ON INDECOMPOSABLE MODULES OVER RINGS WITH MINIMUM CONDITION

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Let \( A \) be an associative ring with left minimum condition and identity. Let \( g(d) \) denote the number of nonisomorphic indecomposable \( A \)-modules which have composition length \( d \), \( d \) a nonnegative integer. If, for each integer \( n \), there exists an integer \( d > n \), such that \( g(d) = \infty \), \( A \) is said to be of strongly unbounded module type.

Assume that the center of the endomorphism ring of each simple (left) \( A \)-module is infinite. The following results concerning the structure of rings of strongly unbounded type are obtained.

I. If the ideal lattice of \( A \) is infinite, then \( A \) is of strongly unbounded module type.

II. If \( A \) is commutative, then \( A \) has only a finite number of (nonisomorphic) finitely generated indecomposable modules if and only if the ideal lattice of \( A \) is distributive. Otherwise, \( A \) is of strongly unbounded module type.

III. If the ideal lattice of \( A \) contains a vertex \( V \) of order greater than three such that, for some primitive idempotent \( e \in A \), the image \( Ve \) of \( V \) is a vertex of order greater than three in the submodule lattice of \( Ae \), then \( A \) is of strongly unbounded module type.

These results are generalizations of earlier ones obtained by J. P. Jans for finite dimensional algebras over algebraically closed fields.

Let \( A \) be an associative ring with left minimum condition and identity. The length, \( c(M) \), of a (left) \( A \)-module \( M \) with composition series is the number of composition factors of \( M \). Let \( g(d) \) denote the number of nonisomorphic indecomposable \( A \)-modules which have length \( d \), \( d \) a nonnegative integer. If \( \sum g(d) < \infty \), \( A \) is said to be of finite module type. If there exists an integer \( n \) such that \( g(d) = 0 \) for all \( d > n \), \( A \) is of bounded module type. If not of bounded module type, \( A \) is of unbounded module type. If for each integer \( n \), there exists \( d > n \) such that \( g(d) = \infty \), \( A \) is of strongly unbounded module type. R. Brauer, J. P. Jans, and R. M. Thrall have conjectured that infinite algebras of unbounded type are of strongly unbounded type, and that algebras of bounded type are of finite type [4]. A discussion of the state of these conjectures may be found in [2] and [4].

J. P. Jans has given sufficient conditions that a finite dimensional algebra over an algebraically closed field be of strongly unbounded type [4]. Through extension and modification of the techniques used
by Jans and by H. Tachikawa [6], some of these results can be obtained for arbitrary rings with minimum condition, provided that the endomorphism rings of the simple $A$-modules have infinite centers.

2. Rings with infinite ideal lattices. Let $A$ be a ring with left minimum condition with the property that the lattice of ideals of $A$ is infinite. H. Tachikawa showed that $A$ is of unbounded type [6]. If $A$ is also a finite dimensional algebra over an algebraically closed field, $A$ is of strongly unbounded type [4]. The following theorem generalizes these results.

**Theorem.** If the center of the endomorphism ring of each simple (irreducible) $A$-module is infinite and if the ideal lattice of $A$ is infinite, then $A$ is of strongly unbounded module type.

**Proof.** Since the ideal lattice of $A$ is infinite, the lattice contains a projective root [1].

Since $A/B$-modules are $A$-modules, we can assume that $B=0$. Also, there exists an $A - A$ isomorphism $\psi: B_1 \cong B_2$. Let $N$ denote the radical of $A$ and define $M = l(N) \cap r(N)$. Since $B_1$ and $B_2$ are simple ideals we have $B_1 + B_2 = B_1 \oplus B_2 \subseteq M$. There exist primitive idempotents $e, f \in A$ such that $fMe \supseteq fB_1e \oplus fB_2e \supseteq (0)$. Choose $u = fue \neq 0$ in $fB_1e$ and let $v = \psi(u)$. Let $A \subseteq fAf$ be a set of representatives for the nonzero distinct cosets of the center of $fAf/fNf$. Evidently, $A$ is infinite. For $\lambda \in A$, define $s(\lambda) = \lambda v - u$. Since $fAv, fAs(\lambda)$, are all nonzero and $u, v, s(\lambda) \in M$, we have $A/fNf \cong Au \cong Av \cong As(\lambda)$.

