Pacific Journal of Mathematics

# SETS OF CONSTANT WIDTH

GULBANK D. CHAKERIAN

Vol. 19, No. 1 May 1966

PACIFIC JOURNAL OF MATHEMATICS
Vol. 19, No. 1, 1966

# SETS OF CONSTANT WIDTH

G. D. CHAKERIAN

**A lower bound, better than those previously known, is given for the volume of a 3-dimensional body of constant width 1. Bounds are also given in the case of $n$-dimensional bodies of constant width 1, $n \geqq 4$. Short proofs of the known sharp bounds for such bodies in the Euclidean and Minkowskian planes are given using properties of mixed areas. An application is made to a measure of outer symmetry of sets of constant width in 2 and 3 dimensions.**

Let $K$ be a convex body in $n$-dimensional Euclidean space $E_n$. For each point $u$ on the unit sphere $S$ centered at the origin, let $b(u)$ be the distance between the two parallel supporting hyperplanes of $K$ orthogonal to the direction. The function $b(u)$ is the "width function" of $K$. If $b(u)$ is constant on $S$, then we say $K$ is a body of constant width.

If $K_1$ and $K_2$ are convex bodies, then $K_1 + K_2$ is the "Minkowski sum" or "vector sum" of $K_1$ and $K_2$ [5, p. 79]. The following useful theorem is well-known.

THEOREM 1. *A convex body $K$ has constant width $b$ if and only if $K + (-K)$ is a spherical ball of radius $b$.*

In the case of $E_2$, a number of special properties of sets of constant width are known-for example, the following theorem of Pàl (see [5, p. 127]).

THEOREM 2. *Any plane convex body $B$ of constant width admits a circumscribed regular hexagon $H$.*

We shall be concerned with the following type of result, due to Blaschke and Lebesgue (see [1], [3], [4], [5, p. 128], [9]).

THEOREM 3. *Any plane convex body $B$ of constant width 1 has area not less than $(\pi - \sqrt{3})/2$, the area of a Reuleaux triangle of width 1.*

The following short proof of Theorem 3 will set the stage for some later arguments.

*Proof of Theorem* 3. Let $A(K)$ denote the area of $K$. The "mixed area" of the plane convex bodies $K_1$ and $K_2$, $A(K_1, K_2)$, can be defined by the fundamental relation [5, p. 48],

$$A(K_1 + K_2) = A(K_1) + 2A(K_1, K_2) + A(K_2) . \tag{1}$$

The mixed area is monotonic in each argument [5, p. 86]. That is, if $K_1 \subset K_2$, then

$$A(K_1, K) \leqq A(K_2, K) . \tag{2}$$

It follows from (1), setting $K_1 = K_2 = K$, that

$$A(K, K) = A(K) . \tag{3}$$

Now let $H$ be a regular hexagon circumscribed about $B$ (Theorem 2). Assume the center of $H$ is the origin, so $H = -H$. Then, using (2) and (3), we obtain

$$A(B, -B) \leqq A(H, -H) = A(H, H) = A(H) . \tag{4}$$

Thus, by (4), (1), and Theorem 1, we have

$$\begin{aligned} \pi &= A(B + (-B)) = 2A(B) + 2A(B, -B) \\ &\leqq 2A(B) + 2A(H) = 2A(B) + \sqrt{3} , \end{aligned} \tag{5}$$

from which the theorem follows.

It has long been conjectured that in $E_3$ any convex body of constant width 1 has volume at least that of a certain "tetrahedron of constant width" $T$ (see [12, p. 81] for the construction of $T$). A computation of the volume of $T$ leads to the conjecture,

*Conjecture* 1. Any 3-dimensional convex body of constant width 1 has volume not less than

$$\frac{2\pi}{3} - \frac{\pi\sqrt{3}}{4} \cos^{-1}(1/3) \approx .42 .$$

In §2 we shall prove that if $B_3$ is a 3-dimensional body of constant width 1, with volume $V(B_3)$, then

$$V(B_3) \geqq \beta = \frac{\pi}{3}(3\sqrt{6} - 7) \approx .365 . \tag{6}$$

Our proof of (6) will depend upon the following theorem of Blaschke [2].

