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**ON THE BOUNDED SLOPE CONDITION**

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Let  $\Omega$  be a bounded open set in  $R^n$  and let  $\varphi(x)$ ,  $x \in \partial\Omega$ , satisfy a "bounded slope condition". The latter reduces to the classical "3-point condition" if  $n=2$  and occurs in papers on partial differential equations. The properties of  $\varphi(x)$  are studied. It is shown, for example, that if  $\partial\Omega \in C^1$  or  $C^{1,\lambda}$ ,  $0 < \lambda \leq 1$ , then  $\varphi(x) \in C^1$  or  $C^{1,\lambda}$ . Hence, if  $\partial\Omega \in C^{1,1}$  is uniformly convex, then  $\varphi(x)$ ,  $x \in \partial\Omega$ , satisfies a bounded slope condition if and only if  $\varphi(x) \in C^{1,1}$ . The proofs use generalized convex functions of Beckenbach and, if  $n > 2$ , the equivalence of the bounded slope condition and an " $(n+1)$ -point condition".

Let  $n \geq 2$ ,  $x = (x^1, \dots, x^n)$  denote a point of  $R^n$  and  $z \in R^1$ , so that  $(x, z) \in R^{n+1}$ . Let  $\Omega$  be a bounded open set in  $R^n$  with boundary  $\Gamma = \partial\Omega$ .

DEFINITION (BSC). A real valued function  $\varphi(x)$  defined for  $x \in \Gamma$  is said to satisfy a bounded slope condition (BSC) [with constant  $K$ ] if, for every  $x_0 \in \Gamma$ , there exist two linear functions  $\lambda_{\pm}(x) = \lambda_{\pm}(x, x_0)$  of  $x$ ,

$$(1.1) \quad \lambda_{\pm}(x, x_0) = a_{\pm} \cdot (x - x_0) + \varphi(x_0) = \sum_{k=1}^n a_{\pm}^k (x^k - x_0^k) + \varphi(x_0),$$

where the constants  $a_{\pm}^k = a_{\pm}^k(x_0)$  depend only on  $x_0$ ,

$$(1.2) \quad \lambda_{-}(x, x_0) \leq \varphi(x) \leq \lambda_{+}(x, x) \quad \text{for } x \in \Gamma,$$

$$(1.3) \quad \|a_{\pm}(x_0)\| = \left( \sum_{k=1}^n |a_{\pm}^k|^2 \right)^{1/2} \leq K \quad \text{for } x_0 \in \Gamma.$$

The definition of a BSC occurs in [4] and is used in [9], [2], [5]. The name "bounded slope condition" was introduced in [9]. This paper is concerned with characterizations and properties of functions  $\varphi$  satisfying a BSC. Section 4 dealing with the smoothness of  $\varphi$  uses generalized convex functions of Beckenbach [1].

It has been pointed out to me by Professor Nirenberg that if  $n = 2$ , a BSC is equivalent to a "3-point condition" occurring in the calculus of variations and the theory of elliptic partial differential equations; cf. [7, 49-51 and 62-63] for references to Hilbert, Lebesgue, Haar, Rado and von Neumann. In Section 3, an " $(n+1)$ -point condition" will be defined and shown to be equivalent to a BSC. This fact will be used in Section 4 on smoothness properties of  $\varphi$ .

Note that, whether or not  $\Omega$  is convex, any linear function

$$(1.4) \quad \varphi(x) = a \cdot x + c = \sum_{k=1}^n a^k x^k + c \quad \text{for } x \in \Gamma$$

satisfies a BSC (with the choices  $\lambda_{\pm}(x, x_0) = a \cdot (x - x_0) + \varphi(x_0) = a \cdot x + c$ ). If however  $\varphi(x)$  satisfies a BSC and is not the restriction of a linear function to  $\Gamma$ , then  $\Omega$  is convex. For, in this case, the linear functions  $\lambda_{\pm}(x, x_0)$  of  $x$  are not identical and (1.1), (1.3) imply that

$$(a_+ - a_-) \cdot (x - x_0) \geq 0 \quad \text{for } x \in \Gamma,$$

hence for  $x \in \Omega$ . Thus, through every boundary point  $x_0$  of  $\Omega$ , there is a supporting plane  $0 \cong (a_+ - a_-) \cdot (x - x_0) = 0$ .

In what follows, it will be assumed that  $\Omega$  is convex. It should be remarked that, even if  $\Omega$  is uniformly convex, it does not follow that  $a_{\pm}$  can be chosen so that  $a_+ \cdot (x - x_0) \geq 0$  [and/or  $a_- \cdot (x - x_0) \leq 0$ ] for  $x \in \Gamma$ . For example, let  $n = 2$ ,  $\Omega$  be the disk  $(x^1)^2 + (x^2 - 1)^2 < 1$  and  $\varphi(x) = x^1$  for  $(x^1, x^2) \in \Gamma: (x^1)^2 + (x^2 - 1)^2 = 1$ . By the remark concerning (1.4),  $\varphi(x) = x^1$  satisfies a BSC. The unique supporting line of  $\Omega$  through the origin is  $x^2 = 0$ . But it is clear that no choice of the constant  $a^2$  satisfies  $a^2 x^2 \geq \varphi(x) = x^1 = \pm [2x^2 - 2(x^2)^2]^{1/2}$  for all  $(x^1, x^2) \in \Gamma$  (e.g., for small  $x^2 > 0$  and  $x^1 = [2x^2 - 2(x^2)^2]^{1/2} > 0$ ).

**2. Characterizations of  $\varphi(x)$ .** Let  $x^* \in \Omega$  and  $z^*$  be a real number. Let  $C(x^*, z^*)$  denote the conical surface consisting of the set of points  $(x, z) \in R^{n+1}$  of the form

$$(2.1) \quad C(x^*, z^*): \quad x = x^* + t(x_0 - x^*), \quad z = z^* + t[\varphi(x_0) - z^*],$$

$t \geq 0$  and  $x_0 \in \Gamma$ , so that  $C(x^*, z^*)$  is the union of the sets of points on the half-lines from  $(x^*, z^*)$  directed towards  $(x_0, \varphi(x_0))$ ,  $x_0 \in \Gamma$ .

**THEOREM 2.1.** *Let  $\Omega \in R^n$ ,  $n \geq 2$ , be a bounded open convex set,  $\Gamma = \partial\Omega$ ,  $x^* \in \Omega$  (fixed), and  $\varphi(x)$  a function defined for  $x \in \Gamma$ . Then  $\varphi(x)$  satisfies a BSC if and only if the conical surface  $C(x^*, z^*)$  bounds a convex set  $\Omega(x^*, z^*) \subset R^{n+1}$  for large  $|z^*|$  (say, for  $|z^*| \geq N$ ; in which case,  $N$  can be chosen independent of  $x^*$ ).*

It will be clear from the proof that  $\varphi(x)$  satisfies a BSC if and only if there exists a convex function  $\rho_-(x)$  and a concave function  $\rho_+(x)$  defined for all  $x \in R^n$  such that the restrictions of  $\rho_{\pm}(x)$  to  $\Gamma = \partial\Omega$  are identical with  $\varphi(x)$ .