**Lemma 1.** If $\lambda \neq \mu \in A, a, b \in A$, and $s(\lambda)a = bs(\mu)$, then $s(\lambda)a = bs(\mu) = 0$.

**Proof.** We may assume that $a \in eAf, b \in fAf$. Since $B_1 \cap B_2 = 0$, we have $\lambda va = b\mu v$ and $ua = bu$. Since $v = \psi(u), va = bv$ so that $\lambda bv = b\mu v$. Thus, since $fAf/fNf$ is a division ring, $\lambda b = b\mu (\text{mod } fNf) = \mu b (\text{mod } fNf)$. Since $\lambda \neq \mu (\text{mod } fNf), b = 0 (\text{mod } fNf)$. Since $v \in M$, the lemma follows.
**Lemma 2.** If \( a, b, c, d \in A \) and \( s(\lambda)a + vb = cs(\lambda) + dv \), then \( va = cv \), \( ua = cu \), and \( vb = dv \).

**Proof.** Since \( B_1 \cap B_2 = 0 \) and \( v = \psi(u) \), we have \( cu = ua \), \( cv = va \), and \( \lambda va + vb - c\lambda v - dv = 0 \). Hence, since \( \lambda c = c\lambda \mod fNf \), \( vb = dv \).

For each positive integer \( n \), let \( X^n \) be the direct sum of \( n \) copies of \( Ae \),

\[
X^n = \bigoplus_{i=1}^n \varepsilon_i(Ae)
\]

and let \( Y^n \) denote the socle of \( X^n \). For \( \lambda \in A \), define

\[
T^n_\lambda = \left\{ \sum_{i=1}^n \varepsilon_i(a_{i-1}v + a_is(\lambda)) : a_0 = 0, a_i \in A \right\} .
\]

Let \( H^n_\lambda = X^n/T^n_\lambda \) and \( S^n_\mu = Y^n/T^n_\mu \). Since the length of \( T^n_\lambda \) is \( n \), the length of \( S^n_\mu \geq 2n - n = n \).

We proceed to show that \( H^n_\lambda \) and \( H^n_\mu \) are not isomorphic, provided \( \lambda \neq \mu \in A \). Suppose \( \theta : H^n_\lambda \cong H^n_\mu \). Since \( X^n \) is projective \([3]\), there exists \( \tilde{\theta} : X^n \to X^n \) such that \( \theta \pi_\lambda = \pi_\mu \tilde{\theta} \), where \( \pi_\lambda, \pi_\mu \) are the natural projections of \( X^n \) onto \( H^n_\lambda, H^n_\mu \), respectively. There exist \( x_1, \ldots, x_n \in eAe \), such that

\[
\tilde{\theta}\varepsilon_n(e) = \sum_{i=1}^n \varepsilon_i(x_i) .
\]

Since \( \pi_\lambda\varepsilon_n s(\lambda) = 0 \), and \( \theta \pi_\lambda = \pi_\mu \tilde{\theta} \), we have \( \pi_\mu \tilde{\theta}\varepsilon_n s(\lambda) = 0 \) and hence \( \tilde{\theta}\varepsilon_n s(\lambda) \in T^n_\mu \). Thus,

\[
\sum_{i=1}^n \varepsilon_i(s(\lambda)x_i) = \tilde{\theta}\varepsilon_n s(\lambda) \in T^n_\mu .
\]

According to the definition of \( T^n_\mu \), there exist \( a_0 = 0, a_1, \ldots, a_n \in A \), such that

\[
s(\lambda)x_i = a_{i-1}v + a_is(\mu) , \quad i = 1, \ldots, n .
\]

Using an induction and Lemma 1, we conclude that \( x_1, \ldots, x_n \in eNe \), and hence

\[
\theta \pi_\lambda \varepsilon_n (v) = \pi_\mu \sum_{i=1}^n \varepsilon_i(vx_i) = 0 .
\]

This contradicts the assumption that \( \theta \) is an isomorphism.