THEOREM 4. *If a 3-dimensional convex body of constant width $b$ has volume $V$ and surface area $S$, then*

$$2V = bS - \frac{2\pi}{3} b^3 . \tag{7}$$

It follows from (7) that Conjecture 1 is equivalent to:

*Conjecture* 1′. Any 3-dimensional convex body of constant width 1 has surface area not less than

$$2\pi - \frac{\pi\sqrt{3}}{2}\cos^{-1}(1/3) .$$

Conjecture 1 can be transformed into still another form using the concept of "mixed surface area." Let $S(K)$ denote the surface area of $K$. If $K_1$ and $K_2$ are 3-dimensional convex bodies, then the surface area of $K_1 + K_2$ can be written in the form

$$S(K_1 + K_2) = S(K_1) + 2S(K_1, K_2) + S(K_2) ,$$

where $S(K_1, K_2)$ is the mixed surface area. Thus, if $K$ has constant width 1, $4\pi = S(K + (-K)) = 2S(K) + 2S(K, -K)$. Hence Conjectures 1 and 1′ are equivalent to:

*Conjecture* 1″. Any 3-dimensional convex body of constant width 1 has mixed surface area not greater than

$$\frac{\pi\sqrt{3}}{2}\cos^{-1}(1/3) .$$

Firey [6] has proved that the volume $V$ of an $n$-dimensional convex body of constant width 1 satisfies

$$V \geqq \frac{\pi - \sqrt{3}}{n!}, \quad n \geqq 2 . \tag{8}$$

In § 2 we give the generally better lower bound,

$$V \geqq \lambda\omega_n \prod_{k=3}^{n}\left(1 - \sqrt{\frac{k}{2k+2}}\right), \quad n \geqq 3 , \tag{9}$$

where $\omega_n$ is the volume of the unit ball in $E_n$, and

$$\lambda = \frac{\pi - \sqrt{3}}{2\pi} .$$

Let $C$ be a centrally symmetric convex body centered at the origin in $E_n$. Then $C$ is the unit sphere for a Minkowskian geometry. We say that a body $K$ has "constant width relative to $C$" if $K + (-K)$ is homothetic to $C$. In particular, one says that $K$ and $C$ are "equivalent in width" in case $K + (-K) = 2C$, since the condition implies that $K$ and $C$ have the same width function. When $C$ is the ordinary unit sphere we obtain the ordinary sets of constant width. Results about plane sets of relative constant width analogous to Theorem 2 and 3 are known (see [8], [10], and [11]). In § 3 we give a proof of the analogue of Theorem 3 in the Minkowski plane, using the same method as in our proof of Theorem 3.

Section 4 is devoted to some results on measures of outer symmetry for sets of constant width.

2. *Proof of* (6). Let $B_3$ be a 3-dimensional convex body of constant width 1. Then the inscribed sphere of $B_3$ has radius $\geqq 1 - \sqrt{3/8}$ (see [5, p. 125]). Assume that the center of the inscribed sphere is the origin. If $p(u)$ is the supporting function of $B_3$, then we have $p(u) \geqq 1 - \sqrt{3/8}$. Hence,

$$(10) \qquad 3V(B_3) = \int_{B_3} p(u)dS(u) \geqq (1 - \sqrt{3/8})S(B_3) ,$$

where $S(B_3)$ is the surface area of $B_3$. Using Theorem 4 in (10), we obtain

$$(11) \qquad 3V(B_3) \geqq (1 - \sqrt{3/8})\left(2V(B_3) + \frac{2\pi}{3}\right) ,$$

and (6) follows upon solving (11) for $V(B_3)$. This completes the proof.