*Proof.* "If". Let  $z^* > 0$  be so large that  $|\varphi(x_0)| < z^*$  for  $x_0 \in \Gamma$  and that  $C(x^*, \pm z^*)$  bound convex sets  $\Omega(x^*, \pm z^*)$ . Let  $z = \lambda_{\pm}(x, x_0)$  be a supporting hyperplane of  $\Omega(x^*, z^*)$  at the point  $(x_0, \varphi(x_0)) \in C(x^*, \pm z^*)$ , corresponding to  $t = 1$  in (2.1). It is clear that  $\lambda_{\pm}(x, x_0)$  are of the

form (1.1) and satisfy (1.2). The conical surfaces  $C(x^*, \pm z^*)$  have representations of the form

$$z = \tau_+(x) \quad \text{and} \quad z = \tau_-(x)$$

defined for all  $x \in R^n$  such that  $-\tau_+(x), \tau_-(x)$  are convex functions. In particular,  $\tau_\pm(x)$  are uniformly Lipschitz continuous on compacts, say, on  $\Omega \cup \Gamma$ . It follows that there exists a constant  $K$  satisfying (1.3).

*Proof. "Only if".* Let  $\varphi(x)$  satisfy a BSC. For fixed  $x_0 \in \Gamma$ , let  $\lambda_\pm(x, x_0)$  be the linear functions of  $x$  in (1.1)—(1.3). Then

$$|\lambda_\pm(x, x_0)| \leq K \|x - x_0\| + |\varphi(x_0)| \leq K \|x\| + K_1,$$

where  $K_1$  is a constant independent of  $x \in R^n$  and  $x_0 \in \Gamma$ . Thus

$$(2.2) \quad \rho_-(x) = \sup \lambda_-(x, x_0), \quad \rho_+(x) = \inf \lambda_+(x, x_0) \quad \text{for } x_0 \in \Gamma$$

exist (finite) for all  $x$  and satisfy

$$(2.3) \quad |\rho_\pm(x)| \leq K \|x\| + K_1 \quad \text{for } x \in R^n,$$

$$(2.4) \quad \rho_\pm(x_0) = \varphi(x_0) \quad \text{for } x_0 \in \Gamma,$$

and  $\mp \rho_\pm(x)$  are convex functions of  $x$ . Since  $\rho_-(x) - \rho_+(x)$  is a convex function and vanishes on  $\Gamma$ .

$$(2.4) \quad \rho_-(x) \leq \rho_+(x) \quad \text{for } x \in \Omega.$$

The convexity of  $\mp \rho_\pm(x)$  and (2.3) imply that  $\rho_\pm(x)$  are uniformly Lipschitz continuous with a Lipschitz constant  $K$  on  $R^n$ .

Let  $\Omega^\pm \in R^{n+1}$  denote the convex sets

$$\Omega^- = \{(x, z) : z > \rho_-(x)\}, \quad \Omega^+ = \{(x, z) : z < \rho_+(x)\}.$$

For  $x_0 \in \Gamma$ , let the linear function  $\lambda^\pm(x, x_0)$  of  $x$ ,

$$(2.5) \quad \lambda^\pm(x, x_0) = a^\pm(x_0) \cdot (x - x_0) + \varphi(x_0),$$

be chosen so that  $z = \lambda^\pm(x, x_0)$  is a supporting plane of  $\Omega^\pm$  at the boundary point  $(x, z) = (x_0, \varphi(x_0))$ . In particular,

$$(2.6) \quad \|a^\pm(x_0)\| \leq K,$$

$$(2.7) \quad \lambda^-(x, x_0) \leq \varphi(x) \leq \lambda^+(x, x_0) \quad \text{for } x \in \Gamma.$$

Let  $\lambda(x, x_0) \equiv a(x_0) \cdot (x - x_0)$  be a linear function of  $x$  such that  $\lambda(x, x_0) = 0$  is a supporting plane for  $\Omega \subset R^n$  with the normalization

$$(2.8) \quad \lambda(x, x_0) > 0 \quad \text{for } x \in \Omega, \quad \|a(x_0)\| = 1.$$

In view of (2.5) and (2.6), there exists numbers  $N > 0$  such that  $|\lambda^\pm(x, x_0)| \leq N$  for  $x \in \Omega, x_0 \in \Gamma$ . Let  $z^* \geq N$  and choose numbers  $\mu^\pm(x_0) \geq 0$  with the property that the linear functions

$$\sigma_\pm(x, x_0) = \lambda^\pm(x, x_0) \pm \mu^\pm(x_0)\lambda(x, x_0)$$

of  $x$  satisfy  $\sigma_\pm(x^*, x_0) = \pm z^*$ . It is clear that  $|\mu^\pm(x_0)\lambda(x^*, x_0)|$  and  $1/|\lambda(x^*, x_0)|$  are bounded for all  $x_0 \in \Gamma$  (and  $x^* \in \Omega$  fixed). Thus if  $\sigma_\pm(x, x_0)$  is written in the form

$$\sigma_\pm(x, x_0) = b^\pm(x_0) \cdot (x - x_0) + \varphi(x_0),$$

there is a constant  $K_0 = K_0(x^*)$  such that

$$\|b^\pm(x_0)\| \leq K_0 \quad \text{for } x_0 \in \Gamma.$$

Also,

$$\sigma_-(x, x_0) \leq \varphi(x) \leq \sigma_+(x, x_0) \quad \text{for } x \in \Gamma.$$

Thus,  $\sigma_\pm(x, x_0)$  satisfy conditions analogous to (1.1) (1.3) with  $\lambda_\pm, a^\pm, K$  replaced by  $\sigma_\pm, b^\pm, K_0$ . Corresponding to (2.2), put

$$(2.9) \quad \tau_-(x) = \sup \sigma_-(x, x_0), \quad \tau_+(x) = \inf \sigma_+(x, x_0) \quad \text{for } x_0 \in \Gamma.$$

The functions  $\mp \tau_\pm(x)$  are convex. Since  $\tau_\pm(x_0) = \varphi(x_0)$  for  $x_0 \in \Gamma, \tau_\pm(x^*) = \pm z^*$ , and for  $x = x^* + t(x - x_0), t \geq 0$ ,

$$\begin{aligned} \sigma_-(x, x_0) &\leq -z^* + t[\varphi(x_0) + z^*], \\ \sigma_+(x, x_0) &\geq z^* + t[\varphi(x_0) - z^*], \end{aligned}$$

it follows that

$$\tau_\pm(x) = z^* + t[\varphi(x_0) \mp z^*] \quad \text{for } x = x^* + t(x_0 - x^*),$$

$t \geq 0$ . Thus  $z = \tau_\pm(x)$  are the conical surfaces  $C(x^*, \pm z^*)$ . Since these surfaces are convex, Theorem 2.1 is proved.

