SETS OF CONSTANT WIDTH

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A lower bound, better than those previously known, is given for the volume of a 3-dimensional body of constant width 1. Bounds are also given in the case of $n$-dimensional bodies of constant width 1, $n \geq 4$. Short proofs of the known sharp bounds for such bodies in the Euclidean and Minkowskian planes are given using properties of mixed areas. An application is made to a measure of outer symmetry of sets of constant width in 2 and 3 dimensions.

Let $K$ be a convex body in $n$-dimensional Euclidean space $E_n$. For each point $u$ on the unit sphere $S$ centered at the origin, let $b(u)$ be the distance between the two parallel supporting hyperplanes of $K$ orthogonal to the direction. The function $b(u)$ is the “width function” of $K$. If $b(u)$ is constant on $S$, then we say $K$ is a body of constant width.

If $K_1$ and $K_2$ are convex bodies, then $K_1 + K_2$ is the “Minkowski sum” or “vector sum” of $K_1$ and $K_2$ [5, p. 79]. The following useful theorem is well-known.

**Theorem 1.** A convex body $K$ has constant width $b$ if and only if $K + (-K)$ is a spherical ball of radius $b$.

In the case of $E_2$, a number of special properties of sets of constant width are known—for example, the following theorem of Pâl (see [5, p. 127]).

**Theorem 2.** Any plane convex body $B$ of constant width admits a circumscribed regular hexagon $H$.

We shall be concerned with the following type of result, due to Blaschke and Lebesgue (see [1], [3], [4], [5, p. 128], [9]).

**Theorem 3.** Any plane convex body $B$ of constant width 1 has area not less than $(\pi - \sqrt{3})/2$, the area of a Reuleaux triangle of width 1.

The following short proof of Theorem 3 will set the stage for some later arguments.

**Proof of Theorem 3.** Let $A(K)$ denote the area of $K$. The “mixed area” of the plane convex bodies $K_1$ and $K_2$, $A(K_1, K_2)$, can be
defined by the fundamental relation [5, p. 48],

\[ A(K_i + K_j) = A(K_i) + 2A(K_i, K_j) + A(K_j) \].

The mixed area is monotonic in each argument [5, p. 86]. That is, if \( K_i \subseteq K_j \), then

\[ A(K_i, K) \leq A(K_j, K) \].

It follows from (1), setting \( K_i = K_j = K \), that

\[ A(K, K) = A(K) \].

Now let \( H \) be a regular hexagon circumscribed about \( B \) (Theorem 2). Assume the center of \( H \) is the origin, so \( H = -H \). Then, using (2) and (3), we obtain

\[ A(B, -B) \leq A(H, -H) = A(H, H) = A(H) \].

Thus, by (4), (1), and Theorem 1, we have

\[ \pi = A(B + (-B)) = 2A(B, -B) \leq 2A(B) + 2A(H) = 2A(B) + \sqrt{3}, \]

from which the theorem follows.

It has long been conjectured that in \( E_3 \) any convex body of constant width 1 has volume at least that of a certain “tetrahedron of constant width” \( T \) (see [12, p. 81] for the construction of \( T \)). A computation of the volume of \( T \) leads to the conjecture,

**Conjecture 1.** Any 3-dimensional convex body of constant width 1 has volume not less than

\[ \frac{2\pi}{3} - \frac{\pi \sqrt{3}}{4} \cos^{-1}(1/3) \approx .42. \]

In §2 we shall prove that if \( B_3 \) is a 3-dimensional body of constant width 1, with volume \( V(B_3) \), then

\[ V(B_3) \geq \beta = \frac{\pi}{3} (3\sqrt{6} - 7) \approx .365. \]

Our proof of (6) will depend upon the following theorem of Blaschke [2].