*Proof of* (9). Define

$$(12) \qquad \lambda_n = \inf V(K) ,$$

as $K$ ranges over all bodies of constant width 1 in $E_n$, and $V(K)$ is the volume of $K$. The Blaschke selection principle implies that there exist bodies of constant width 1 having volume $\lambda_n$. Let $B$ be such a body, and let $p(u)$ be the support function of $B$ with the center of its inscribed sphere as origin. Then, by [5, p. 125],

$$p(u) \geqq 1 - \sqrt{\frac{n}{2n+2}} .$$

Denoting the area element of $B$ by $dS(u)$, we have,

$$(13) \qquad n\lambda_n = nV(B) = \int_B p(u)dS(u) \geqq \left(1 - \sqrt{\frac{n}{2n+2}}\right)S(B) ,$$

where $S(B)$ is the surface area of $B$. If we denote by $B_u$ the projection of $B$ onto a hyperplane orthogonal to $u$, then (see [5, p. 89])

$$(14) \qquad S(B) = \frac{1}{\omega_{n-1}} \int V(B_u)du ,$$

where $V(B_u)$ is the $(n-1)$-dimensional volume of $B_u$ and the integration is over the surface of the unit sphere in $E_n$. Since $B_u$ is an $(n-1)$-dimensional body of constant width 1, we have by (12) that $V(B_u) \geqq \lambda_{n-1}$. Hence

$$S(B) \geqq \frac{n\omega_n\lambda_{n-1}}{\omega_{n-1}} . \tag{15}$$

Combined with (13), this yields

$$\lambda_n \geqq \left(1 - \sqrt{\frac{n}{2n+2}}\right) \frac{\omega_n}{\omega_{n-1}} \lambda_{n-1} , \tag{16}$$

from which (9) follows. This completes the proof.

3. In this section, $C$ is a centrally symmetric plane convex body centered at the origin 0. $C$ admits an inscribed affine regular hexagon $H$ (i.e., the affine image of a regular hexagon) having a side parallel to any specified direction [10]. Let the vertices of $H$ be labelled $P_1, P_2, \cdots, P_6$ on the boundary of $C$ traversed in the positive direction. A "relative Reuleaux triangle" is obtained by attaching arcs $P_1P_2$, $P_3P_4$, and $P_5P_6$ of the boundary of $C$ to the respective sides $P_1P_2$, $P_20$, $0P_1$ of the triangle $0P_1P_2$. With $H$ as above, a centrally symmetric hexagon circumscribed about $C$ and touching $C$ at $P_i$, $1 \leqq i \leqq 6$, is called a "$C$-hexagon." In fact, any hexagon homothetic to such a hexagon will be called a $C$-hexagon. Note that it case $C$ is a circle, any $C$-hexagon is just a regular hexagon. One then sees that the following theorem from [10] is a Minkowskian geometry analogue of Theorem 2.

THEOREM 2′. *Let $K$ be equivalent in width to $C$. Then $K$ admits a circumscribed $C$-hexagon.*

Let $H$ be a $C$-hexagon circumscribed about $C$. Let $H'$ be the corresponding affine regular hexagon inscribed in $C$ with its vertices on $H$. Then we shall show that

$$A(H) \leqq 4/3\, A(H') . \tag{17}$$

This follows from the following general lemma.

LEMMA 1. *Let $H'$ be an affine regular hexagon inscribed in a centrally symmetric plane convex body $K$. Then*

$$A(K) \leqq 4/3\, A(H') . \tag{18}$$

*Proof.* By considering the support lines of $K$ through the vertices of $H'$, one sees that it suffices to prove (18) for $K$ a centrally

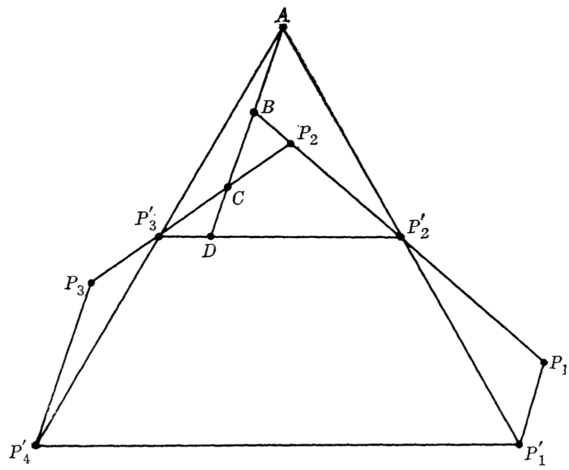


Figure 1.