Next, suppose that \( H^n_\lambda \) decomposes. Let \( \eta \) be the idempotent endomorphism of \( H^n_\lambda \) associated with an indecomposable direct summand of \( H^n_\lambda \) such that \( \eta \pi_\lambda \varepsilon_n (v) \neq 0 \).
LEMMA 3. The restriction of \( \eta \) to \( S^n_k \) is a monomorphism.

Proof. Since \( X^n \) is projective, \( \eta \) may be lifted to an endomorphism \( \tilde{\eta} \) of \( X^n \). There exist \( y_{ij} \in eAe \) such that

\[
\tilde{\eta} \varepsilon_j (e) = \sum_{i=1}^{n} \varepsilon_i (y_{ij}) , \quad j = 1, \ldots, n .
\]

From the definition of \( T^n_k \), we have that

\[
\tilde{\eta} (\varepsilon_{j-1} (s(\lambda)) + \varepsilon_j (v)) \in T^n_k , \quad j = 2, \ldots, n .
\]

and \( \tilde{\eta} \varepsilon_n (s(\lambda)) \in T^n_k \). Thus,

\[
\tilde{\eta} (\varepsilon_{j-1} (s(\lambda)) + \varepsilon_j (v)) = \sum_{i=1}^{n} \varepsilon_i (s(\lambda) y_{i,j-1} + vy_{ij}) \in T^n_k ,
\]

for \( j = 2, \ldots, n \), and

\[
\tilde{\eta} \varepsilon_n (s(\lambda)) = \sum_{i=1}^{n} \varepsilon_i (s(\lambda) y_{in}) \in T^n_k .
\]

Hence, there exist \( a_{ij} \in fAf \) such that

\[
s_{\lambda} y_{i,j-1} + vy_{ij} = a_{i,j-1}s_{\lambda} ,
\]

\[
s_{\lambda} y_{i,j-1} + vy_{ij} = a_{i,j-1}s_{\lambda} + a_{i-1,j}v ,
\]

\[
s_{\lambda} y_{in} = a_{in}s_{\lambda} ,
\]

and

\[
s_{\lambda} y_{in} = a_{in}s_{\lambda} + a_{i-1,n}v ,
\]

for \( i, j = 2, 3, \ldots, n \).

Since \( f_{\varepsilon_n} e = s_{\lambda} \) and \( f_{\varepsilon} e = v \), we may assume that \( a_{ij} \in fAf \), \( i, j = 1, 2, \ldots, n \). Applying Lemma 2, we obtain,

\[
uy_{ij} = a_{ij} u ,
\]

and

\[
v y_{ij} = a_{ij} v , \quad i, j = 1, 2, \ldots, n ;
\]

\[
v y_{ij} = a_{i,j-1} v , \quad i, j = 2, 3, \ldots, n ;
\]

and

\[
y_{i-1,n} = 0 (\text{mod } eNe) , \quad i = 2, 3, \ldots, n .
\]

Suppose \( i < j \). Then we have

\[
v y_{ij} = a_{ij} v = vy_{i+1,j+1} = \cdots = vy_{i+n-j,n} = 0 .
\]

Therefore, \( y_{ij} = 0 (\text{mod } eNe) \). Suppose \( i > j \). Then
vy_{ij} = a_{i-1,j-1}v = vy_{i-1,j-1} = \cdots = vy_{i-j+1,1}.

Also,

\[ vy_{kk} = vy_{nn}, \quad k = 1, 2, \ldots, n. \]

Since \( \gamma \pi_{\lambda}(\varepsilon_{n}(v)) = \pi_{\lambda} \varepsilon_{n}(vy_{nn}) \neq 0 \), we have \( y_{nn} \neq 0 \) (mod \( eNe \)). From these equations and the idempotence of \( \gamma \) it follows that

\[ y_{ij} = \begin{cases} e \pmod{eNe}, & \text{if } i = j, \\ 0 \pmod{eNe}, & \text{if } i < j, \\ y_{i-j+1,1} \pmod{eNe}, & \text{if } i > j. \end{cases} \]