**Theorem 4.** If a 3-dimensional convex body of constant width \( b \) has volume \( V \) and surface area \( S \), then

\[ 2V = bS - \frac{2\pi}{3} b^3. \]
It follows from (7) that Conjecture 1 is equivalent to:

Conjecture 1'. Any 3-dimensional convex body of constant width 1 has surface area not less than

$$2\pi - \frac{\pi \sqrt{3}}{2} \cos^{-1}(1/3).$$

Conjecture 1 can be transformed into still another form using the concept of "mixed surface area." Let $S(K)$ denote the surface area of $K$. If $K_1$ and $K_2$ are 3-dimensional convex bodies, then the surface area of $K_1 + K_2$ can be written in the form

$$S(K_1 + K_2) = S(K_1) + 2S(K_1, K_2) + S(K_2),$$

where $S(K_1, K_2)$ is the mixed surface area. Thus, if $K$ has constant width 1, $4\pi = S(K + (-K)) = 2S(K) + 2S(K, -K)$. Hence Conjectures 1 and 1' are equivalent to:

Conjecture 1''. Any 3-dimensional convex body of constant width 1 has mixed surface area not greater than

$$\frac{\pi \sqrt{3}}{2} \cos^{-1}(1/3).$$

Firey [6] has proved that the volume $V$ of an $n$-dimensional convex body of constant width 1 satisfies

$$V \geq \frac{\pi - \sqrt{3}}{n!}, \quad n \geq 2.$$  \hspace{1cm} (8)

In §2 we give the generally better lower bound,

$$V \geq \lambda \omega_n \prod_{k=3}^n \left(1 - \sqrt{\frac{k}{2k + 2}}\right), \quad n \geq 3,$$  \hspace{1cm} (9)

where $\omega_n$ is the volume of the unit ball in $E_n$, and

$$\lambda = \frac{\pi - \sqrt{3}}{2\pi}.$$  

Let $C$ be a centrally symmetric convex body centered at the origin in $E_n$. Then $C$ is the unit sphere for a Minkowskian geometry. We say that a body $K$ has "constant width relative to $C" if $K + (-K)$ is homothetic to $C$. In particular, one says that $K$ and $C$ are "equivalent in width" in case $K + (-K) = 2C$, since the condition implies that $K$ and $C$ have the same width function. When $C$ is the ordinary unit sphere we obtain the ordinary sets of constant width. Results about
plane sets of relative constant width analogous to Theorem 2 and 3 are known (see [8], [10], and [11]). In § 3 we give a proof of the analogue of Theorem 3 in the Minkowski plane, using the same method as in our proof of Theorem 3.

Section 4 is devoted to some results on measures of outer symmetry for sets of constant width.

2. Proof of (6). Let $B_3$ be a 3-dimensional convex body of constant width 1. Then the inscribed sphere of $B_3$ has radius $\geq 1 - \sqrt{3/8}$ (see [5, p. 125]). Assume that the center of the inscribed sphere is the origin. If $p(u)$ is the supporting function of $B_3$, then we have $p(u) \geq 1 - \sqrt{3/8}$. Hence,

$$ 3V(B_3) = \int_{S_3} p(u)dS(u) \geq \left(1 - \sqrt{3/8}\right)S(B_3), $$

where $S(B_3)$ is the surface area of $B_3$. Using Theorem 4 in (10), we obtain

$$ 3V(B_3) \geq \left(1 - \sqrt{3/8}\right)\left(2V(B_3) + \frac{2\pi}{3}\right), $$

and (6) follows upon solving (11) for $V(B_3)$. This completes the proof.

Proof of (9). Define

$$ \chi_\alpha = \inf V(K), $$

as $K$ ranges over all bodies of constant width 1 in $E_\alpha$, and $V(K)$ is the volume of $K$. The Blaschke selection principle implies that there exist bodies of constant width 1 having volume $\chi_\alpha$. Let $B$ be such a body, and let $p(u)$ be the support function of $B$ with the center of its inscribed sphere as origin. Then, by [5, p. 125],

$$ p(u) \geq 1 - \sqrt{\frac{n}{2n + 2}}. $$

Denoting the area element of $B$ by $dS(u)$, we have,

$$ n\chi_\alpha = nV(B) = \int_B p(u)dS(u) \geq \left(1 - \sqrt{\frac{n}{2n + 2}}\right)S(B), $$

where $S(B)$ is the surface area of $B$. If we denote by $B_u$ the projection of $B$ onto a hyperplane orthogonal to $u$, then (see [5, p. 89])

$$ S(B) = \frac{1}{\omega_n} \int V(B_u)du, $$

(14)
where $V(B_u)$ is the $(n - 1)$-dimensional volume of $B_u$ and the integration is over the surface of the unit sphere in $E_n$. Since $B_u$ is an $(n - 1)$-dimensional body of constant width 1, we have by (12) that $V(B_u) \geq \lambda_{n-1}$. Hence

\begin{equation}
S(B) \geq \frac{n\omega_n \lambda_{n-1}}{\omega_{n-1}} .
\end{equation}