For applications, it will be convenient to reformulate Theorem 2.1 in different terms. Let  $x^* \in \Omega$  be fixed and  $x_0, x_1 \in \Gamma$ . Suppose that the half-lines

$$(2.10) \quad x^* + tx_0 \quad \text{and} \quad x^* + tx_1 \quad \text{for } t \geq 0$$

in  $R^n$  are not on the same line and so determine a 2-dimensional plane  $\pi_2(x_0, x_1) \in R^n$  and a convex sector  $S(x_0, x_1)$  of  $\pi_2(x_0, x_1)$  with vertex at  $x^*$ . Let  $\Gamma_{01}$  be the 2-dimensional plane convex curve  $\Gamma_{01} = \pi_2(x_0, x_1) \cap \Gamma$ . By a point  $x_{01}$  of  $\Gamma$  between  $x_0$  and  $x_1$  is meant a point  $x_{01}$  of the arc  $\Gamma(x_0, x_1) = S(x_0, x_1) \cap \Gamma_{01}$ . Introduce rectangular coordinates  $(\xi, \eta)$  in the plane  $\pi_2(x_0, x_1)$  with  $x^*$  as origin such that the  $\xi$ -axis,  $\eta$ -axis, and the half-line  $(x, z) = (x^*, t), t \geq 0$ , form a right-hand system. It will be supposed that the enumeration of  $x_0, x_1$  is chosen so that the arc

$\Gamma(x_0, x_1)$  in  $\pi_2(x_0, x_1)$  is positively oriented in going from  $x_0$  to  $x_1$ . Let  $(\xi_0, \eta_0), (\xi_1, \eta_1), (\xi_{01}, \eta_{01})$  be the  $(\xi, \eta)$ -coordinates of  $x_0, x_1, x_{01}$ , respectively.

When the half-lines (2.10) are on the same line in  $R^n$ , the notion of a point  $x_{01}$  between  $x_0$  and  $x_1$  will not be defined.

**COROLLARY 2.1.** *Let  $\Omega \subset R^n$  be a bounded open convex set,  $\varphi(x)$  a function defined for  $x \in \Gamma = \partial\Omega$ , and  $x^* \in \Omega$ . Then  $\varphi(x)$  satisfies a BSC if and only if there exists a number  $N$  such that, for  $|z^*| \geq N$ , the inequality*

$$(2.11) \quad z^* \begin{Bmatrix} \xi_0 & \eta_0 & \varphi(x_0) - z^* \\ \xi_{01} & \eta_{01} & \varphi(x_{01}) - z^* \\ \xi_1 & \eta_1 & \varphi(x_1) - z^* \end{Bmatrix} \leq 0$$

holds for all points  $x_0, x_1 \in \Gamma$  and points  $x_{01} \in \Gamma$  between them.

See Lemma 3.1 and part (b) of the proof of Theorem 3.1 for analogous necessary and sufficient conditions.

*Proof.* It has to be verified that (2.11) is equivalent to the ‘‘convexity’’ of the cones (2.1). As the case  $z^* < 0$  is similar to that of  $z^* > 0$ , consider only the latter. For  $z^* > 0$ , it will be shown that (2.11) is equivalent to the concavity of  $z$  in (2.1) as a function of  $x$ .

To verify that  $z$  is concave (i.e., that  $-z$  is convex), it suffices to consider the situation when  $x$  varies along a line in  $R^n$ . If  $x$  varies along a line which passes through  $x^*$ , the concavity of the function  $z$  is clear. Consider a line  $L$  in  $R^n$  which does not pass through  $x^*$ . After a suitable translation and rotation of coordinates in the  $x$ -space, it can be supposed that  $x^* = 0$  and that the line  $L$  and the point  $x^* = 0$  are in the  $(x^1, x^2)$ -plane,  $x^3 = \dots = x^n = 0$ . We now ignore the trivial coordinates  $x^3 = \dots = x^n = 0$  and write  $(\xi, \eta)$  in place of  $(x^1, x^2)$ .

It can be supposed that  $L$  is the line  $L: \xi = c > 0$ . Consider two points  $\pi_0 = (c, u_0), \pi_1 = (c, u_1)$  on  $L, u_0 < u_1$ , and the condition

$$(2.12) \quad z(\pi_{01}) \geq \theta z(\pi_0) + (1 - \theta)z(\pi_1)$$

for  $z$  to be a concave function of  $\pi_{01} = (c, u_{01}), u_{01} = \theta u_0 + (1 - \theta)u_1, 0 < \theta < 1$ .

Let the half-line from  $x^*$  toward  $\pi_0, \pi_1, \pi_{01}$  meet  $\Gamma$  at  $x_0 = (\xi_0, \eta_0), x_1 = (\xi_1, \eta_1), x_{01} = (\xi_{01}, \eta_{01})$ , respectively, and let  $t_0, t_1, t_{01}$  denote the unique positive numbers such that

$$(2.13) \quad \pi_j = t_j x_j, \text{ i.e., } (c, u_j) = t_j(\xi_j, \eta_j), \text{ for } j = 0, 1, \text{ and } 01.$$

Correspondingly,

$$(2.14) \quad z(\pi_j) = z^* + t_j[\varphi(x_j) - z^*] \text{ for } j = 0, 1, \text{ and } 01 .$$

From (2.13),  $t_j = c/\xi_j$ , so that, by (2.14), (2.12) is equivalent to

$$(2.15) \quad [\varphi(x_{01}) - z^*]/\xi_{01} \geq \theta[\varphi(x_0) - z^*]/\xi_0 + (1 - \theta)[\varphi(x_1) - z^*]/\xi_1$$

Also,  $u_j = t_j\eta_j = c\eta_j/\xi_j$  and  $u_{01} = \theta(u_0 - u_1) + u_1$ , so that

$$\begin{aligned} \theta &= (\eta_1/\xi_1 - \eta_{01}/\xi_{01})/(\eta_1/\xi_1 - \eta_0/\xi_0) , \\ 1 - \theta &= (\eta_{01}/\xi_{01} - \eta_0/\xi_0)/(\eta_1/\xi_1 - \eta_0/\xi_0) . \end{aligned}$$

Since  $\xi_0\eta_1 - \xi_1\eta_0 > 0$ , (2.15) is equivalent to

$$\begin{aligned} &(\xi_0\eta_1 - \xi_1\eta_0)[\varphi(x_{10}) - z^*] \\ &\geq (\xi_{01}\eta_1 - \xi_1\eta_{01})[\varphi(x_0) - z^*] - (\xi_{01}\eta_0 - \xi_0\eta_{01})[\varphi(x_1) - z^*] \end{aligned}$$

which, in turn, is equivalent to (2.11) when  $z^* > 0$ . This completes the proof.

**3. The  $(n + 1)$ -point condition.** Let  $n \geq 2$  and  $\Omega \subset R^n$  be a bounded open convex set and  $\varphi(x)$  a function defined for  $x \in \Gamma = \partial\Omega$ .

**DEFINITION (I).**  $(n + 1)$ -point condition.  $\varphi(x)$  is said to satisfy an  $(n + 1)$ -point condition [with constant  $K$ ] if, for every set of  $n + 1$  points  $x_0, \dots, x_n$  of  $\Gamma$ , there is a hyperplane

$$(3.1) \quad z = a \cdot x + c = \sum_{h=1}^n a^h x^h + c$$

in  $R^{n+1}$  which passes through the points  $(x, z) = (x_j, \varphi(x_j))$  for  $j = 0, 1, \dots, n$  and satisfies

$$(3.2) \quad \|a\| = \left(\sum_{k=1}^n |a^k|^2\right)^{1/2} \leq K .$$

In deciding whether or not  $\varphi$  satisfies an  $(n + 1)$ -point condition, continuity considerations show that it suffices to consider only sets of  $n + 1$  points  $x_0, \dots, x_n$  of  $\Gamma$  such that  $(x, z) = (x_j, \varphi(x_j))$ ,  $j = 0, \dots, n$ , determine a unique hyperplane in  $R^{n+1}$ . In particular, if  $x_0, \dots, x_n$  are on an  $(n - 1)$ -dimensional plane  $\pi_{n-1} \subset R^n$ , then the restriction of  $\varphi(x)$  to  $\Gamma \cap \pi_{n-1}$  is the restriction of a linear function of  $x$ .