Next assume that \( x \in Y^{*} \) and \( \gamma \pi_{\lambda}(x) = 0 \). Then \( \overline{\gamma}(x) \in T_{n}^{*} \). There exist elements \( r_{j} \) of the socle of \( Ae \) such that \( x = \sum_{j=1}^{n} \varepsilon_{j} r_{j} \), from which the equation

\[ \overline{\gamma}(x) = \sum_{i=1}^{n} \varepsilon_{i} \left( r_{i} + \sum_{j=1}^{i-1} r_{j} y_{i-j+1,1} \right) \]

follows. Since \( \overline{\gamma}(x) \in T_{n}^{*} \), there exist \( b_{0} = 0, b_{1}, \ldots, b_{n} \in Ae \) such that

\[ \sum_{j=1}^{i-1} r_{j} y_{i-j+1,1} + r_{i} = b_{i}s(\lambda) + b_{i-1}v, \quad i = 2, \ldots, n. \]

Defining

\[ \alpha_{0} = 0, \]
\[ \alpha_{1} = b_{1}, \]
\[ \alpha_{k} = b_{k} - \sum_{j=1}^{k-1} \alpha_{j} a_{k-j+1,1}, \quad k = 2, \ldots, n. \]

it follows that

\[ r_{k} = \alpha_{k}s(\lambda) + \alpha_{k-1}v, \quad k = 1, \ldots, n. \]

Thus, \( x \in T_{n}^{*} \) and \( \pi_{\lambda} x = 0 \). This proves Lemma 3.

From Lemma 3, we conclude that \( S_{n}^{*} \) is contained in an indecomposable direct summand \( V_{\lambda} \) of \( H_{n}^{*} \). Calculation of \( H_{n}^{*}/S_{n}^{*} \cong X^{*}/Y^{*} \) shows that every direct summand of \( H_{n}^{*} \) not equal to \( V_{\lambda} \) is isomorphic to \( Ae/S(Ae) \), \( S(Ae) \) the socle of \( Ae \). Thus, \( V_{\lambda} \cong V_{\mu} \) if and only if \( H_{n}^{*} \cong H_{n}^{*} \) and hence \( V_{\lambda} \not\cong V_{\mu} \) if \( \lambda \not\in \mu \in A \). This completes the proof of the theorem.

3. Commutative rings.

**Theorem.** If \( A \) is commutative, then \( A \) is of finite type if and only if the ideal lattice of \( A \) is distributive. Otherwise, \( A \) is of
unbounded type, strongly so if the endomorphism ring of each simple A-module is infinite.

Proof. It is sufficient to show that, if the ideal lattice of A is distributive, A is generalized uni-serial (see [5]). Let e be a primitive idempotent in A and consider the lattice of submodules of Ae. Since A is commutative, these submodules are ideals in A. Suppose the lattice contains a vertex

where we assume, without loss of generality, that the lattice from (0) to \( L_0 \) is a chain. Then \( L_0 = N^{k+1}e \) for some \( k \), and \( L_1 + L_2 \subseteq N^k e \). Choose \( \alpha_i \in L_i - L_0, i = 1, 2 \), and define

\[
L_3 = Ae(\alpha_1 + \alpha_2) + L_0.
\]

The mapping \( ae \rightarrow ae(\alpha_1 + \alpha_2) + L_0 \) induces an isomorphism \( L_0/L_0 \cong Ae/Ne \) so that we have \( L_0 \subseteq L_3 \subseteq L_1 + L_2 \). Since \( L_1 \cap L_2 = L_0 \), it follows directly that \( L_3 \cap L_1 = L_3 \cap L_2 = L_0 \). Clearly \( L_1 + L_2 = L_1 + L_2 = L_1 + L_2 = L_1 + L_2 \). Hence the ideal lattice of A contains the projective root

which contradicts the assumption that the lattice is distributive. Thus, A is generalized uni-serial and of finite type.

4. Lattices with vertex of order four. In this section we assume that the center of the endomorphism ring of each simple A-module is infinite.

