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A CLASS OF BISIMPLE INVERSE SEMIGROUPS

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The purpose of this paper is to study a certain generalization of the bicyclic semigroup and to determine the structure of some classes of bisimple (inverse) semigroups mod groups.

Let S be a bisimple semigroup and let E_S denote the collection of idempotents of S . E_S is said to be integrally ordered if under its natural order it is order isomorphic to I^0 , the nonnegative integers, under the reverse of their usual order. E_S is lexicographically ordered if it is order isomorphic to $I^0 \times I^0$ under the order $(n, m) < (k, s)$ if $k < n$ or $k = n$ and $s < m$. If \mathcal{H} is Green's relation and E_S is lexicographically ordered, $S/\mathcal{H} \cong (I^0)^4$ under a simple multiplication. A generalization of this result is given to the case where E_S is n -lexicographically ordered. The structure of S such that E_S is integrally ordered and the structure of a class of S such that E_S is lexicographically ordered are determined mod groups. These constructions are special cases of a construction previously given by the author. This paper initiates a series of papers which take a first step beyond the Rees theorem in the structure theory of bisimple semigroups.

The theory of bisimple inverse semigroups has been investigated by Clifford [2] and Warne [7], [8], and [9].

If S is a bisimple semigroup such that E_S is lexicographically ordered, S/\mathcal{H} is shown to be isomorphic to the semigroup obtained by embedding the bicyclic semigroup C in a simple semigroup with identity by means of the Bruck construction [1]. We denote this semigroup by CoC . An interpretation of this construction introduced by the author in [10] is used.

In [2, p. 548, main theorem], Clifford showed that S is a bisimple inverse semigroup with identity if and only if $S \cong \{(a, b): a, b \in P\}$, where P is a certain right cancellative semigroup with identity isomorphic to the right unit subsemigroup of S , under a suitable multiplication and definition of equality. In the special case \mathcal{L} (Green's relation) is a congruence on P (equivalently, \mathcal{H} is a congruence on S), Warne showed [8, p. 1117, Theorem 2.1; p. 1118, Theorem 2.2 and first remark] that $P \cong U \times P/\mathcal{L}$, where U is the group of units of P (of S), under a Schreier multiplication or equivalently, $S \cong \{(a, b), (c, d): a, c \in U, b, d \in P/\mathcal{L}\}$. Warne also notes [8, p. 1118, second remark and p. 1121, Example 2] that a class of semigroups

¹ Some of the results given here have been stated in a research announcement in the Bull. Amer. Math. Soc. [12].

studied by Rees [6, p. 108, Theorem 3.3] may be substituted as a class of P in the above construction (here, $P/\mathcal{L} \cong (I^0, +)$, [8, p. 1118, Equation 2.9]). By [2, p. 553, Theorem 3.1], this substitution will yield the multiplication for the class of bisimple (inverse with identity) semigroups such that E_s is integrally ordered in terms of ordered quadruples. We carry out the indicated calculations, which are routine, in detail here to yield Equation 3.4, which with the equality definition $((g, n), (h, m)) = ((g_1, n_1), (h_1, m_1))$ if $gg_1^{-1} = hh_1^{-1}$, $n = n_1$ and $m = m_1$, is the structure theorem in terms of ordered quadruples. (The author was aware of this result in the spring of 1963.)

N. R. Reilly informed us he had a multiplication for these semigroups (*, p. 572) in terms of ordered triples. His elegant formulation follows from our quadruple formulation by an application of [2, p. 548, Equation 1.2]. A still more convenient formulation is $S \cong U \times C$ with a suitable multiplication.²

Next, it is shown that for a class of bisimple semigroups S such that E_s is lexicographically ordered, $S \cong GX(CoC)$, where G is a certain group, under a suitable multiplication. The above techniques of [8] are again utilized here. The greater generality achieved in the integrally ordered case appears to arise from the fact that in this case P is a splitting extension of U by I^0 (i.e., in notation of [8, p. 1117], $a^b = e$, the identity of U for all $a, b \in I^0$).³

These structure theorems resemble the Rees theorem for completely simple semigroups [3] in that they completely describe the structure or certain classes of bisimple semigroups mod groups.

\mathcal{R} , \mathcal{L} , \mathcal{H} , and \mathcal{D} will denote Green's relations [3, p. 47]. R_a denotes the equivalence class containing the element a . Unless otherwise stated, the definitions and terminology of [3] will be used.

1. Preliminary discussion. We first summarize the construction of Clifford referred to in the introduction.

Let S be a bisimple inverse semigroup with identity. Such semigroups are characterized by the following conditions [8, p. 1111; 3, 4, 2 are used].

- A1: S is bisimple.
- A2: S has an identity element.
- A3: Any two idempotents of S commute.

It is shown by Clifford [2] that the structure of S is determined by that of its right unit semigroup P and that P has the following properties:

- B1: The right cancellation law holds in P .
- B2: P has an identity element

² See p. 576, (2).

³ See p. 576, (3), (5).

