequences of adding sentences $S$ undecidable in a theory $T$ as new axioms, we see that one undecidable sentence is as good as another insofar as representation of sets of distinct degree of unsolvability by the same formula is concerned.

## REFERENCES

1. M. Davis, *Computability and Unsolvability*, McGraw Hill, 1958.
2. H. Putnam and R. Smullyan, *Exact separation of recursively enumerable sets within theories*, Proc. Amer. Math. Soc. **11** (1960), 574-577.

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