**THEOREM 3.1.** Let  $\Omega \subset R^n$  be a bounded open convex set and  $\varphi(x)$  a function defined for  $x \in \Gamma = \partial\Omega$ . Then  $\varphi(x)$  satisfies a BSC if and only if  $\varphi$  satisfies an  $(n + 1)$ -point condition.

In the proof, it will be convenient to have the following auxiliary definition.

DEFINITION (II). Let  $n \geq 2$ ,  $\Omega \subset R^n$  a bounded, open convex set,  $\Gamma = \partial\Omega$ ,  $\varphi(x)$  a function on  $\Gamma$ , and  $2 \leq m \leq n$ . The function  $\varphi$  is said to satisfy an  $(m + 1)$ -point condition with constant  $K$  if, for every  $m$ -dimensional plane  $\pi_m \subset R^n$  containing an interior point of  $\Omega$ , the restriction of  $\varphi(x)$  to the boundary of  $\Omega \cap \pi_m$  satisfies an  $(m + 1)$ -point condition with a constant  $K$  (in the sense of Definition (I) where  $n = m$ ).

The proof of Theorem 3.1 will be given in several steps: (a), Lemma 3.1, (b), (c), (d), in which  $\Omega, \Gamma, \varphi$  are as in Theorem 3.1.

(a)  $\varphi(x)$  satisfies an  $(n + 1)$ -point condition if and only if there exists a number  $N$  with the property that, for every set of  $n + 1$  points  $x_0, \dots, x_n$  of  $\Gamma$ , there is a hyperplane (3.1) passing through  $(x, z) = (x_j, \varphi(x_j))$  for  $j = 0, \dots, n$  and satisfying

$$(3.3) \quad |a \cdot x + c| \leq N \quad \text{for } x \in \Omega .$$

In fact, if (3.1) is the hyperplane satisfying (3.1) and (3.2), then, for  $x \in \Omega$ ,

$$|a \cdot x + c| = |a \cdot (x - x_0) + \varphi(x_0)| \leq K \text{ diam } \Omega + \text{const} .$$

Conversely, if (3.1) is a hyperplane satisfying (3.3) and  $a \neq 0$  then there is a number  $c_0 > 0$  (independent of a) and a pair of points  $y_0, y_1 \in \Omega$  such that

$$y_0 - y_1 = ta / \|a\| , \quad t = c_0 > 0 .$$

Thus, from

$$a \cdot (y_0 - y_1) = (a \cdot y_0 + c) - (a \cdot y_1 + c)$$

and (3.3),  $|a \cdot (y_0 - y_1)| \leq 2N$ , and so  $\|a\| \leq 2N/c_0$ .

LEMMA 3.1. Let  $\Omega, \Gamma, \varphi$  be as in Theorem 3.1. Let  $x_j = (x_j^1, \dots, x_j^n)$  for  $j = 0, 1, \dots, n$  be  $n + 1$  points of  $\Gamma$ ,

$$(3.4) \quad \Delta_0(x_0, \dots, x_n) = \begin{vmatrix} x_0^1 & \dots & x_0^n & 1 \\ x_1^1 & \dots & x_1^n & 1 \\ \dots & \dots & \dots & \dots \\ x_n^1 & \dots & x_n^n & 1 \end{vmatrix} ,$$

$$(3.5) \quad \Delta(x, z) = \begin{vmatrix} x^1 & \dots & x^n & z & 1 \\ x_0^1 & \dots & x_0^n & \varphi(x_0) & 1 \\ x_1^1 & \dots & x_1^n & \varphi(x_1) & 1 \\ \dots & \dots & \dots & \dots & \dots \\ x_n^1 & \dots & x_n^n & \varphi(x_n) & 1 \end{vmatrix} .$$



Then  $\varphi$  satisfies an  $(n + 1)$ -point condition if and only if there exists a number  $N$  such that

$$(3.6) \quad (-1)^n z \Delta(x, z) \Delta_0(x_0, \dots, x_n) \geq 0 \quad \text{for } |z| \geq N,$$

$x \in \Omega$ , and all sets of  $n + 1$  points  $x_0, \dots, x_n$  of  $\Gamma$ .

In fact, if there is a unique hyperplane of the form (3.1) passing through  $(x_j, \varphi(x_j))$  for  $j = 0, 1, \dots, n$ , then

$$\Delta(x, z) = (-1)^n \Delta_0(x_0, \dots, x_n) [z - (a \cdot x + c)].$$

Thus (3.6) for  $|z| \geq N, x \in \Omega$ , is equivalent to (3.3).

(b) Let  $m, 2 \leq m \leq n$ , be fixed. If  $\varphi(x)$  satisfies an  $(n + 1)$ -point condition with constant  $K$  in the sense of Definition (I), then it satisfies an  $(m + 1)$ -point condition with constant  $K$  in the sense of Definition (II).

This is clear. Theorem 3.1 and its proof will show that the converse is correct.

(c) BSC  $\Rightarrow$   $(n + 1)$ -point condition.

Let  $\varphi(x)$  satisfy a BSC and let  $x_0, \dots, x_n$  be  $n + 1$  points of  $\Gamma$  such that there is a unique hyperplane passing through the points  $(x_j, \varphi(x_j))$  for  $j = 0, \dots, n$ . This hyperplane necessarily has an equation of the form (3.1). It will be shown that there exists a number  $N$  satisfying (3.3).

Let  $N$  be so large that the conical surfaces  $C(x^*, z^*)$  in Theorem 2.1 bound open convex sets  $\Omega(x^*, z^*)$  for every  $x^* \in \Omega$  and  $|z^*| \geq N$ . It will be shown that (3.3) holds for the arbitrary (but fixed) point  $x = x^* \in \Omega^*$ .

Suppose first that  $x^*$  is in the convex closure of the set of points  $x_0, \dots, x_n$ . Consider a supporting hyperplane  $\pi^+ : z = a_+ \cdot x + c_+$  of the convex set  $\Omega(x^*, N)$  through the boundary point  $(x_0, \varphi(x_0)) \in C(x^*, N)$ . Then  $(x, z) \in C(x^*, N)$  implies that  $z \leq a_+ \cdot x + c_+$ . Hence

$$(3.7) \quad a \cdot x + c \leq a_+ \cdot x + c_+$$

holds for  $x = x_0, \dots, x_n$  and hence for all  $x$  in the convex closure of the set of point  $x_0, \dots, x_n$ . In particular,  $a \cdot x^* + c \leq a_+ \cdot x^* + c_+ = N$ . Similarly,  $a \cdot x^* + c \geq -N$ .