B3: The intersection of two principal left ideals of P is a principal left ideal of P .

Let P be any semigroup satisfying B1, B2 and B3. From each class of \mathcal{L} -equivalent elements of P , let us pick a fixed representative. B3 states that if a and b are elements of P , there exists c in P such that $Pa \cap Pb = Pc$. c is determined by a and b to within \mathcal{L} -equivalence. We define avb to be the representative of the class to which c belongs. We observe also that

$$(1.1) \quad a \vee b = b \vee a .$$

We define a binary operation $*$ by

$$(1.2) \quad (a*b)b = a \vee b$$

for each pair of elements a, b of P .

Now let $P^{-1}oP$ denote the set of ordered pairs (a, b) of elements of P with quality defined by

$$(1.3) \quad (a, b) = (a', b') \text{ if } a' = ua \text{ and } b' = ub \text{ where } u \text{ is a unit in } P \text{ (} u \text{ has a two sided inverse with respect to } 1, \text{ the identity of } P \text{)} .$$

We define product in $P^{-1}oP$ by

$$(1.4) \quad (a, b)(c, d) = ((c*b)a, (b*c)d) .$$

Clifford's main theorem states: Starting with a semigroup P satisfying B1, 2, 3, Equations (1.2), (1.3), and (1.4) define a semigroup $P^{-1}oP$ satisfying A1, 2, 3. P is isomorphic with the right unit subsemigroup of $P^{-1}oP$ (the right unit subsemigroup of $P^{-1}oP$ is the set of elements of $P^{-1}oP$ having a right inverse with respect to 1; this set is easily shown to be a semigroup). Conversely, if S is a semigroup satisfying A1, 2, 3, its right unit subsemigroup P satisfies B1, 2, 3 and S is isomorphic to $P^{-1}oP$.

The following results are also obtained:

LEMMA 1.1 [2]. *For a, b in P and u, v in U , the group of units of P , we have $(ua*vb)v = a*b$. The unit group of P is equal to the unit group of S . $a\mathcal{H}b$ (in S) if and only if $a\mathcal{L}b$ (in P). $a\mathcal{L}b$ (in P) if and only if $a = ub$ for some u in U .*

THEOREM 1.1 [2]. *Let S be a semigroup satisfying A1, 2, 3, and let P be its right unit subsemigroup. Then P satisfies B3 (as well as B1 and B2), and the semi-lattice of principal left ideals of P under intersection is isomorphic with the semi-lattice of idempotent*

elements of S .

We now briefly review the work of Rédei [5] on the Schreier extension theory for semigroups (we actually give the right-left dual of his construction) and we also present some pertinent material from [8]. Let G be a semigroup with identity e . We consider a congruence relation ρ on G and call the corresponding division of G into congruence classes a compatible class division of G . The class H containing the identity is said to be the main class of the division. H is easily shown to be a subsemigroup of G . The division is called right normal if and only if the classes are of the form,

$$(1.5) \quad Ha_1, Ha_2, \dots (a_i = e)$$

and $h_1a_i = h_2a_i$ with h_1, h_2 in H implies $h_1 = h_2$. The system (1.5) is shown to be uniquely determined by H . H is then called a right normal divisor of G and G/ρ is denoted by G/H .

Let G, H , and S be semigroups with identity. Then, if there exists a right normal divisor H' of G such that $H \cong H'$ and $S \cong G/H'$, G is said to be a Schreier extension of H by S .

Now, let H and S be semigroups with identities E and e respectively. Consider HXS under the following multiplication:

$$(1.6) \quad (A, a)(B, b) = (AB^a a^b, ab)(A, B \text{ in } H; a, b \text{ in } S) \\ a^b, B^a(\text{in } H)$$

designate functions of the arguments a, b and B, a respectively, and are subject to the conditions

$$(1.7) \quad a^e = E, e^a = E, B^e = B, E^a = E.$$

We call $H \times S$ under this multiplication a Schreier product of H and S and denote it by HoS .

Rédei's main theorem states:

THEOREM 1.2 (Redei). *A Schreier product $G = HoS$ is a semigroup if and only if*

$$(1.8) \quad (AB)^c = A^c B^c (A, B \text{ in } H; c \text{ in } S)$$

$$(1.9) \quad (B^a)^c c^a = c^a B^{ca} (B \text{ in } H; a, c \text{ in } S)$$

$$(1.10) \quad (a^b)^c c^{ab} = c^a (ca)^b (a, b, c \text{ in } S)$$

are valid. These semigroups (up to an isomorphism) are all the Schreier extensions of H by S and indeed the elements (A, e) form a right normal divisor H' of G for which

$$(1.11) \quad \begin{aligned} G/H' &\cong S(H'(E, a) \rightarrow a) \\ H' &\cong H((A, e) \rightarrow A) \end{aligned}$$

are valid.