Theorem. If the ideal lattice of A contains a vertex V of order greater than three such that for some primitive idempotent \( e \in A \), the image \( Ve \) of V is a vertex of order greater than three in the submodule lattice of \( Ae \), then A is of strongly unbounded module type.
Proof. There exists an ideal $B_0 \subseteq A$ with distinct covers $B_1, B_2, B_3, B_4$ such that $B_i \cap B_j = 0$, $i = 1, 2, 3, 4$. Since $A/B_0$ modules are $A$-modules we can assume that $B_0 = 0$. Because of the theorem of §1, we assume that the ideal lattice of $A$ is distributive and hence that

$$\sum_{i=1}^{4} B_i = \bigoplus_{i=1}^{4} B_i .$$

There exist primitive idempotents $f_i \in A$ such that $f_i B_i \cap B_j = 0$, $i = 1, 2, 3, 4$. Let $A \subseteq e Ae$ be a set of representatives for the nonzero cosets of the center of $e Ae$. Choose $u_i = f_i u_i \neq 0 \in B_i e$, $i = 1, 2, 3, 4$. For $\lambda \in A$ we have $Af_i / Nf_i \cong Au_i \cong Au_i \lambda$, $i = 1, 2, 3, 4$. For each positive integer $n$ define

$$X^n = \bigoplus_{i=1}^{2n} e_i(Ae)$$

and denote the socle of $X^n$ by $Y^n$. Define

$$T^n_\lambda = \left\{ \sum_{i=1}^{n} e_i (a_i u_i + c_i u_3 + d_i u_4 + d_{i-1} u_i) + e_{i+n} (b_i u_2 + c_i u_3 + d_i u_4) : d_0 = 0, a_i, b_i, c_i, d_i \in A, i = 1, \ldots, n. \right\},$$

$$H^n_\lambda = X^n / T^n_\lambda ,$$

and

$$S^n_\lambda = Y^n / T^n_\lambda .$$

Since the composition length of $T^n_\lambda$ is equal to $4n$ and the composition length of $Y^n$ is greater than or equal to $8n$, the composition length of $S^n_\lambda$ increases without bound as $n$ increases.

Let $\lambda \neq \mu$ be elements of $A$. We next prove that $H^n_\lambda$ and $H^n_\mu$ are not isomorphic. Suppose $\theta$ is an isomorphism from $H^n_\lambda$ onto $H^n_\mu$. Since $X^n$ is projective, $\theta$ can be lifted to an endomorphism $\tilde{\theta}$ of $X^n$. There exist $x_1, \cdots, x_{2n}, y_1, \cdots, y_{2n}$ in $e Ae$ such that

$$\tilde{\theta} e_{x_i}(e) = \sum_{i=1}^{2n} e_i(x_i)$$

and

$$\tilde{\theta} e_{y_i}(e) = \sum_{i=1}^{2n} e_i(y_i) .$$

Since, $\theta \pi_\lambda e_\mu(u_i) \neq 0$, we have

$$\pi_\mu \left( \sum_{i=1}^{2n} e_i(u_i y_i) \right) = \theta \pi_\lambda e_\mu(u_i) \neq 0 .$$
Thus, since $u_4 \in M$, there exists $k$, $1 \leq k \leq 2n$, such that

$$y_k \in eNe.$$  

Since $u_i y_i \in Au_2 + Au_3 + Au_4$ for $i > n$, we have $u_i y_i = 0$ for $i > n$, and hence, since $eAe/eNe$ is a division ring, $y_i \in eNe$, for $i > n$. Similarly, $\partial e_{2n}(u_n) \in T^a_\mu$ implies $x_i \in eNe$, for $i \leq n$. It follows that

$$\bar{\partial}(e_n u_3 + e_{2n} u_2) = \sum_{i=1}^{n} \epsilon_i(u_3 y_i) + \sum_{i=n+1}^{2n} \epsilon_i(u_2 x_i) \in T^a_\mu.$$  

Therefore, $u_i y_i = u_i x_{i+n}$ for $i = 1, \ldots, n$, and hence,

$$y_i = x_{i+n}, \ (mod\ eNe) \quad i = 1, \ldots, n.$$  

From this we obtain

$$\bar{\partial}(e_n (u_{1n}) + e_{2n} (u_{1})) = \sum_{i=1}^{n} \epsilon_i(u_{1n} y_i) + \sum_{i=n+1}^{2n} \epsilon_i(u_{1} y_{i-n}) \in T^a_\mu.$$  

Hence, using the definition of $T^a_\mu$ there exist $d_1, \ldots, d_n \in A$ such that

$$u_{i} y_1 = d_1 u_i y_1,$$

$$u_{i} y_j = d_j u_i y_j + d_{j-1} u_i, \quad j = 2, \ldots, n,$$

and

$$u_{i} y_j = d_j u_i, \quad j = 1, \ldots, n.$$  

Replacing $d_j u_i$ by $u_i y_j$ in these equations, we have

$$u_{i} y_1 = u_i y_i \mu$$

and

$$u_{i} y_j = u_i y_j \mu + u_i y_{j-1}, \quad j = 2, \ldots, n.$$  

Since $u_i \in M$, a simple induction shows that

$$y_i \in eNe, \quad i = 1, \ldots, n.$$  

We conclude that $H^a_\lambda$ and $H^a_\mu$ are not isomorphic.

