ON SETS REPRESENTED BY THE SAME FORMULA IN
DISTINCT CONSISTENT AXIOMATIZABLE ROSSER
THEORIES

ROBERT ARNOLD DI PAOLA
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ROBERT A. DiPAOLA

In this note a theorem is proved which includes the following: if \( T \) is a consistent, axiomatizable Rosser theory in which all recursive functions of one argument are definable and \( S \) is any sentence undecidable in \( T \), then given any pair \( (d_1, d_2) \) of \( \text{re} \) (recursively enumerable) degrees, there is a formula \( F \) which represents a set of degree \( d_1 \) in \( T \) and of degree \( d_2 \) in \( T' = T(S) \), the theory obtained from \( T \) by adjoining \( S \) as a new axiom.

For the theory of recursive functions, we follow [1]. If \( T \) is a theory and \( S \) a sentence undecidable in \( T \), we write \( T(S) \) for the theory obtained by adding \( S \) to \( T \) as a new axiom.

**THEOREM.** If \( T \) is a consistent, axiomatizable theory in which all recursive functions of one argument are definable, and in which some \( \text{EI} \) (effectively inseparable) pair of \( \text{re} \) sets is separable, and \( S \) is any sentence undecidable in \( T \), then if \( (A, B) \) is any pair of \( \text{re} \) sets with \( A \subset R \subset B \), where \( R \) is recursive, there is a formula which represents \( A \) in \( T \) and \( B \) in \( T(S) \).

**Proof.** The quite simple proof proceeds by way of two lemmas.

**LEMMA 1.** If \( T \) and \( S \) are as in the theorem, \( A \) is an \( \text{re} \) set and \( R \) is a recursive subset of \( A \), there is a formula which represents \( R \) in \( T \) and \( A \) in \( T(S) \).

**Proof.** We take formulas \( F(x) \) and \( G(x) \) such that \( F(x) \) represents \( A \) in \( T(S) \) and \( G \) defines \( R \) in \( T \) and hence in \( T(S) \). The formula \( H(x) = (F(x) \land S) \lor G(x) \) represents \( R \) in \( T \) and \( A \) in \( T(S) \).

**LEMMA 2.** If \( T \) and \( S \) are as above and \( A \) is any \( \text{re} \) set, there is a formula which represents \( A \) in \( T \) and the set \( I \) of nonnegative integers in \( T(S) \).

**Proof.** Consider an \( \text{re} \) \( \text{EI} \) pair \( (U_1, U_2) \) and a formula \( F(x) \) which separates \( (U_1, U_2) \) in \( T \). The formula \( F(x) \lor S \) represents \( I \) in \( T(S) \); it represents in \( T \) a superset of \( U_1 \) disjoint from \( U_2 \), and consequently represents a creative set \( C \) in \( T \). Using a well-known theorem of Myhill,
we take a recursive function $f$ such that $A = f^{-1}(C)$. Using an argument similar to that of Lemma 1 of [2], we can find a formula $G(X)$ which represents $A = f^{-1}(C)$ in $T$ and $I = f^{-1}(I)$ in $T(S)$. The lemma is proved.

To complete the proof of the theorem, we take $F(x)$ representing $A$ in $T$ and the set $I$ in $T(S)$, by Lemma 2, and $G(x)$ representing $R$ in $T$ and $B$ in $T(S)$. The formula $H(x) = F(x) \land G(x)$ represents $A$ in $T$ and $B$ in $T(S)$.

If $(d_1, d_2)$ is any pair of re degrees, we can find re sets $A$ and $B$, with $A \subseteq R \subseteq B$, where $R$ is recursive, such that $A$ is of degree $d_1$ and $B$ of degree $d_2$. We consequently have:

**Corollary.** If $T$ and $S$ are as in the theorem and $(d_1, d_2)$ is any pair of re degrees, there is a formula $F$ which represents a set of degree $d_1$ in $T$ and of degree $d_2$ in $T(S)$.

Thus, with regard to the consequences of adding sentences $S$ undecidable in a theory $T$ as new axioms, we see that one undecidable sentence is as good as another insofar as representation of sets of distinct degree of unsolvability by the same formula is concerned.

**References**


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