THEOREM 1.3 [8]. *Let U be a group with identity E and let S be a semigroup satisfying B1 and B2 (denote its identity by e) and suppose S has a trivial group of units. Then every Schreier extension $P = UoS$ of U by S satisfied B1 and B2 (the identity is (E, e)) and the group of units of P is $U' \cong \{(A, e) : A \text{ in } U\} \cong U$. Furthermore \mathcal{L} is a congruence relation on P and $P/\mathcal{L} \cong S$. P satisfies B3 if and only if S satisfies B3.*

Conversely, let P be a semigroup satisfying B1 and B2 on which \mathcal{L} is a congruence relation. Let U be the group of units of P . Then U is a right normal divisor of P and $P/U \cong P/\mathcal{L}$. Thus, P is a Schreier extension of U by P/\mathcal{L} . P/\mathcal{L} satisfies B1 and B2 and has a trivial group of units.

The following statements are valid for any semigroup obeying the conditions of Theorem 1.3 (i.e. semigroups satisfying B1, B2 on which \mathcal{L} is a congruence).

$$(1.12) \quad P(A, a) = \{(C, ba) : C \text{ in } U, b \text{ in } S\}.$$

$$(1.13) \quad (A, a)L(B, b) \text{ if and only if } a = b.$$

As remarked in [8], the semigroups considered by Rees (Theorem 1.5 below) fall into this category.

Now, Rees defines a *right normal divisor* in a different manner than Rédei. He says that V is a right normal divisor of a semigroup P satisfying B1 and B2 if V is a subgroup of the unit group U of P and $aU \subseteq Ua$ for all a in P . However, let us show that the Rees definition is just a specialization of the Rédei definition to the case where the main class is a group and the semigroup we are dealing with satisfies B1 and B2. In this case, suppose that V is a right normal divisor in the sense of Rédei. Then, clearly, V is a subgroup of U . The congruence class containing a is just Va . Let u in V . Then, $u\rho 1$. Thus, $au\rho a$, i.e., au in Va . Conversely, suppose V is a right normal divisor in the sense of Rees. Let us define $a\rho b$ if and only if $Va = Vb$. It is easily seen that ρ is a congruence on P with main class V , i.e., V is a right normal divisor in the sense of Rédei.

Let us now briefly review the theory of Rees [6]. Let P be a semigroup satisfying B1 and B2. The partially ordered system of principal left ideals of P , ordered by inclusion, will be denoted by $O(P)$ and termed the ideal structure of P . If (O, \supseteq) is a partially

ordered set, we denote the set of all elements x of O satisfying $x \leq a$ by O_a and term such a set a section of O . Then we take as $P(O)$ the set of all order isomorphic mappings γ of $O(P)$ onto sections of $O(P)$. If U is the group of units of S , $M = (g \text{ in } U/xg \text{ in } Ux \text{ for all } x \text{ in } P)$ is the greatest right normal divisor of P .

The following theorems are established.

THEOREM 1.4 [6]. *If P has an ideal structure $O(P)$ and M is the right normal divisor just described, then there is a subsemigroup P' of $P(O)$ isomorphic to P/M . Further, every principal left ideal of $P(O)$ has a generator in P' .*

THEOREM 1.5 [6]. *A semigroup P satisfying B1 and B2 whose ideal structure is isomorphic with ϑ (the ideal structure of $(I^0, +)$) and whose group of units is isomorphic with a given group G is isomorphic with a semigroup $T = G \times I^0$ under the following multiplication (1.14) $(g, m)(h, n) = (g(h\alpha^m), m + n)$, g, h in G , m, n in I^0 , α being an endomorphism of G , α^0 being interpreted as the identity transformation of G and conversely T has the above properties.*

LEMMA 1.2. *Let S be a bisimple inverse semigroup with identity with right unit subsemigroup P . U , the group of units of P , is a right normal divisor of P if and only if \mathcal{L} is a congruence on S .*

Proof. Let U be a right normal divisor of P . Let $(a, b), (c, d)$ be in S and suppose that $(a, b)\mathcal{L}(c, d)$. Now $(a, b)\mathcal{R}(c, d)$ if and only if $a = uc$ where u in U and $(a, b)\mathcal{L}(c, d)$ if and only if $b = vd$ where v in U . I will prove the first. Suppose that $(a, b)\mathcal{R}(c, d)$. Then there exists $(x, y), (w, z)$ in S such that $(a, b) = (c, d)(x, y)$ and $(c, d) = (a, b)(w, z)$. Thus, by 1.3 and 1.4 $a = p(x*d)c$ and $c = q(w*b)a$ where p, q in U . Thus, by B1 and B2 $a = uc$ for some u in U by B1 and B2. Now suppose that $a = u'c$ for some u' in U . We note first that $(b*b)b = b \vee b = ub$ for some u in U by 1.2, the definition of \vee , and Lemma 1.1. Thus, $b*b = u$ by B1.

Now $(a, b)(b, u'd) = (ua, uu'd) = (u'^{-1}a, d) = (c, d)$ by (1.3). Similarly $(c, d)(d, u'^{-1}b) = (a, b)$, i.e., $(a, b)\mathcal{R}(cd)$.

