

Pacific Journal of Mathematics

**ON SETS REPRESENTED BY THE SAME FORMULA IN
DISTINCT CONSISTENT AXIOMATIZABLE ROSSER
THEORIES**

ROBERT ARNOLD DI PAOLA

ON SETS REPRESENTED BY THE SAME FORMULA IN DISTINCT CONSISTENT AXIOMATIZABLE ROSSER THEORIES

ROBERT A. DI PAOLA

In this note a theorem is proved which includes the following: if T is a consistent, axiomatizable Rosser theory in which all recursive functions of one argument are definable and S is any sentence undecidable in T , then given any pair (d_1, d_2) of *re* (recursively enumerable) degrees, there is a formula F which represents a set of degree d_1 in T and of degree d_2 in $T' = T(S)$, the theory obtained from T by adjoining S as a new axiom.

For the theory of recursive functions, we follow [1]. If T is a theory and S a sentence undecidable in T , we write $T(S)$ for the theory obtained by adding S to T as a new axiom.

THEOREM. *If T is a consistent, axiomatizable theory in which all recursive functions of one argument are definable, and in which some *EI* (effectively inseparable) pair of *re* sets is separable, and S is any sentence undecidable in T , then if (A, B) is any pair of *re* sets with $A \subset R \subset B$, where R is recursive, there is a formula which represents A in T and B in $T(S)$.*

Proof. The quite simple proof proceeds by way of two lemmas.

LEMMA 1. *If T and S are as in the theorem, A is an *re* set and R is a recursive subset of A , there is a formula which represents R in T and A in $T(S)$.*

Proof. We take formulas $F(x)$ and $G(x)$ such that $F(x)$ represents A in $T(S)$ and G defines R in T and hence in $T(S)$. The formula $H(x) = (F(x) \wedge S) \vee G(x)$ represents R in T and A in $T(S)$.

LEMMA 2. *If T and S are as above and A is any *re* set, there is a formula which represents A in T and the set I of nonnegative integers in $T(S)$.*

Proof. Consider an *re EI* pair (U_1, U_2) and a formula $F(x)$ which separates (U_1, U_2) in T . The formula $F(x) \vee S$ represents I in $T(S)$; it represents in T a superset of U_1 disjoint from U_2 , and consequently represents a creative set C in T . Using a well-known theorem of Myhill,

we take a recursive function f such that $A = f^{-1}(C)$. Using an argument similar to that of Lemma 1 of [2], we can find a formula $G(X)$ which represents $A = f^{-1}(C)$ in T and $I = f^{-1}(I)$ in $T(S)$. The lemma is proved.

To complete the proof of the theorem, we take $F(x)$ representing A in T and the set I in $T(S)$, by Lemma 2, and $G(x)$ representing R in T and B in $T(S)$. The formula $H(x) = F(x) \wedge G(x)$ represents A in T and B in $T(S)$.

If (d_1, d_2) is any pair of *re* degrees, we can find *re* sets A and B , with $A \subset R \subset B$, where R is recursive, such that A is of degree d_1 and B of degree d_2 . We consequently have:

COROLLARY. *If T and S are as in the theorem and (d_1, d_2) is any pair of *re* degrees, there is a formula F which represents a set of degree d_1 in T and of degree d_2 in $T(S)$.*

Thus, with regard to the consequences of adding sentences S undecidable in a theory T as new axioms, we see that one undecidable sentence is as good as another insofar as representation of sets of distinct degree of unsolvability by the same formula is concerned.

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1. M. Davis, *Computability and Unsolvability*, McGraw Hill, 1958.
2. H. Putnam and R. Smullyan, *Exact separation of recursively enumerable sets within theories*, Proc. Amer. Math. Soc. **11** (1960), 574-577.

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