Next, suppose that $H^a_\lambda$ decomposes and let $\gamma$ be an idempotent endomorphism of $H^a_\lambda$ such that $\gamma x_\lambda(e_n(u_n)) \neq 0$. Since $X$ is projective, $\gamma$ can be lifted to an endomorphism $\bar{\gamma}$ of $X^a$. There exist $y_{ij} \in eAe$ such that $\bar{\gamma}(\bar{\epsilon}_j(e)) = \sum_{i=1}^{2n} \epsilon_i(y_{ij})$. If $j \leq n$, we have

$$\bar{\gamma}(\bar{\epsilon}_j(u_i)) = \sum_{i=1}^{2n} \epsilon_i(u_i y_{ij}) \in T^a_\lambda$$

and hence

$$y_{ij} = 0, \ (mod\ eNe) \quad 1 \leq i \leq n, n + 1 \leq j \leq 2n.$$
For $j \leq n$, we have,
\[ \bar{\eta}(\varepsilon_j(u_3) + \varepsilon_{j+n}(u_3)) = \sum_{i=1}^{n} \varepsilon_i(u_3y_{i,j}) + \sum_{i=n+1}^{2n} \varepsilon_i(u_3y_{i,j+n}) \in T^n_{\lambda}. \]

Thus, by the definition of $T^n_{\lambda}$,
\[ y_{ij} = y_{i+n,j+n} \pmod{eNe}, \quad 1 \leq i, j \leq n. \]

We infer that
\[ \bar{\eta}(\varepsilon_n(u_\lambda) + \varepsilon_{2n}(u_\lambda)) = \sum_{i=1}^{n} \varepsilon_i(u_\lambda y_{i,n}) + \sum_{i=n+1}^{2n} \varepsilon_i(u_\lambda y_{i-n,n}) \in T^n_{\lambda}. \]

Hence, there exist $d_0, \ldots, d_n \in A$, $d_0 = 0$, such that
\[ u_\lambda y_{jn} = d_j u_\lambda + d_{j-1} u_4, \]
and
\[ u_4 y_{jn} = d_j u_4, \quad j = 1, \ldots, n. \]

Replacing $d_j u_4$ by $u_4 y_{jn}$, we have
\[ u_\lambda y_{jn} = u_4 y_{jn} \lambda, \]
and
\[ u_\lambda y_{jn} = u_4 y_{jn} \lambda + u_4 y_{j-1,n}, \quad j = 2, \ldots, n. \]

Hence, for $i < n$ we obtain $y_{in} = 0 \pmod{eNe}$. And, since $\gamma$ is idempotent and $eAe/eNe$ is a division ring, $y_{nn} = e \pmod{eNe}$. Now suppose $k < n$. Then
\[ \bar{\eta}(\varepsilon_k(u_\lambda) + \varepsilon_{k+1}(u_4)) + \varepsilon_{k+n}(u_4)) \]
\[ = \sum_{i=1}^{n} \varepsilon_i(u_\lambda y_{ik} + u_4 y_{i,k+1}) + \sum_{i=n+1}^{2n} \varepsilon_i(u_4 y_{i-n,k}) \in T^n_{\lambda}. \]

Hence, there exist $d^*_1, d^*_2, \ldots, d^n_k \in A$, $d^*_0 = 0$, such that
\[ u_\lambda y_{ik} + u_4 y_{i,k+1} = d^*_j u_\lambda + d^*_j u_4, \]
and
\[ u_4 y_{jk} = d^*_j u_4, \quad j = 1, \ldots, n. \]