Let (p, q) be in S . Then by (1.4),

$$\begin{aligned} (a, b)(p, q) &= ((p*b)a, (b*p)q) \\ (c, d)(p, q) &= ((p*d)c, (d*p)q) \end{aligned}$$

Since $(a, b)\mathcal{L}(c, d)$ there exists u, v in U such that $a = uc, b = vd$. Thus, by Lemma 1.1 and the fact that U is a right normal divisor

$$(p*b)a = (p*vd)uc = (1p*vd)vv^{-1}uc = (p*d)v^{-1}uc = t(p*d)c ,$$

where t is in U .

Thus, $(a, b)(p, q)\mathcal{R}(c, d)(p, q)$ and \mathcal{R} is a right congruence. Since \mathcal{R} is always a left congruence, it is a congruence. One shows similarly that \mathcal{L} is a congruence. Thus, \mathcal{H} is a congruence relation on S .

Suppose \mathcal{H} is a congruence on S . Let a, b in P and suppose $a\mathcal{L}b$ (in P). By Lemma 1.1 $a\mathcal{H}b$ (in S). Thus c in P implies $ca\mathcal{H}cb$ (in S) and $ca\mathcal{L}cb$ (in P) by Lemma 1.1. Hence \mathcal{L} is a congruence on P and U is a right normal divisor of P by Theorem 1.3.

2. **The Bruck product.** Let S be an arbitrary semigroup and C be the bicyclic semigroup ([3], p. 43), i.e., C is the set of all pairs of nonnegative integers with multiplication given by $(m, n)(m', n') = (m + m' - \min(n, m'), n + n' - \min(n, m'))$. Consider $W = C \times S$ with multiplication given by $((m, n), s)((m', n'), s') = ((m, n)(m', n'), f(n, m'))$ where $f(n, m') = s, ss',$ or s' according to whether $n > m', n = m',$ or $n < m'$. We call W the Bruck product of C and S and write $W = CoS$. I used a special case of this product in [10]. CoC is easily shown to be a bisimple inverse semigroup with identity for which E_s is lexicographically ordered. If S is an arbitrary semigroup, let S^1 be S with an appended identity [3, p. 4]. One can show that CoS^1 is a simple semigroup with identity containing S as a subsemigroup. Since this is equivalent to the construction of R. H. Bruck [1] for embedding an arbitrary semigroup in a simple semigroup with identity, we call o a Bruck product.

THEOREM 2.1 [8]. *Let S and S^* be bisimple inverse semigroups with identity with right unit subsemigroups P and P^* respectively. $S \cong S^*$ if and only if $P \cong P^*$.*

THEOREM 2.2. *Let S be a bisimple (inverse) semigroup. E_s is lexicographically ordered if and only if \mathcal{H} is a congruence on S and $S/\mathcal{H} \cong CoC$ where CoC denotes the Bruck product of C by C .*

Proof. First we suppose that E_s is lexicographically ordered. Clearly S has an identity. For let e be the largest element of the lexicographic chain. If a in S , a is in R_f for some f in E_s since S is regular. Then, $ea = efa = fa = a$. Similarly, $ae = a$. Let P be the right unit subsemigroup of S . Then by Theorem 1.1, we may write the ideal structure of $P, O(P)$ as follows:

$$\begin{aligned} (0, 0) &> (0, 1) > (0, 2) > (0, 3) > \\ (1, 0) &> (1, 1) > (1, 2) > (1, 3) > \\ (2, 0) &> (2, 1) > (2, 2) > (2, 3) > \end{aligned}$$

$$\begin{aligned} (3, 0) &> (3, 1) > (3, 2) > (3, 3) > \\ (4, 0) &> (4, 1) > (4, 2) > (4, 3) > \end{aligned}$$

If we define for (m, k) in $O(P)$

$$\begin{aligned} (n, s)t_{(m,k)} &= (n + m, s) \text{ if } n > 0 \\ &= (m, s + k) \text{ if } n = 0 \end{aligned}$$

we easily see that $t_{(m,k)}$ is an order isomorphism of $O(P)$ onto the section of $O(P)$ determined by (m, k) . In fact all order isomorphisms of $O(P)$ onto sections of $O(P)$ are of this form.

Clearly $P(O) \cong I^0XI^0$ under the multiplication

$$\begin{aligned} (n, s)(m, k) &= (n + m, s) \text{ if } n > 0 \\ &= (m, s + k) \text{ if } n = 0. \end{aligned}$$

Thus, the only subsemigroup of $P(O)$ containing a generator of every principal left ideal of $P(O)$ is $P(O)$ itself. This follows since $P(O)(n, k) = ((u + n, v): u, v \text{ in } I^0, u > 0)U((n, v + k): v \text{ in } I^0)$. The unit group of $P(O)$ is trivial (note the identity of $P(O)$ is $(0, 0)$).