Consider now the case where  $x^*$  is not in the convex closure of the set of points  $x_0, \dots, x_n$ . Let  $B$  denote the convex closure of  $x^*$  and  $x_0, \dots, x_n$ , so that  $B$  is bounded by a polyhedron. Since  $x_0, \dots, x_n$  are not contained in an  $(n - 1)$ -dimensional plane  $\pi_{n-1}$ , the set  $B \subset R^n$  has interior points. Thus there are  $n$  edges on the boundary of  $B$  terminating at  $x^*$ . Suppose that the enumeration of  $x_0, \dots, x_n$  is such that the line segments  $[x^* x_j]$ , where  $j = 1, \dots, n$ , are on the

boundary of  $B$ . Thus  $B$  contains the closed "simplex"  $B^*$  with vertices  $x^*, x_1, \dots, x_n$ .

Suppose, if possible, that  $x_0 \in B^*$ . Let  $\pi_{n-1}$  be a supporting  $(n-1)$ -dimensional plane (in  $R^n$ ) of  $\Omega$  through the point  $x_0$ . Then the face  $x_1, \dots, x_n$  of  $B^*$  is not on  $\pi_{n-1}$  (for otherwise  $x_0, \dots, x_n \in \pi_{n-1}$ ). Also,  $x^* \notin \pi_{n-1}$  since  $x^* \in \Omega$ . Thus no face of  $B^*$  is on  $\pi_{n-1}$  and, since  $x_0$  is not a vertex of  $B^*$ ,  $\pi_{n-1}$  is not a supporting plane of  $B^*$ . Hence  $\pi_{n-1}$  separates at least one pair of vertices of  $B^*$ . But this is impossible since  $\pi_{n-1}$  supports  $\Omega$ . Hence  $x_0 \notin B^*$ .

Consequently,  $B$  is the union of two simplices,  $B^*$  with vertices  $x^*, x_1, \dots, x_n$  and  $B_0$  with vertices  $x_0, x_1, \dots, x_n$ , with the common face  $x_1, \dots, x_n$ . Thus the diagonal  $[x_0x^*]$  of  $B$  meets the face  $x_1, \dots, x_n$  of  $B$  at some point. Consequently,  $x^*$  is in the convex closure of the set of points on the  $n$  half-lines  $x_0 + t(x_j - x_0)$ , where  $t \geq 0$  and  $j = 1, \dots, n$ . Let  $\pi^+ : z = a_+ \cdot x + c_+$  be a supporting hyperplane of  $\Omega(x^*, N)$  through the boundary point  $(x_0, \varphi(x_0)) \in C(x^*, N)$ . Then (3.7) holds for  $x = x_0, \dots, x_n$ , hence on the half-lines  $x = x_0 + t(x_j - x_0)$ ,  $t \geq 0$  and  $j = 1, \dots, n$ , and consequently for all points (including  $x = x^*$ ) in the convex closure of the set of points on these half-lines. Thus, as before,  $a \cdot x^* + c \leq a_+ \cdot x^* + c_+ = N$ . Similarly  $a \cdot x^* + c \geq -N$ . By (a), this proves that  $\varphi$  satisfies an  $(n+1)$ -point condition.

(d)  $(n+1)$ -point condition  $\Rightarrow$  BSC.

Let  $\varphi(x)$  satisfy an  $(n+1)$ -point condition with a constant  $K$ . Then  $\varphi(x)$  satisfies a 3-point condition with constant  $K$  by (b). Let  $\pi_2 \subset R^n$  be a 2-dimensional plane containing an interior point  $x^* \in \Omega$ , and  $(\xi, \eta)$  rectangular coordinates on  $\pi_2$ . Let  $(\xi_0, \eta_0), (\xi_1, \eta_1), (\xi_{01}, \eta_{01})$  be the  $(\xi, \eta)$ -coordinates of points  $x_0, x_1, x_{01}$  of  $I' \cap \pi_2$ ,

$$(3.8) \quad \delta_0(x_0, x_{01}, x_1) = \begin{vmatrix} \xi_0 & \eta_0 & 1 \\ \xi_{01} & \eta_{01} & 1 \\ \xi_1 & \eta_1 & 1 \end{vmatrix},$$

$$(3.9) \quad \delta(\xi, \eta, z) = \begin{vmatrix} \xi_0 - \xi & \eta_0 - \eta & \varphi(x_0) - z \\ \xi_{01} - \xi & \eta_{01} - \eta & \varphi(x_{01}) - z \\ \xi_1 - \xi & \eta_1 - \eta & \varphi(x_1) - z \end{vmatrix}.$$

Then, by Lemma 3.1, there exists a constant  $N$  such that

$$(3.10) \quad z\delta_0(x_0, x_{01}, x_1)\delta(\xi, \eta, z) \leq 0$$

for  $|z| \geq N$  and all points  $(\xi, \eta) \in \Omega \cap \pi_2$ . It follows from Corollary 2.1 that  $\varphi(x)$  satisfies a BSC (for if the origin of the  $(\xi, \eta)$ -coordinate system is chosen at  $x^* \in \Omega \cap \pi_2$ , then (2.11) and (3.10) with  $(\xi, \eta) = 0$  are equivalent). This completes the proof.

4. Smoothness of  $\varphi(x)$ . It will be shown that if  $\varphi$  satisfies a BSC, then its smoothness (in some sense) is similar to that of a convex function.

COROLLARY 4.1. *Let  $\Omega \subset R^n$  be a bounded open convex set and  $\varphi(x)$  a function on  $\Gamma = \partial\Omega$  satisfying a BSC. Let  $\pi_2$  be a 2-dimensional plane in  $R^n$  containing an interior point of  $\Omega$ ,  $\Gamma_{01} = \pi_2 \cap \Gamma$ , and*

$$(4.1) \quad \Gamma_{01} : x = x(s)$$

*an arclength parametrization of  $\Gamma_{01}$ . Then*

$$(4.2) \quad \psi(s) = \varphi(x(s))$$

*has a derivative  $d\psi/ds$  except on a set of  $s$ -values which is at most countable.*

It turns out that when one imposes additional smoothness conditions on  $\Gamma$ , the required smoothness on a function  $\varphi$  satisfying a BSC is correspondingly increased.

COROLLARY 4.2. *Let  $\Omega \subset R^n$  be a bounded open convex set and  $\varphi(x)$  a function on  $\Gamma = \partial\Omega$  satisfying a BSC.*

- (i) *If  $\Gamma \in C^1$ , then  $\varphi(x) \in C^1$ .*
- (ii) *If  $\Gamma \in C^{1,\lambda}$ , then  $\varphi(x) \in C^{1,\lambda}$ .*

A function on an open set  $A \subset R^n$  is said to be of class  $C^{1,\lambda}$  if it has continuous, first order, partial derivatives which are uniformly Hölder [or Lipschitz] continuous of order  $\lambda$ ,  $0 < \lambda < 1$  [or  $\lambda = 1$ ] on closed spheres in  $A$ . The definition of a hypersurface  $\Gamma \subset R^{n+1}$  of class  $C^{1,\lambda}$  or of a function  $\varphi(x)$  on  $\Gamma$  of class  $C^{1,\lambda}$  is analogous.