Replacing $d^*_j u_4$ by $u_4 y_{jk}$ we obtain $u_4 y_{i,k+1} = 0$, and $u_4 y_{j,k+1} = u_4 y_{j-1,k}$, $j = 2, \ldots, n$, $k = 1, \ldots, n - 1$. It follows from these equations that $y_{ik} = 0 \pmod{eNe}$ for $k = 2, \ldots, n$, and $y_{jk} = y_{j-1, k+1} \pmod{eNe}$, $j, k = 1, \ldots, n - 1$. If $i < j \leq n$, then
\[ y_{ij} = y_{i-1,j-1} = \cdots = y_{i,j-i+1} = 0 \pmod{eNe}. \]
And, if \( n \geq i \geq j \),
\[
y_{ij} = y_{i-1,j-1} = \cdots = y_{i-j+1,1} \pmod{eNe}.
\]
These results imply
\[
y_{ij} = \begin{cases} 
0 \pmod{eNe}, & \text{if } i < j, \text{ or } j \leq n < i, \\
\varepsilon \pmod{eNe}, & \text{if } i = j, \\
y_{i-j+1,1} \pmod{eNe}, & \text{if } j < i \leq n, \text{ or } n < j < i.
\end{cases}
\]

We shall now show that the restriction of \( \eta \) to \( S^*_n \) is a monomorphism and that \( \eta(S^*_n) = S^*_n \). Suppose that \( x \in Y^n \) is such that \( \pi_\lambda(s) \) is an element of the kernel of \( \eta \),
\[
x = \sum_{i=1}^{2n} \varepsilon_i(x_i).
\]
We have \( \eta \pi_\lambda(x) = \pi_\lambda \tilde{\eta}(x) = 0 \), and so
\[
\tilde{\eta}(x) \in T^*_n.
\]
\[
\tilde{\eta}(x) = \sum_{j=1}^{2n} \tilde{\eta}_j(x_j)
\]
\[
= \sum_{j=1}^{2n} \sum_{i=1}^{2n} \varepsilon_i(x_jy_{ij})
\]
\[
= \sum_{j=1}^{2n} \sum_{i=1}^{2n} \varepsilon_i(x_jy_{i-j+1,1}) + \sum_{j=n+1}^{2n} \sum_{i=1}^{2n} \varepsilon_i(x_jy_{i-j+1,1}).
\]
Thus, there exist \( a_i, b_i, c_i, d_i, i = 1, \ldots, n \) in \( A \), \( d_0 = 0 \) such that
\[
\sum_{j=1}^{i} x_jy_{ij} = a_i u_1 + c_i u_2 + d_i u_3 + d_{i-1} u_4,
\]
and
\[
\sum_{j=1}^{i} x_{n+j}y_{ij} = b_i u_2 + c_i u_3 + d_i u_4, \quad \text{for } i = 1, 2, \ldots, n.
\]
Using the definition of \( T^*_n \), it follows that
\[
x_j = \alpha_j u_1 + \gamma_j u_2 + \delta_j u_3 + \delta_{j-1} u_4,
\]
and
\[
x_{j+n} = \beta_j u_2 + \gamma_j u_3 + \delta_j u_4, \quad j = 1, \ldots, n - 1,
\]
where \( \alpha_i = a_i, \beta_i = b_i, \gamma_i = c_i, \delta_0 = 0, \delta_1 = d_1, \) and
\[
\alpha_k u_1 = a_k u_1 - \alpha_{k-1} u_j y_{21} - \cdots - \alpha_i u_i y_{k1},
\]
\[
\beta_k u_2 = b_k u_2 - \beta_{k-1} u_j y_{21} - \cdots - \beta_i u_i y_{k1},
\]
\[
\gamma_k u_3 = c_k u_3 - \gamma_{k-1} u_j y_{21} - \cdots - \gamma_i u_i y_{k1},
\]
\[
\delta_k u_4 = d_k u_4 - \delta_{k-1} u_j y_{21} - \cdots - \delta_i u_i y_{k1}, \quad \text{for } k > 1.
\]
Hence, $\pi_\lambda(x) = 0$, and the restriction of $\eta$ to $S^\lambda_\alpha$ is a monomorphism. The proof can now be completed as in §1.

**BIBLIOGRAPHY**


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