Combined with (13), this yields

\begin{equation}
\lambda_n \geq \left(1 - \sqrt{\frac{n}{2n+2}} \right) \frac{\omega_n}{\omega_{n-1}} \lambda_{n-1} ,
\end{equation}

from which (9) follows. This completes the proof.

3. In this section, $C$ is a centrally symmetric plane convex body centered at the origin 0. $C$ admits an inscribed affine regular hexagon $H$ (i.e., the affine image of a regular hexagon) having a side parallel to any specified direction [10]. Let the vertices of $H$ be labelled $P_1, P_2, \ldots, P_6$ on the boundary of $C$ traversed in the positive direction. A “relative Reuleaux triangle” is obtained by attaching arcs $P_1P_2, P_2P_3$, and $P_5P_6$ of the boundary of $C$ to the respective sides $P_1P_2, P_2P_3, 0P_1$ of the triangle $OP_1$. With $H$ as above, a centrally symmetric hexagon circumscribed about $C$ and touching $C$ at $P_i$, $1 \leq i \leq 6$, is called a “$C$-hexagon.” In fact, any hexagon homothetic to such a hexagon will be called a $C$-hexagon. Note that if case $C$ is a circle, any $C$-hexagon is just a regular hexagon. One then sees that the following theorem from [10] is a Minkowskian geometry analogue of Theorem 2.

**Theorem 2'.** Let $K$ be equivalent in width to $C$. Then $K$ admits a circumscribed $C$-hexagon.

Let $H$ be a $C$-hexagon circumscribed about $C$. Let $H'$ be the corresponding affine regular hexagon inscribed in $C$ with its vertices on $H$. Then we shall show that

\begin{equation}
A(H) \leq \frac{4}{3} A(H') .
\end{equation}

This follows from the following general lemma.

**Lemma 1.** Let $H'$ be an affine regular hexagon inscribed in a centrally symmetric plane convex body $K$. Then

\begin{equation}
A(K) \leq \frac{4}{3} A(H') .
\end{equation}

**Proof.** By considering the support lines of $K$ through the vertices of $H'$, one sees that it suffices to prove (18) for $K$ a centrally
symmetric hexagon $H$. Since the problem is affine invariant, one may even assume $H'$ is a regular hexagon, although this does not really simplify matters. In Figure 1, $P'_1, P'_2, P'_3, P'_4$ are consecutive vertices of $H'$, and $P_1, P_2, P_3$ are vertices of $H$. $AD$ is drawn parallel to $P'_1P_3$, which is parallel to $P'_3P'_1$ (the degenerate cases, where $P_3 = P'_1$ or $P_1 = P'_1$ are easily disposed of and will not be dwelt upon here). $B$ is the intersection of $P_1P_2$ with $AD$, and $C$ is the intersection of $P_3P_4$ with $AD$. Triangle $P'_1P_3P'_2$ is congruent to $P'_1CA$, and $P'_1P_3P'_2$ is congruent to $P'_1BA$. Hence the area of the pentagon $P'_1P_3P_2P_4P'_1$ is not greater than the area of triangle $P'_1AP'_1$, so the area of $H$ is not greater than twice the area of $P'_1AP'_1$, which is precisely $4/3 A(H')$. This completes the proof.

**Theorem 3'**. Any plane convex body $K$ which is equivalent in width to $C$ has area not less than that of some relative Reuleaux triangle equivalent in width to $C$.