symmetric hexagon $H$. Since the problem is affine invariant, one may even assume $H'$ is a regular hexagon, although this does not really simplify matters. In Figure 1, $P_1'$, $P_2'$, $P_3'$, $P_4'$ are consecutive vertices of $H'$, and $P_1$, $P_2$, $P_3$ are vertices of $H$. $AD$ is drawn parallel to $P_4'P_3$, which is parallel to $P_1'P_1$ (the degenerate cases, where $P_3 = P_4'$ or $P_1 = P_1'$ are easily disposed of and will not be dwelt upon here). $B$ is the intersection of $P_1P_2$ with $AD$, and $C$ is the intersection of $P_3P_2$ with $AD$. Triangle $P_4'P_3P_3'$ is congruent to $P_3'CA$, and $P_1'P_1P_2'$ is congruent to $P_2'BA$. Hence the area of the pentagon $P_1'P_1P_2P_3P_4'$ is not greater than the area of triangle $P_1'AP_4'$, so the area of $H$ is not greater than twice the area of $P_1'AP_4'$, which is precisely $4/3\,A(H')$. This completes the proof.

THEOREM 3′. *Any plane convex body $K$ which is equivalent in width to $C$ has area not less than that of some relative Reuleaux triangle equivalent in width to $C$.*

*Proof.* It is easy to check that the area of any relative Reuleaux triangle $T$ equivalent in width to $C$ is given by

$$A(T) = 2A(C) - 4/3\,A(H)\,, \tag{19}$$

where $H$ is the affine regular hexagon inscribed in $C$ on which the construction of $T$ is based. Let $H'$ be a $C$-hexagon circumscribed about $K$ (Theorem 2′), let $H''$ be the translate of $H'$ circumscribed about $C$, and let $H$ be the corresponding affine regular hexagon inscribed in $C$ with its vertices on $H''$. Let the center of $H'$ be at the origin (which can be assumed by translating $K$) so $H' = -H'$. Then, proceeding as in the proof of Theorem 3, and using (17), we have

$$\begin{aligned} 4A(C) &= A(K + (-K)) = 2A(K) + 2A(K, -K) \\ &\leqq 2A(K) + 2A(H', -H') = 2A(K) + 2A(H') \\ &= 2A(K) + 2A(H'') \leqq 2A(K) + 8/3\, A(H) . \end{aligned} \tag{20}$$

Hence,

$$A(K) \geqq 2A(C) - 4/3\, A(H) = A(T) . \tag{21}$$

This completes the proof.

To prove that a relative Reuleaux triangle is the only body equivalent in width to $C$ with minimum area requires a little more argument. A sketch of the proof is as follows. If $K$ is such a body of minimum area, then equality must hold throughout (20). This means that $A(K, -K) = A(H')$ for a $C$-hexagon $H'$ circumscribed about $K$. It follows that $A(-K, K) = A(H', K)$. If we let $p_1(\theta), p_2(\theta)$ be the support functions of $K$ and $H'$ respectively, with origin at the center of $H'$, and let $s_1$ denote arclength along $K$, the last equation implies that

$$\int p_1(\theta + \pi) ds_1 = \int p_1(\theta) ds_1 . \tag{22}$$

Equation (22) implies that $K$ must pass through 3 alternate vertices of $H'$, from which readily follows the fact that $K$ is a relative Reuleaux triangle.

4. For any $n$-dimensional convex body $K$ we define a "coefficient of outer symmetry," $\mu(K)$, as follows. Let $S$ be a centrally symmetric convex body of minimum volume containing $K$. Then

$$\mu(K) = \frac{V(K)}{V(S)} , \tag{23}$$

Thus $\mu(K) \leqq 1$, and $\mu(K) = 1$ if and only if $K$ is centrally symmetric. Sharp lower bounds for $\mu(K)$ are not known for $n \geqq 3$; however, it is known that $\mu(K) \geqq 1/2$ if $K$ is 2-dimensional, with equality holding if and only if $K$ is a triangle. A standing conjecture is that in $E_n$, $n \geqq 3$, $\mu(K) \geqq \mu(T)$, where $T$ is a simplex.

