

Pacific Journal of Mathematics

CONDITIONS IMPLYING NORMALITY IN HILBERT SPACE

MARY RODRIGUEZ EMBRY

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The problem with which this paper is concerned is that of finding new conditions which imply the normality of an operator on a complete inner product space S . Each such condition, presented in this paper, involves the commutativity of certain operators, associated with a given operator A . Theorem 1 states the equivalence of the following conditions: (i) A is normal, (ii) each of AA^* and A^*A commutes with $\operatorname{Re} A$, (iii) AA^* commutes with $\operatorname{Re} A$ and A^*A commutes with $\operatorname{Im} A$. Theorem 2 states that A is normal if AA^* and A^*A commute and $\operatorname{Re} A$ is nonnegative definite. Finally, Theorem 3 states that if AA^* commutes with each of A^*A and $\operatorname{Re} A$, then AA^* commutes with A . In this case, if A is reversible, then A is normal.

The notation and terminology used will be as follows. S is a complex, linear space and Q is an inner product for S , such that S is complete with respect to the norm N , induced by Q . T is the space of linear operators on S to S , continuous with respect to N . If A is in T , A^* is the adjoint of A with respect to Q , $\operatorname{Re} A = (A + A^*)/2$, and $\operatorname{Im} A = (A - A^*)/2i$. An element A of T is nonnegative definite if $Q(Ax, x) \geq 0$ for each x in S , Hermitian if $A = A^*$, normal if $AA^* = A^*A$, reversible if A is one-to-one, and invertible if A is one-to-one and onto.

The following special notation will be used throughout the paper. Let $B^2 = AA^*$ and $C^2 = A^*A$, where B and C are nonnegative definite. b and c will denote the spectral resolutions of B^2 and C^2 , respectively (1, pp. 114-116). These spectral resolutions will be taken to be continuous from the right at each point.

One can see from the following example that relatively strong hypotheses on operators associated with A are necessary in order that A be normal. Let A be the operator on l_2 , defined by $A = \{a_{i,j}\}_{i,j=1}^{\infty}$ where $a_{i,i+1} = 1$ and $a_{i,j} = 0$ for $j \neq i + 1$. Then $B = 1$ and $C = P$, where P is a certain projection not equal to 1 or 0. Since $B = 1$, then B commutes with C , $\operatorname{Re} A$, $\operatorname{Im} A$, and even with A itself. However, A is not normal.

2. Commutativity relations concerning B and C .

THEOREM 1. *The following are equivalent:*

- (i) A is normal,

- (ii) each of B and C commutes with $\operatorname{Re} A$,
- (iii) B commutes with $\operatorname{Re} A$ and C commutes with $\operatorname{Im} A$.

Proof. That (i) implies (ii) and (iii) is obvious. Let $H = \operatorname{Re} A$ and $K = \operatorname{Im} A$.

(ii) \Rightarrow (i). If $HB^2 = B^2H$ and $HC^2 = C^2H$, then one has

$$(1) \quad A(B^2 - C^2) = (B^2 - C^2)A^*$$

$$(2) \quad \text{and } A^*(B^2 - C^2) = (B^2 - C^2)A.$$

Multiplying (1) on the left by A^* and using (2), one finds that

$$(3) \quad C^2(B^2 - C^2) = (B^2 - C^2)B^2.$$

Multiplying (2) on the left by A and using (1), one has

$$(4) \quad B^2(B^2 - C^2) = (B^2 - C^2)C^2.$$

Subtracting (4) from (3), one sees that $(B^2 - C^2)^2 = -(B^2 - C^2)^2$. Therefore, $B^2 = C^2$, and A is normal.

(iii) \Rightarrow (i). If $KC^2 = C^2K$, then

$$(5) \quad (B^2 - C^2)A = -A^*(B^2 - C^2).$$

Multiplying (5) on the left by A and using (1), one has $-B^2(B^2 - C^2) = (B^2 - C^2)C^2$. Therefore, $B^4 = C^4$, and $B^2 = C^2$ (2, p. 262).

LEMMA 2.1. (i) $A \left[\int f(t)dc \right] = \left[\int f(t)db \right] A$, for each continuous complex-valued function on the real line.