*Proof.* It is easy to check that the area of any relative Reuleaux triangle $T$ equivalent in width to $C$ is given by

$$A(T) = 2A(C) - 4/3 A(H),$$

where $H$ is the affine regular hexagon inscribed in $C$ on which the construction of $T$ is based. Let $H'$ be a $C$-hexagon circumscribed about $K$ (Theorem 2'), let $H''$ be the translate of $H'$ circumscribed about $C$, and let $H$ be the corresponding affine regular hexagon inscribed in $C$ with its vertices on $H''$. Let the center of $H'$ be at the origin (which can be assumed by translating $K$) so $H' = -H'$. Then,
proceeding as in the proof of Theorem 3, and using (17), we have
\[
4A(C) = A(K + (-K)) = 2A(K) + 2A(-K)
\]
\[
\leq 2A(K) + 2A(H', -H') = 2A(K) + 2A(H')
\]
\[
= 2A(K) + 2A(H'') \leq 2A(K) + 8/3 A(H). 
\]
Hence,
\[
A(K) \geq 2A(C) - 4/3 A(H) = A(T). 
\]
This completes the proof.

To prove that a relative Reuleaux triangle is the only body equivalent in width to \( C \) with minimum area requires a little more argument. A sketch of the proof is as follows. If \( K \) is such a body of minimum area, then equality must hold throughout (20). This means that \( A(K, -K) = A(H') \) for a \( C \)-hexagon \( H' \) circumscribed about \( K \). It follows that \( A(-K, K) = A(H', K) \). If we let \( p_i(\theta), p_o(\theta) \) be the support functions of \( K \) and \( H' \) respectively, with origin at the center of \( H' \), and let \( s_i \) denote arc length along \( K \), the last equation implies that
\[
\int p_i(\theta + \pi)ds_i = \int p_i(\theta)ds_i. 
\]
Equation (22) implies that \( K \) must pass through 3 alternate vertices of \( H' \), from which readily follows the fact that \( K \) is a relative Reuleaux triangle.

4. For any \( n \)-dimensional convex body \( K \) we define a "coefficient of outer symmetry," \( \mu(K) \), as follows. Let \( S \) be a centrally symmetric convex body of minimum volume containing \( K \). Then
\[
\mu(K) = \frac{V(K)}{V(S)}, 
\]
Thus \( \mu(K) \leq 1 \), and \( \mu(K) = 1 \) if and only if \( K \) is centrally symmetric. Sharp lower bounds for \( \mu(K) \) are not known for \( n \geq 3 \); however, it is known that \( \mu(K) \geq 1/2 \) if \( K \) is 2-dimensional, with equality holding if and only if \( K \) is a triangle. A standing conjecture is that in \( E_n, n \geq 3, \mu(K) \geq \mu(T) \), where \( T \) is a simplex.

**Theorem 5.** Let \( B \) be a plane convex body of constant width 1. Then \( \mu(B) \geq \mu(R) \), where \( R \) is a Reuleaux triangle, and equality holds only if \( B \) is a Reuleaux triangle.

**Proof.** Let \( H \) be a regular hexagon circumscribed about \( B \).
Then, using Theorem 3, we have

\begin{equation}
\mu(B) \geq \frac{A(B)}{A(H)} \geq \frac{A(R)}{A(H)} = \frac{\pi - \sqrt{3}}{\sqrt{3}} = 0.81 \ldots .
\end{equation}

where \( R \) is a Reuleaux triangle of width 1. On the other hand, any centrally symmetric convex set \( S \) containing \( R \) must also contain an equilateral triangle \( T \) of side 1 and thus has area \( \geq 2A(T) = A(H) \). Hence

\begin{equation}
\frac{A(R)}{A(H)} = \mu(R).
\end{equation}

Equality can hold in (24) only if \( A(B) = A(R) \), which happens only if \( B \) is a Reuleaux triangle (see end of §3). This completes the proof.

It is known that any set \( K \) of constant width in \( E_3 \) admits a regular circumscribed octahedron \( J \) (see [7]). Suppose \( K \) has constant width 1, and let \( S \) be a centrally symmetric set of minimum volume containing \( K \). Then, using (6),

\begin{equation}
\mu(K) = \frac{V(K)}{V(S)} \geq \frac{\beta}{V(J)} = \frac{2\beta}{\sqrt{3}} \approx 0.42.
\end{equation}

Clearly one can obtain crude lower bounds, in this same fashion, in terms of \( \lambda \) and the volume of some centrally symmetric “covering body” \( J_n \) (one could, for example, use for \( J_n \) a sphere of radius \( \sqrt{n}/(2n+2) \)).

**References**


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