REMARK. Let  $\varphi(x)$  satisfy a BSC and let the conical surfaces  $C(x^*, z^*)$ ,  $z^* = \pm N$ , have the equations

$$C(x^*, \pm N) : z = \tau_{\pm}(x) \quad \text{for all } x$$

[cf. the proof of Theorem 2.1]; so that  $\tau_{\pm}(x) = \varphi(x)$  for  $x \in \Gamma$ . Then, in case (i) of Corollary 4.2,  $\tau_{\pm}(x)$  has continuous partial derivatives except at  $x = x^*$ ; in case (ii), these partial derivatives are uniformly Hölder continuous of order  $\lambda$  on compacts not containing  $x = x^*$ . Thus suitable modifications of  $\tau_{\pm}(x)$  near  $x = x^*$  give functions on  $R^n$  which are respectively of class  $C^1, C^{1,\lambda}$  and which are identical with  $\varphi$  on  $\Gamma$ .

The arguments in [5] show that if  $\Omega$  is uniformly convex (whether or not  $\Gamma \in C^{1,1}$ ) and if  $\varphi$  is the restriction to  $\Gamma$  of a function on  $R^n$  of class  $C^{1,1}$ , then  $\varphi$  satisfies a BSC; cf. [8, 625-628] and [2], where

$\Gamma$  and  $\varphi$  are of class  $C^2$ . Conversely, if  $\Gamma \in C^{1,1}$  (whether or not  $\Omega$  is uniformly convex), then, by Corollary 4.2 (ii), a necessary condition for  $\varphi$  to satisfy a BSC in that  $\varphi \in C^{1,1}$ . Thus we have

**COROLLARY 4.3.** *Let  $\Omega \in R^n$  be a bounded, open set with a uniformly convex boundary  $\Gamma = \partial\Omega$  of class  $C^{1,1}$ . Then a necessary and sufficient condition for a function  $\varphi(x), x \in \Gamma$ , to satisfy a BSC is that  $\varphi(x) \in C^{1,1}$ .*

$\Gamma$  is called *uniformly convex* if there is a constant  $c > 0$  such that through every  $x_0 \in \Gamma$ , there is a hyperplane  $\pi_{n-1} \subset R^n$  satisfying  $\text{dist}(x, \pi_{n-1}) \geq c \|x - x_0\|^2$  for  $x \in \Gamma$ .

The “sufficiency” does not hold if  $\Omega$  is not uniformly convex, but is only strictly convex. For example, let  $n = 2$  and let the “lower” portion of  $\Gamma$  be on the curve  $x^2 = (x^1)^4$  near the origin and let  $\varphi(x) = (x^1)^2$  for  $(x^1, x^2) \in \Gamma$ . Then there is no choice of constants  $a^1, a^2$  such that  $a^1x^1 + a^2x^2 \geq (x^1)^2$  for small  $|x^1|$  and  $(x^1)^2 = |x^2|^{1/2} \geq 0$ .

The following remark will not be used below but it may be of interest to note that if  $\Gamma \in C^2, \varphi \in C^2$  and if  $\Gamma_{01}: \xi = \xi(s), \eta = \eta(s)$  and  $\psi(s)$  are in the proof of Corollary 4.1 below, then condition (2.11) is equivalent to

$$z^* \begin{vmatrix} \psi(s) - z^* & \xi(s) & \eta(s) \\ \psi'(s) & \xi'(s) & \eta'(s) \\ \psi''(s) & \xi''(s) & \eta''(s) \end{vmatrix} \leq 0$$

for all 2-dimensional plane sections  $\Gamma_{01}$  of  $\Gamma$ . This fact makes it clear, for example, that if  $\Gamma \in C^2, \varphi \in C^2$  and  $\Omega$  is uniformly convex, then  $\varphi$  satisfies a BSC.

*Proof of Corollary 4.1.* Choose a coordinate system in  $R^n$  such that  $\pi_2$  is the plane  $x^3 = \dots = x^n = 0$  and with the origin at a point  $x^*$  in  $\pi_2 \cap \Omega$ . Write  $(\xi, \eta)$  in place of  $(x^1, x^2)$ . Let  $\xi = \xi(s), \eta = \eta(s)$ , where  $0 \leq s \leq s_0$ , be an arclength parametrization of  $\Gamma_{01}$ .

Choose an  $s$ -interval, say  $0 \leq s \leq \alpha < s_0$ , such that the radius vector [the line from the origin to  $(\xi(s), \eta(s))$ ] moves through an angle less than  $\pi$  as  $s$  varies from 0 to  $\alpha$ . Then, if  $\xi, \eta$  is a pair of arbitrary numbers and  $0 \leq s_1 < s_2 \leq \alpha$ , the linear equations

$$c_1\xi(s_1) + c_2\eta(s_1) = \xi, \quad c_1\xi(s_2) + c_2\eta(s_2) = \eta$$

have a unique solution for  $c_1, c_2$ . In the terminology of Beckenbach [1], this means that the linear family  $F$  of functions  $c_1\xi(s) + c_2\eta(s)$  is a 2-parameter family on the interval  $0 \leq s \leq \alpha$ .

Let  $\psi(s)$  be defined by (4.2), i.e.,

$$\psi(s) = \varphi(\xi(s), \eta(s), 0, \dots, 0) .$$

Then (2.11) implies that  $\psi(s) - z^*$  is  $F$ -concave and  $\psi(s) + z^*$  is  $F$ -convex, where  $z^* = N > 0$ . In other words, if

$$(4.3) \quad f_{\pm}(s) = c_{1\pm}\xi(s) + c_{2\pm}\eta(s)$$

is a linear combination of  $\xi(s), \eta(s)$  such that  $f_{\pm}(s) = \psi(s) \mp z^*$  at  $s = s_1, s_2$  where  $0 \leq s_1 < s_2 \leq \alpha$ , then  $f_+(s) \geq \psi(s) - z^*, f_-(s) \leq \psi(s) + z^*$  for  $s_1 \leq s \leq s_2$ ; cf. the proof of Corollary 4.2,  $n > 2$ , below.

Let  $0 < s_0 < \alpha$ . By [6], there exist elements (4.3) of  $F$  which support  $\psi(s) \mp z^*$  in the sense that

$$(4.4) \quad f_{\pm}(s_0) = \psi(s_0) \mp z^* ,$$

$$(4.5) \quad f_-(s) - z^* \leq \psi(s) \leq f_+(s) + z^* \quad \text{for } 0 \leq s \leq \alpha ;$$

see also [3] for generalizations and references to Bonsall, J. W. Green and Reid.

Since  $\Gamma_{01}$  is a (plane) convex curve,  $\xi(s)$  and  $\eta(s)$  are differentiable except possibly on a countable set of  $s$ -values. Choose  $s = s_0$  so that  $\xi' = d\xi/ds, \eta' = d\eta/ds$  exist at  $s = s_0$ . Note that

$$[f_+(s) + z^*] - [f_-(s) - z^*]$$

is nonnegative for  $0 \leq s \leq \alpha$  and vanishes at  $s = s_0$ . Hence  $f'_+(s_0) = f'_-(s_0)$ . Consequently (4.5) implies that  $\psi'(s_0)$  exists (and is  $f'_+(s_0) = f'_-(s_0)$ ). This proves Corollary 4.1.