(ii) $AC^n = B^nA$, for each positive integer n .

(iii) $Ac(t) = b(t)A$, for each value of t .

Proof. (i) By definition of B^2 and C^2 , $AC^{2n} = B^{2n}A$ for each positive integer n . Therefore, $A \left[\int t^n dc \right] = \left[\int t^n db \right] A$ for each non-negative integer n . The desired result follows by use of the Weierstrass approximation theorem. (ii) and (iii) are both special cases of (i).

THEOREM 2. If $BC = CB$ and $\operatorname{Re} A$ is nonnegative definite, then A is normal.

Proof. Let t be a real number and let $H = \operatorname{Re} A$ and $K = \operatorname{Im} A$. Define $k(t) = [1 - c(t)]Ac(t)$ and $n(t) = c(t)A[1 - c(t)]$. Then, using Lemma 2.1, one finds that $Ak(t)^*(S) \subset k(t)(s)$ and $An(t)^*(S) \subset n(t)(S)$. Since $k(t)^2 = 0$ and $n(t)^2 = 0$, $k(t)Ak(t)^* = 0$ and $n(t)An(t)^* = 0$.

Therefore, $k(t)Hk(t)^* = 0$ and $n(t)Hn(t)^* = 0$. Since H is nonnegative definite, then $Hk(t)^* = Hn(t)^* = 0$. Substituting for $k(t)$ and $n(t)$, one sees that

$$(1) \quad H[1 - c(t)]A^*c(t) = 0 \quad \text{and}$$

$$(2) \quad Hc(t)A^*[1 - c(t)] = 0 .$$

Subtracting (2) from (1) gives

$$(3) \quad \begin{aligned} &H[A^*c(t) - c(t)A^*] = 0, \text{ so that} \\ &HA^*[c(t) - b(t)] = 0 \text{ by Lemma 2.1.} \end{aligned}$$

In an analogous fashion, using $p(t) = [1 - b(t)]A^*b(t)$ and $q(t) = b(t)A^*[1 - b(t)]$, one arrives at

$$(4) \quad HA[b(t) - c(t)] = 0 .$$

Combining (3) and (4), one finds that $HK[b(t) - c(t)] = 0$ and $H^2[b(t) - c(t)] = 0$. Then $H[b(t) - c(t)] = 0$. A simple calculation shows that $B^2 - C^2 = 2i(KH - HK)$. Combining these last three equations, one has $(B^2 - C^2)(b(t) - c(t)) = 0$. Since t was arbitrary, then $(B^2 - C^2)^2 = B^2 - C^2 = 0$ and A is normal.

THEOREM 3. *If B commutes with each of C and $\text{Re } A$, then B commutes with A . Moreover, in this case, if A is reversible, then A is normal.*

Indication of proof. The final conclusion follows easily from Lemma 2.1. Again let $H = \text{Re } A$ and $K = \text{Im } A$. By use of the hypotheses, Lemma 2.1, and certain algebraic manipulations, one can show the following:

$$(1) \quad (B - C)CH = 0$$

$$(2) \quad (B - C)H(B - C) = 0$$

$$(3) \quad C(CH - HC)C = 0$$

$$(4) \quad AHA^*B = BAHA^* \quad \text{and}$$

$$(5) \quad A(B^2 - C^2)B^2 = AC^2(B^2 - C^2) = 0 .$$

This final equation then implies that $A(B^2 - C^2) = 0$. Therefore, by Lemma 2.1, $AB^2 = B^2A$. Since B^2 commutes with A , so does B (2, p. 260).

In concluding this paper, I should like to note that the proofs of Theorems 2 and 3 can be made much simpler algebraically, if it is assumed that A is invertible. However, it seemed reasonable to make the extra effort to prove the theorems without this added hypothesis.

I should also like to note that Lemma 2,1 appeared in my doctoral thesis at the University of North Carolina. Theorems 2 and 3 appeared in the same thesis with the added hypothesis of invertibility of A . Again I would like to thank Dr. J. S. Mac Nerney of the Department of Mathematics of the University of North Carolina for the direction of my doctoral thesis.

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