By Theorem 1.4, $P/M \cong P(O)$. Since the unit group of $P(O)$ is trivial, $M = U$. Thus, again by Theorem 1.4, U is a right normal divisor of P . Thus, \mathcal{H} is a congruence on S by Lemma 1.2. Since ([8], p. 1111) any homomorphic image of a bisimple inverse semigroup with identity is a bisimple inverse semigroup with identity, S/\mathcal{H} is such a semigroup.

Let $a \rightarrow \bar{a}$ denote the natural homomorphism of S onto S/\mathcal{H} . If \bar{a} is a right unit of S/\mathcal{H} there exists \bar{x} in S/\mathcal{H} such that $\bar{a}\bar{x} = \bar{1}$, where 1 is the identity of S . Thus, $ax\mathcal{H} = 1$ and there exists y in S such that $axy = 1$, i.e., a in P . Now, if a in P , $ax = 1$ for some x in S . Thus, $\bar{a}\bar{x} = \bar{1}$ and \bar{a} is in the right unit subsemigroup of S/\mathcal{H} . Hence the right unit subsemigroup of S/\mathcal{H} is $P/\mathcal{H} = P/\mathcal{L} \cong P(O)$ by Lemma 1.1. Now, as noted above CoC is a bisimple inverse semigroup with identity. It is easily seen that the right unit subsemigroup of CoC is isomorphic to $P(O)$. Thus, by Theorem 2.1 $S/\mathcal{H} \cong CoC$. The converse is clear.

COROLLARY 2.1. *S is a bisimple (inverse) semigroup with trivial unit group and E_s is lexicographically ordered if and only if S is isomorphic to CoC .*

Proof. This follows from Theorem 2.3 of [3].

LEMMA 2.1. *Let S be a bisimple (inverse) semigroup. E_S is integrally ordered if and only if \mathcal{H} is a congruence on S and $S/\mathcal{H} \cong C$.*

Proof. \mathcal{H} is a congruence on S by ([8], p. 1118) and Lemma 1.2. By Theorem 1.1, Theorem 1.5, 1.14, and 1.13, $P/\mathcal{L} \cong I^0$, where I^0 is the nonnegative integers under addition. But, as above, P/\mathcal{L} is the right unit subsemigroup of S/\mathcal{H} . Hence $S/\mathcal{H} \cong C$ by Theorem 2.1. The converse is clear.

LEMMA 2.2. *S is a bisimple (inverse) semigroup with trivial unit group and E_S integrally ordered if and only if $S \cong C$.*

Let S be a semigroup. We say E_S is n -lexicographically ordered if and only if E_S is order isomorphic to $\underbrace{I^0 \times I^0 \times \dots \times I^0}_n$ under the order

$$(k_1, k_2, \dots, k_n) < (s_1, s_2, \dots, s_n)$$

if $k_i > s_i$ or $k_i = s_i, k_2 > s_2$ or $k_i = s_i (i = 1, 2, j - 1), k_j > s_j$ or $k_i = s_i (i = 1, 2, n - 1), k_n > s_n$. E_S is 2-lexicographically ordered if and only if E_S is lexicographically ordered. E_S is 1-lexicographically ordered if and only if E_S is integrally ordered.

We will define the n -dimensional bicyclic semigroup C_n as follows: $C_1 = C$ and $C_n = (Co \dots o(Co(Co(CoC))))$ for $n > 1$ where o is the Bruck product (there are $n - 1$ o 's).

C_n is a bisimple inverse semigroup with E_{o_n} n -lexicographically ordered. The 1-dimensional bicyclic semigroup is the bicyclic semigroup. The 2-dimensional bicyclic semigroup is the Bruck product CoC of C and C .

The following theorem and corollary are obtained by employing the techniques used in the proofs of Theorem 2.1 and Corollary 2.1 respectively.

THEOREM 2.3. *S is a bisimple (inverse) semigroup with E_S n -lexicographically ordered if and only if \mathcal{H} is a congruence on S and $S/\mathcal{H} \cong C_n$.*

COROLLARY 2.2. *S is a bisimple (inverse) semigroup with E_S n -lexicographically ordered and trivial unit group if and only if $S \cong C_n$.*

3. Multiplications on two classes of bisimple inverse semigroups.

THEOREM 3.1. *S is a bisimple (inverse) semigroup such that E_s is integrally ordered if and only if $S \cong G \times C$ where G is a group and C is the bicyclic semigroup under the multiplication:*

$$(3.1) \quad (z, n, m)(z', n_1, m_1) = (z\alpha^{n_1-r}z'\alpha^{m-r}, (n, m)(n_1, m_1))$$

where $r = \min(m, n_1)$, α an endomorphism of G , α^0 is the identity transformation of G and juxtaposition is multiplication in G and C .