*Proof of Corollary 4.2,  $n = 2$ .* It is clear from the proof of Corollary 4.1 that if  $\Gamma \in C^1$ , then  $\psi'(s)$  exists for all  $s$ . It is also clear that the coefficients  $c_{1\pm}, c_{2\pm}$  in (4.3)–(4.5) are determined by the linear equations

$$\begin{aligned} c_{1\pm}\xi(s_0) + c_{2\pm}\eta(s_0) &= \psi(s_0) \mp z^* , \\ c_{1\pm}\xi'(s_0) + c_{2\pm}\eta'(s_0) &= \psi'(s_0) . \end{aligned}$$

The determinant  $\xi(s_0)\eta'(s_0) - \xi'(s_0)\eta(s_0)$  is bounded away from zero for  $0 \leq s_0 \leq \alpha$ . Also  $\psi(s_0) \mp z^*, \psi'(s_0)$  are bounded (in fact, the boundedness of  $\psi'(s_0)$  follows from the fact that  $\varphi(x)$  is uniformly Lipschitz continuous). Thus there exists a constant  $M$  such that the functions  $|c_{1\pm}|, |c_{2\pm}|$  of  $s_0$  are majorized by  $M$ .

For  $\delta > 0$ , let

$$\omega(\delta) = \sup (|\xi'(s) - \xi'(s_1)| + |\eta'(s) - \eta'(s_1)|)$$

for  $|s - s_1| \leq \delta, 0 \leq s < s_1 \leq \alpha$ . Thus

$$|f'_\pm(s) - f'_\pm(s_0)| \leq M\omega(|s - s_0|).$$

Consequently, by (4.5),

$$|\psi(s) - \psi(s_0) - \psi'(s_0)(s - s_0)| \leq M\omega(|s - s_0|) |s - s_0|.$$

Interchanging  $s, s_0$  and adding gives

$$|\psi'(s) - \psi'(s_0)| \leq 2M\omega(|s - s_0|).$$

This proves Corollary 4.2 if  $n = 2$ .

*Proof of Corollary 4.2,  $n > 2$ .* When  $n > 2$ , it is necessary to estimate the degree of continuity of the directional derivatives of  $\varphi$  not only in the direction of the derivative but also in directions orthogonal to it.

It suffices to deal with  $\varphi(x)$  in a neighborhood of a given point of  $\Gamma$ . Choose a coordinate system in  $R^n$  with the origin at such a point and such that  $\Omega$  is in the half-space  $x^n \geq 0$ . Then, in the neighborhood of the origin,  $\Gamma$  has a parametrization of the form

$$(4.6) \quad x^n = \zeta(x^1, \dots, x^{n-1}) \quad \text{for} \quad |x^j| \leq \varepsilon, \quad j = 1, \dots, n-1,$$

where  $\zeta$  is of class  $C^1$  or  $C^{1,\lambda}$  in case (i) or (ii).

Write  $\xi$  for  $(\xi^1, \dots, \xi^{n-1})$ , where  $\xi^i = x^i$  for  $i = 1, \dots, n-1$  and  $\psi(\xi) = \varphi(x) = \varphi(\xi, \zeta(\xi))$  for  $x \in \Gamma$ . It has to be shown that  $\psi$  is correspondingly of class  $C^1$  or  $C^{1,\lambda}$ . The proof of Corollary 4.1 shows that  $\psi_{\xi_i} = \partial\psi/\partial\xi^i, i = 1, \dots, n-1$ , exist at every point.

Let  $\gamma > 0$  be chosen so that

$$(4.7) \quad x^* = (0, \dots, 0, \gamma) \in \Omega.$$

Let  $x_j = (\xi_j, \zeta(\xi_j)) = (x_j^1, \dots, x_j^n)$ , where  $j = 0, 1, \dots, n$ , be  $n+1$  points of  $\Gamma$ . The analogue of (3.4) is

$$(4.8) \quad \Delta_0(x_0, \dots, x_n) = (-1)^n \begin{vmatrix} x_1^1 - x_0^1 & \dots & x_1^n - x_0^n \\ x_2^1 - x_0^1 & \dots & x_2^n - x_0^n \\ \dots & \dots & \dots \\ x_n^1 - x_0^1 & \dots & x_n^n - x_0^n \end{vmatrix}$$

and that of (3.5) is

$$(4.9) \quad \Delta(x^*, z) = (-1)^{n+1} \begin{vmatrix} x_0^1 & x_0^2 & \dots & x_0^{n-1} & x_0^n - \gamma & \tau \\ x_1^1 & x_1^2 & \dots & x_1^{n-1} & x_1^n - \gamma & \varphi(x_1) - z \\ \dots & \dots & \dots & \dots & \dots & \dots \\ x_n^1 & x_n^2 & \dots & x_n^{n-1} & x_n^n - \gamma & \varphi(x_n) - z \end{vmatrix},$$

where  $\tau = \varphi(x_0) - z$  and  $x^* = (0, \dots, 0, \gamma)$ . Thus (3.6) holds for  $z =$

$\pm N, |x_j^k| \leq \varepsilon$ , and  $j, k = 0, 1, \dots, n - 1$ .

In (4.8) and (4.9), consider  $\gamma, z, x_1, \dots, x_n$  as fixed for the moment and  $x_0$  (or rather  $\xi_0 = (\xi_0^1, \dots, \xi_0^{n-1})$ ) as variable. In (4.9), let  $\tau$  be replaced by

$$(4.10) \quad f_{\pm}(\xi_0) = \sum_{k=1}^{n-1} c_{k\pm} \xi_0^k + c_{n\pm} [\zeta(\xi_0) - \gamma],$$

where  $c_{1\pm}, \dots, c_{n\pm}$  are chosen, if possible, so that

$$(4.11) \quad f_{\pm}(\xi_j) = \varphi(x_j) - z, \quad z = \pm N, \quad j = 1, \dots, n.$$

Then the analogue of the determinant (4.9) vanishes and so,  $\Delta(x^*, z)$  is not changed if the last column is replaced by  $\tau - f_{\pm}(\xi_0), 0, \dots, 0$ . In this case, we conclude from (3.6) that  $\tau = \varphi(x_0) \mp N$  satisfies

$$\mp N \Delta_0(x_0, x_1, \dots, x_n) \Delta_0(x^*, x_1, \dots, x_n) [(\varphi(x_0) \mp N) - f_{\pm}(\xi_0)] \geq 0.$$

Thus according as

$$(4.12) \quad \Delta_0(x_0, x_1, \dots, x_n) \Delta_0(x^*, x_1, \dots, x_n) > 0 \text{ or } < 0,$$

we have

$$(4.13) \quad \varphi(x_0) - N \leq f_+(\xi_0) \quad \text{or} \quad \varphi(x_0) - N \geq f_+(\xi_0)$$

and

$$(4.14) \quad \varphi(x_0) + N \geq f_-(\xi_0) \quad \text{or} \quad \varphi(x_0) + N \leq f_-(\xi_0).$$

If the points  $x_1, \dots, x_n$  are not in an  $(n - 2)$ -dimensional plane  $\pi_{n-2}$ , then  $\Delta_0(x^*, x_1, \dots, x_n) \neq 0$ . It will be supposed that  $x_1, \dots, x_n$  are enumerated so that

$$(4.15) \quad \Delta_0(x^*, x_1, \dots, x_n) > 0.$$

Then the coefficients  $c_{k\pm}$  of (4.10) can be uniquely determined so that (4.11) holds. The alternative (4.12) is now equivalent to

$$(4.16) \quad \Delta_0(x_0, x_1, \dots, x_n) > 0 \quad \text{or} \quad < 0.$$

This, in turn, is equivalent to

$$(4.17) \quad \begin{array}{l} \text{the line segment } [x^*x_0] \text{ does not or does meet the} \\ \pi_{n-1} \text{ determined by } x_1, \dots, x_n; \text{ i.e., } x_0 \text{ is or is not} \\ \text{on the same side of } \pi_{n-1} \text{ as } x^*. \end{array}$$