THEOREM 5. *Let $B$ be a plane convex body of constant width* 1. *Then $\mu(B) \geqq \mu(R)$, where $R$ is a Reuleaux triangle, and equality holds only if $B$ is a Reuleaux triangle.*

*Proof.* Let $H$ be a regular hexagon circumscribed about $B$.

Then, using Theorem 3, we have

$$\mu(B) \geqq \frac{A(B)}{A(H)} \geqq \frac{A(R)}{A(H)} = \frac{\pi - \sqrt{3}}{\sqrt{3}} = .81 \cdots . \tag{24}$$

where $R$ is a Reuleaux triangle of width 1. On the other hand, any centrally symmetric convex set $S$ containing $R$ must also contain an equilateral triangle $T$ of side 1 and thus has area $\geqq 2A(T) = A(H)$. Hence

$$\frac{A(R)}{A(H)} = \mu(R) . \tag{25}$$

Equality can hold in (24) only if $A(B) = A(R)$, which happens only if $B$ is a Reuleaux triangle (see end of § 3). This completes the proof.

It is known that any set $K$ of constant width in $E_3$ admits a regular circumscribed octahedron $J$ (see [7]). Suppose $K$ has constant width 1, and let $S$ be a centrally symmetric set of minimum volume containing $K$. Then, using (6),

$$\mu(K) = \frac{V(K)}{V(S)} \geqq \frac{\beta}{V(J)} = \frac{2\beta}{\sqrt{3}} \approx .42 . \tag{26}$$

Clearly one can obtain crude lower bounds, in this same fashion, in terms of $\lambda_n$ and the volume of some centrally symmetric "covering body" $J_n$ (one could, for example, use for $J_n$ a sphere of radius $\sqrt{n/(2n+2)}$).

## References

1. A. S. Besicovitch, *Minimum area of a set of constant width,* Proceedings of Symposia in Pure Mathematics, Vol. 7, *Convexity* (Amer. Math. Soc., 1963), 13-14.

2. W. Blaschke, *Einige Bemerkungen über Kurven und Flächen von konstanter Breite,* Ber. Verh. sächs. Akad. Leipzig **67** (1915), 290-297.

3. ———, *Konvexe Bereiche gegebener konstanter Breite und kleinsten Inhalts,* Math. Annalen **76** (1915), 504-513.

4. H. G. Eggleston, *A proof of Blaschke's theorem on the Reuleaux Triangle,* Quart. J. Math. **3** (1952), 296-7.

5. ———, *Convexity,* Cambridge Univ. Press, Cambridge, 1958.

6. W. J. Firey, *Lower bounds for volumes of convex bodies,* Archiv der Math. **16** (1965), 69-74.

7. D. Gale, *On inscribing n-dimensional sets in a regular n-simplex,* Proc. Amer. Math. Soc. **4** (1953), 222-225.

8. T. Kubota and D. Hemmi, *Some problems of minima concerning the oval,* Jour. Math. Soc. Japan **5** (1953), 372-389.

9. H. Lebesgue, *Sur le problème des isopérimètres et sur les domaines de largeur constante,* Bull. Soc. Math. France, C. R. (1914), 72-76.

10. D. Ohmann, *Extremal probleme für konvexe Bereiche der euklidischen Ebene,* Math. Z. **55** (1952), 346-352.

11. M. Sholander, *On certain minimum problems in the theory of convex curves*, Trans. Amer. Math. Soc. **73** (1952), 139–173.
12. M. Yaglom and V. G. Boltyanskii, *Convex figures*, GITTL, Moscow, 1951 (Russian); English transl., Holt, Rinehart and Winston, New York, 1961.

Received September 29, 1965.

UNIVERSITY OF CALIFORNIA, DAVIS

Printed in Japan by International Academic Printing Co., Ltd., Tokyo Japan

**Pacific Journal of Mathematics**
Vol. 19, No. 1 May, 1966