Proof. As in the proof of Theorem 1.7, S is a bisimple inverse semigroup with identity. By Theorem 1.1, Clifford's main theorem, and Theorem 1.5, $P \cong U \times I^0$ where U is the group of units of S under the multiplication 1.14 if and only if E_s is integrally ordered. The \mathcal{L} -classes of P are $L_0, L_1, L_2 \cdots L_n \cdots$ where $L_n = ((g, n): g \text{ in } U)$ by 1.13. Let (e, n) where e is the identity of U be a representative element of L_n . Thus, $(e, n) \vee (e, m) = (e, \max(n, m))$ by 1.12 and the definition of \vee . Using (1.2) by a routine calculation, we have

$$(3.2) \quad \begin{aligned} (e, n) * (e, m) &= (e, n - m) \text{ if } n \geq m \\ &= (e, o) \quad \text{if } m \geq n \end{aligned}$$

Using Lemma 1.1, (1.14), and Theorem 1.3, we obtain

$$(3.3) \quad \begin{aligned} (g, n) * (h, m) &= (h^{-1}\alpha^{n-m}, n - m) \text{ if } n \geq m \\ &= (h^{-1}, o) \quad \text{if } m \geq n \end{aligned}$$

Now using (1.14) (1.4), and (3.3), we obtain

$$(3.4) \quad \begin{aligned} &((g, n), (h, m))((g_1, n_1), (h_1, m_1)) \\ &= ((h^{-1}g)\alpha^{n_1-r}, n_1 + n - r, (g_1^{-1}h_1)\alpha^{m-r}, m + m_1 - r). \end{aligned}$$

Now, by (1.3) and (3.4), we have

$$\begin{aligned} &(e, n, g^{-1}h, m)(e, n_1, g_1^{-1}h_1, m_1) \\ &= (e, n_1 + n - r, (g^{-1}h)\alpha^{n_1-r} (g_1^{-1}h_1)\alpha^{m-r}, m + m_1 - r) \end{aligned}$$

Let $z = g^{-1}h$ and $z' = g_1^{-1}h_1$. Then

$$*(n, z, m)(n_1, z', m_1) = (n + n_1 - r z\alpha^{n_1-r}, z'\alpha^{m-r}, m + m_1 - r)$$

or

$$(z, n, m)(z', n_1, m_1) = (z\alpha^{n_1-r}z'\alpha^{m-r}, (n, m)(n_1, m_1)).$$

The converse follows by Clifford's theorem.

To actually determine the multiplication on S , one determines P (we are actually given P here) and then places P in the Clifford construction. However, after one ascertains the multiplication, a very short proof of the fact can be given by the use of Theorem 1.6.

Alternative proof of Theorem 2.1. Let $S^* = G \times C$ be a groupoid with multiplication (3.1). We can show that S^* is a bisimple inverse semigroup with identity by routine calculation (we must go through this to prove the converse anyway). It is easily seen that the right unit subsemigroup P^* of S^* is isomorphic to P . Thus, $S \cong S^*$ by Theorem 2.1.

A semigroup with zero, 0 , is said to be 0-right cancellative if a, b, c in $S, c \neq 0, ac = bc$ implies that $a = b$. If G is a group, let $\epsilon(G)$ denote the semigroup of endomorphisms of G .

A nontrivial group G is said to be a $*$ -group if

- (1) Every nontrivial endomorphism of G maps G onto G .
- (2) $\epsilon(G)$ is 0-right cancellative. ((1) \rightarrow (2) if G is an abelian group).⁴

The $*$ -groups include all cyclic groups of prime order, all groups of type p^∞ , and the additive group of rational numbers.⁵

If S is a semigroup with identity 1 and a, x in S with $ax = 1$, we write $x = a^{-1}$.

THEOREM 3.2. *S is a bisimple (inverse) semigroup such that (1) E_s is lexicographically ordered, (2) U is a $*$ -group, (3) $aa^{-1} = 1$ implies that $Ua \subseteq aU$, if and only if $S \cong GX(\text{Co}C)$ where G is a $*$ -group, C is the bicyclic semigroup, o is the Bruck product, with the multiplication,*

$$(g, (n, k), (m, l))(h, (n_1, k_1), (m_1, l_1)) \\ = (g\alpha^{n_1-r}h\alpha^{k-r}, ((n, k), (m, l))((n_1, k_1), (m_1, l_1)))$$

where $r = \min(n_1, k)$ and α is a nontrivial endomorphism of G α^0 denotes the identity transformation, and juxtaposition denotes multiplication in G and $\text{Co}C$.

Proof. Let P be the right unit subsemigroup of S . If U is a right normal divisor of P , then clearly \mathcal{L} is a congruence on P . Thus by Theorem 2.2 Lemma 1.2, and Theorem 1.3, P is a Schreier extension of U by $P/U(=P/\mathcal{L})$. Now, the semigroup of right units P^* of $\text{Co}C$ is easily seen to be isomorphic to $I^0 \times I^0$ under the multiplication

$$(n, m)(p, q) = (n + p, m) \text{ if } n > 0 \\ (n + p, m + q) \text{ if } n = 0$$

Now $a = (1, o)$ and $b = (o, 1)$ are generators of P^* and $ab = a$. Now, as remarked in the proof of Theorem 2.2 the right unit subsemigroup

⁴ (1) \rightarrow (2) also if G is simple or finite.