Let  $x_1 = (\xi_1, \zeta(\xi_1))$  be fixed and  $h > 0$  small. Choose  $\xi_j = h e_{j-1} + \xi_1$ , for  $j = 2, \dots, n$ , where  $e_j = (0, \dots, 0, 1, 0, \dots, 0)$  and the 1 is in the  $j$ -th place. Correspondingly,  $x_j = (\xi_j, \zeta(\xi_j))$ . Then (4.15) holds (e.g., if  $x_1 = 0$ , then  $\Delta(x^*, x_1, \dots, x_n)$  reduces to  $\gamma h^{n-1} > 0$ ). The equations

(4.11) for  $c_{k\pm}$  are equivalent to

$$(4.18) \quad \sum_{k=1}^{n-1} c_{k\pm} \xi_1^k + c_{n\pm} [\zeta(\xi_1) - \gamma] = \psi(\xi_1) \mp N,$$

$$c_{j\pm} h + c_{n\pm} [\zeta(\xi_{j+1}) - \zeta(\xi_1)] = \psi(\xi_{j+1}) - \psi(\xi_1),$$

for  $j = 1, \dots, n - 1$ . For these choices of  $c_{k\pm}$ , the first inequality in both (4.13), (4.14) hold if the segment  $[x^*x_0]$  does not meet  $\pi_{n-1}$  containing  $x_1, \dots, x_n$ .

If the last  $n - 1$  equations of (4.18) are divided by  $h$  and  $h \rightarrow 0$ , it follows that the solutions  $c_{1\pm}, \dots, c_{n\pm}$  of (4.18) tend to the unique solutions of the equations

$$(4.19) \quad \sum_{k=1}^{n-1} c_{k1} \xi_1^k + c_{n\pm} [\zeta(\xi_1) - \gamma] = \psi(\xi_1) \mp N,$$

$$c_{j\pm} + c_{n\pm} \zeta_j(\xi_1) = \psi_j(\xi_1),$$

for  $j = 1, \dots, n - 1$ , where  $\zeta_j = \partial\zeta/\partial\xi^j$ . Also, the  $\pi_{n-1}$  containing  $x_1, \dots, x_n$  tends to the tangent plane of  $\Gamma$  at  $x_1$ .

Thus, if  $c_{1\pm}, \dots, c_{n\pm}$  are chosen as the solution of (4.19), then, in addition, to (4.19),

$$(4.20) \quad f_-(\xi) - N \leq \psi(\xi) \leq f_+(\xi) + N$$

for all  $\xi$ ,  $|\xi^j| \leq \varepsilon$ . Actually, one first obtains (4.20) for all  $\xi$  such that  $x = (\xi, \zeta(\xi))$  is not on the tangent plane  $\pi_{n-1}$  to  $\Gamma$  at  $x_1$ . By continuity considerations, (4.20) holds also for the limits of such points. On the other hand, if  $\pi_{n-1} \cap \Gamma$  contains interior points, then  $\varphi(x)$  is a linear function of  $x \in \pi_{n-1} \cap \Gamma$  and (4.20) is trivial for those  $\xi$  for which  $x = (\xi, \zeta(\xi)) \in \pi_{n-1} \cap \Gamma$ .

Let  $\omega(\delta)$  be a monotone majorant for the degree of continuity of  $\zeta_j = \partial\zeta/\partial\xi^j$ ,  $j = 1, \dots, n - 1$ . Then arguing as at the end of the proof of the case  $n = 2$ , it is seen that there is a constant  $M$  such that the degree of continuity of the partial derivatives of  $f_{\pm}(\xi) - N$  is majorized by  $M\omega(\delta)$ . (For in the matrix of coefficients of (4.19), the first row is the vector  $x_1 - x^*$  from the point  $x^*$  to the point  $x_1 \in \Gamma$ , the second row is the vector  $(1, 0, \dots, 0, \zeta_1(\xi_1))$  which is a tangent vector to  $\Gamma$  at  $x_1$ , etc., so that the determinant of this matrix is bounded away from zero). Thus,

$$(4.21) \quad \left| \psi(\xi) - \psi(\xi_1) - \sum_{k=1}^{n-1} \psi_k(\xi_1)(\xi^k - \xi_1^k) \right| \leq M\omega(\delta)\delta,$$

where

$$(4.22) \quad \delta = \max(|\xi^1 - \xi_1^1|, |\xi^2 - \xi_1^2|, \dots, |\xi^{n-1} - \xi_1^{n-1}|).$$

As in the proof in the case  $n = 2$ , this implies that



$$\left| \sum_{k=1}^{n-1} [\psi_k(\xi) - \psi_k(\xi_1)](\xi^k - \xi_1^k) \right| \leq 2M\omega(\delta)\delta.$$

In particular,

$$(4.23) \quad |\psi_k(\xi) - \psi_k(\xi + \delta e_k)| \leq 2M\omega(\delta).$$

The relations (4.19), (4.20) show that

$$(4.24) \quad |[f_+(\xi) + N] - [f_-(\xi) - N]| \leq 2M\omega(\delta)\delta.$$

Let  $k \neq j$  and let

$$\xi_1, \xi_2 = \xi_1 + \delta e_j, \quad \xi_3 = \xi_1 + \delta e_j + \delta e_k, \quad \xi_4 = \xi_1 + \delta e_k$$

be the vertices of a square. By (4.20), the quantity

$$\psi(\xi_1) + \psi(\xi_3) - \psi(\xi_2) - \psi(\xi_4) = \sum_{m=1}^4 (-1)^{m+1} \psi(\xi_m)$$

is bounded from above by

$$[f_+(\xi_1) + N] + [f_+(\xi_3) + N] - [f_-(\xi_2) - N] - [f_-(\xi_4) - N]$$

and there is an analogous bound from below. Hence, by (4.24),

$$\left| \sum_{m=1}^4 (-1)^{m+1} \psi(\xi_m) \right| \leq \left| \sum_{m=1}^4 (-1)^{m+1} [f_+(\xi_m) + N] \right| + 4M\omega(2\delta)\delta.$$

Since (4.23) implies that

$$\begin{aligned} |\psi(\xi_4) - \psi(\xi_1) - \delta \psi_k(\xi_1)| &\leq 2M\omega(\delta)\delta \\ |\psi(\xi_3) - \psi(\xi_1) - \delta \psi_k(\xi_2)| &\leq 2M\omega(\delta)\delta \end{aligned}$$

and similar relations hold if  $\psi$  is replaced by  $f_+$ , it follows that

$$\delta |\psi_k(\xi_1) - \psi_k(\xi_2)| \leq \delta |f_{+k}(\xi_1) - f_{+k}(\xi_2)| + 16M\omega(2\delta)\delta.$$

Consequently

$$|\psi_k(\xi_1) - \psi_k(\xi_1 + \delta e_j)| \leq 18M\omega(2\delta) \quad k \neq j.$$

This, together with (4.23), proves Corollary 4.2 for  $n > 2$ .

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