⁵ The $*$ -groups also include all nontrivial finite simple groups.

of $S/\mathcal{H} \cong CoC$ (Theorem 2.2) is P/\mathcal{L} . Thus, we may label the \mathcal{L} -classes of P as $\{L_{(n,k)} : n, k \text{ in } I^0\}$. Now let a' in $L_{(1,0)}$ and b^* in $L_{(0,1)}$. Thus, $a'b^* = ua'$ for some u in U . Thus by (3) $ua' = a'v$ for some v in U . Hence, $a'b^* = a'v, a'b^*v^{-1} = a'$. Let $b^*v^{-1} = b'$. Now, since U is a right normal divisor of $P, b^*v^{-1} = wb^*$ for some w in U and b' in $L_{(0,1)}$. Thus, $\{b'^k a'^s, k, s \text{ in } I^0\}$ form a complete system of representative elements (5) which is also a semigroup. Thus the factors c^d of (1.6) are all equal to E , the identity of U . Thus, (1.6) becomes

$$(3.5) \quad (A, n, k)(B, m, l) = (AB^{(n,k)}, (n, k)(m, l))$$

where A, B in $U, (n, k), (m, l)$ in P/\mathcal{L} and juxtaposition is multiplication in U and P/\mathcal{L} . Now let $a = (1, 0)$ and $b = (0, 1)$, and let $e = (0, 0)$, the identity of P/\mathcal{L} . Then $(E, a)(g, e) = (g\alpha, e)(E, a)$ (α , fixed), α a transformation of U , since U is a right normal divisor of P and $\{(g, e) : g \text{ in } U\}$ is isomorphic to U (Theorem 1.3). Now $(E, a)(g, e) = (g^a, a)$ by 1.6. Hence $g^a = g\alpha$. Similarly, $g^b = g\beta$. By (1.8) α and β are endomorphisms of U . By (1.9), $(g^b)^a = g^{ab} = g^a$ (g in U). Thus $g\alpha = g\beta\alpha, g$ in U , i.e., $\alpha = \beta\alpha$. Let us first suppose that $\alpha \neq 0$ in $\varepsilon(U)$. Then since $\varepsilon(U)$ is 0-right cancellative β is the identity automorphism of U . Now, by 1.9, $g^{(n,k)} = g^{(0,1)^k(1,0)^n} = (g^{(1,0)^n})^{(0,1)^k} = g\alpha^n\beta^k = g\alpha^n$ and (3.5) becomes

$$(A, n, k)(B, m, l) = (A(B\alpha^n), (n, k)(m, l))$$

By routine calculation, we can show that $S^* = Ux(CoC)$ under the multiplication

$$(g, (n, k), (m, l))(h, (n_1, k_1), (m_1, l_1)) = (g\alpha^{n_1-r}h\alpha^{k-r}, (((n, k), (m, l))(n_1, k_1), (m_1, l_1))),$$

where $r = \min(n_1, k)$ and α is an endomorphism of U , is a bisimple inverse semigroup with identity. To show associativity is straight forward, but tedious. Now,

$$(g, (n, k), (m, l))\mathcal{S}(h, (n_1, k_1), (m_1, l_1)) \text{ if and only if } n = n_1 \text{ and } m = m_1$$

and

$$(g, (n, k), (m, l))\mathcal{L}(h, (n_1, k_1), (m_1, l_1)) \text{ if and only if } k = k_1 \text{ and } l = l_1$$

Thus, if

$$(g, (n, k), (m, l)), (h, (u, v), (r, s)) \text{ in } S^*, \\ (g, (n, k), (m, l))\mathcal{S}(g, (n, v), (m, s))\mathcal{L}(h, (u, v), (r, s))$$

and S^* is bisimple. $(E, (0, 0), (0, 0))$ where E is the identity of U is the identity of S^* .

The idempotents of S^* are $\{(E, (n, n), (k, k)), n, k \text{ in } I^0\}$. It is easily seen that these commute.

Thus, S^* is a bisimple inverse semigroup with identity, [8, p. 1111].

The right unit subsemigroup P^* of S^* is $\{(g, 0, n, 0, k) : n, k \text{ in } I^0, g \in G\}$. It is seen immediately that P^* is isomorphic to P and hence $S \cong S^*$ by Theorem 2.1. Let us give the converse of this case. Now it is quite easily seen that the unit group of S is $\{g, (0, 0), (0, 0)\} \cong G$. (the unit group is $H_{((0,0),(0,0))}$). Thus, U is a $*$ -group.

The right unit subsemigroup P of S is $\{(g, n, k) : n, k \text{ in } I^0\}$ under the multiplication

$$\begin{aligned} (g, n, k)(h, m, s) &= (g(h\alpha^n), n + m, k) \text{ if } n > 0 \\ (g, 0, k)(h, m, s) &= (gh, m, k + s) \end{aligned}$$

Let $(g, 0, 0) \in U$ and $(h, m, s), m > 0$ be in P . Since G is a $*$ -group, there exists g' in G such that $h^{-1}gh = g'\alpha^m$ (since α is nontrivial, α^m is nontrivial) as $\varepsilon(G)$ is 0-right cancellative). Thus

$$(g, 0, 0)(h, m, s) = (gh, m, s) = (h(g'\alpha^m), m, s) = (h, m, s)(g', 0, 0).$$

Next, we consider $(h, 0, m)$. Now, let $g' = h^{-1}gh$. Then,

$$(g, 0, 0)(h, 0, m) = (gh, 0, m) = (hg', 0, m) = (h, 0, m)(g', 0, 0)$$

Hence, U satisfies (3).

$$E_S = \{E, (n, n), (k, k) : n, k \text{ in } I^0\}$$

and multiplication in E_S is given by

$$\begin{aligned} (n, k)(m, l) &= (n, k) \text{ if } n > m \\ &= (n, k) \text{ if } n = m \text{ and } k > l. \end{aligned}$$

Thus (1) is satisfied.

Next, suppose α is the zero of $\varepsilon(U)$, i.e., $g\alpha = E, g \text{ in } U$. This means $g^\alpha = E, g \text{ in } U$. Now $g^{(n,k)} = g^{(0,1)^k(1,0)^n} = (g^{(1,0)^n})^{(0,1)^k} = (E)^{(0,1)^k} = E$ if $n \neq 0$. If $n = 0, g^{(n,k)} = g^{(0,k)} = g\beta^k$. Thus, our multiplication (3.5) becomes $(A, n, k)(B, m, s) = (A, n + m, k)$ if $n \neq 0,$

$$(A, 0, k)(B, m, s) = (A(B\beta^k), m, k + s) .$$

Now, by (3), if $(g, 0, 0)$ in U , there exists $(g', 0, 0)$ in U such that if $m \neq 0$

$$(g, 0, 0)(B, m, s) = (gB, m, s) = (B, m, s)(g', 0, 0) = (B, m, s) .$$

Hence, $gB = B$ and $g = E$. Since g was arbitrary, U is a trivial group and we have a contradiction. Thus α cannot be a trivial endomorphism.

EXAMPLE. Let G be a $*$ -group, C be the bicyclic semigroup, and \circ be the Bruck product. If we let α be the trivial endomorphism of G in the 1-dimensional (2-dimensional) case, S is a bisimple inverse semigroup with E_s integrally (lexicographically) ordered and with group of units a $*$ -group. However (3) of Theorem (3.2) is not satisfied. $S = CoG$ is the 1-dimensional case.

Added in proof. (1) A nontrivial group is called an e -group if every nontrivial endomorphism of G is an epimorphism. The following theorem has a proof similar to that of Theorem 3.2.

THEOREM. In Theorem 3.2, replace $*$ -group by e -group and the multiplication given there by

$$(g, (n, k), (m, 1))(h, (r, s), (u, v)) \\ = (g\alpha^{r-\delta}\beta^{u-\gamma_1(r,k)}h\alpha^{k-\delta}\beta^{1-\gamma_2(r,k)}, ((n, k), (m, 1))((r, s), (u, v)))$$

where if $r > k$, $\gamma_1(r, k) = 0$, $\gamma_2(r, k) = 1$; if $k > r$, $\gamma_1(r, k) = u$, $\gamma_2(r, k) = 0$; if $k = r$, $\gamma_1(r, k) = \gamma_2(r, k) = \min(u, 1)$, $\delta = \min(k, r)$ and α, β are nontrivial endomorphisms of G such that $\beta\alpha = \alpha$.

(2) N. R. Reilly [11] has determined a structure theorem equivalent to Theorem 3.1 by different methods. According to his terminology, a bisimple semigroup S is called a bisimple ω -semigroup if E_s is integrally ordered. If E_s is lexicographically ordered we will call S an L -bisimple semigroup.

(3) A bisimple semigroup S is L_n -bisimple (I -bisimple, I - ω -bisimple) if E_s is n -lexicographically ordered (is order isomorphic to Z under the reverse of the usual order, is order isomorphic to ZXI^0 under the usual lexicographic order [Van der Waerden, Vol. 1, p. 81]). We describe the structure of these classes of semigroups completely mod groups in [12], [13], and [16]. The structure theorem for L -bisimple semigroups generalizes Theorem 3.2. We investigate several of the properties of L -bisimple, I -bisimple and I - ω -bisimple semigroups, such as homomorphisms, congruences, and (ideal) extensions in [12], [13], [14], [17], and [18]. The method of attack- initiated here- which readily allows applications of results of [7]-[9] is used throughout.

(4) We will also call the n -dimensional bicyclic semigroup the $2n$ -cyclic semigroup in future papers.

(5) We have also studied some of the properties of the semigroups whose structure has been given here in [13] and [15].

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