Pacific Journal of Mathematics

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Vol. 19, No. 1 May 1966

PACIFIC JOURNAL OF MATHEMATICS
Vol. 19, No. 1, 1966

ON INDECOMPOSABLE MODULES OVER RINGS WITH MINIMUM CONDITION

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Let A be an associative ring with left minimum condition and identity. Let $g(d)$ denote the number of nonisomorphic indecomposable A-modules which have composition length d, d a nonnegative integer. If, for each integer n, there exists an integer $d > n$, such that $g(d) = \infty$, A is said to be of strongly unbounded module type.

Assume that the center of the endomorphism ring of each simple (left) A-module is infinite. The following results concerning the structure of rings of strongly unbounded type are obtained.

I. If the ideal lattice of A is infinite, then A is of strongly unbounded module type.

II. If A is commutative, then A has only a finite number of (nonisomorphic) finitely generated indecomposable modules if and only if the ideal lattice of A is distributive. Otherwise, A is of strongly unbounded module type.

III. If the ideal lattice of A contains a vertex V of order greater than three such that, for some primitive idempotent $e \in A$, the image Ve of V is a vertex of order greater than three in the submodule lattice of Ae, then A is of strongly unbounded module type.

These results are generalizations of earlier ones obtained by J. P. Jans for finite dimensional algebras over algebraically closed fields.

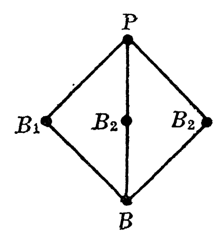
Let A be an associative ring with left minimum condition and identity. The *length*, $c(M)$, of a (left) A-module M with composition series is the number of composition factors of M. Let $g(d)$ denote the number of nonisomorphic indecomposable A-modules which have length d, d a nonnegative integer. If $\sum_d g(d) < \infty$, A is said to be of *finite module type*. If there exists an integer n such that $g(d) = 0$ for all $d > n$, A is of *bounded module type*. If not of bounded module type, A is of *unbounded module type*. If for each integer n, there exists $d > n$ such that $g(d) = \infty$, A is of *strongly unbounded module type*. R. Brauer, J. P. Jans, and R. M. Thrall have conjectured that infinite algebras of unbounded type are of strongly unbounded type, and that algebras of bounded type are of finite type [4]. A discussion of the state of these conjectures may be found in [2] and [4].

J. P. Jans has given sufficient conditions that a finite dimensional algebra over an algebraically closed field be of strongly unbounded type [4]. Through extension and modification of the techniques used by Jans and by H. Tachikawa [6], some of these results can be obtained for arbitrary rings with minimum condition, provided that the endomorphism rings of the simple A-modules have infinite centers.

2. Rings with infinite ideal lattices. Let A be a ring with left minimum condition with the property that the lattice of ideals of A is infinite. H. Tachikawa showed that A is of unbounded type [6]. If A is also a finite dimensional algebra over an algebraically closed field, A is of strongly unbounded type [4]. The following theorem generalizes these results.

THEOREM. *If the center of the endomorphism ring of each simple (irreducible) A-module is infinite and if the ideal lattice of A is infinite, then A is of strongly unbounded module type.*

Proof. Since the ideal lattice of A is infinite, the lattice contains a projective root [1].



Since A/B-modules are A-modules, we can assume that $B=0$. Also, there exists an $A-A$ isomorphism $\psi\colon B_1 \cong B_2$. Let N denote the radical of A and define $M = l(N) \cap r(N)$. Since B_1 and B_2 are simple ideals we have $B_1 + B_2 = B_1 \oplus B_2 \subseteqq M$. There exist primitive idempotents $e, f \in A$ such that $fMe \supseteqq fB_1e \oplus fB_2e \supset (0)$. Choose $u = fue \neq 0$ in fB_1e and let $v = \psi(u)$. Let $\Lambda \subset fAf$ be a set of representatives for the nonzero distinct cosets of the center of fAf/fNf. Evidently, Λ is infinite. For $\lambda \in \Lambda$, define $s(\lambda) = \lambda v - u$. Since fAu, $fAv, fAs(\lambda)$, are all nonzero and $u, v, s(\lambda) \in M$, we have $Af/Nf \cong Au \cong Av \cong As(\lambda)$.

LEMMA 1. *If $\lambda \neq \mu \in \Lambda$, $a, b \in A$, and $s(\lambda)a = bs(\mu)$, then $s(\lambda)a = bs(\mu) = 0$.*

Proof. We may assume that $a \in eAe, b \in fAf$. Since $B_1 \cap B_2 = 0$, we have $\lambda va = b\mu v$ and $ua = bu$. Since $v = \psi(u)$, $va = bv$ so that $\lambda bv = b\mu v$. Thus, since fAf/fNf is a division ring, $\lambda b = b\mu \pmod{fNf} = \mu b \pmod{fNf}$. Since $\lambda \neq \mu \pmod{fNf}$, $b = 0 \pmod{fNf}$. Since $v \in M$, the lemma follows.

LEMMA 2. *If* $a, b, c, d \in A$ *and* $s(\lambda)a + vb = cs(\lambda) + dv$, *then* $va = cv$, $ua = cu$, *and* $vb = dv$.

Proof. Since $B_1 \cap B_2 = 0$ and $v = \psi(u)$, we have $cu = ua$, $cv = va$, and $\lambda va + vb - c\lambda v - dv = 0$. Hence, since $\lambda c = c\lambda \pmod{fNf}$, $vb = dv$.

For each positive integer n, let X^n be the direct sum of n copies of Ae,

$$X^n = \bigoplus \sum_{i=1}^{n} \varepsilon_i(Ae) ,$$

and let Y^n denote the socle of X^n. For $\lambda \in \Lambda$, define

$$T_\lambda^n = \left\{ \sum_{i=1}^{n} \varepsilon_i(a_{i-1}v + a_i s(\lambda)) \colon a_0 = 0,\ a_i \in A \right\} .$$

Let $H_\lambda^n = X^n/T_\lambda^n$ and $S_\lambda^n = Y^n/T_\lambda^n$. Since the length of T_λ^n is n, the length of $S_\lambda^n \geqq 2n - n = n$.

We proceed to show that H_λ^n and H_μ^n are not isomorphic, provided $\lambda \neq \mu \in \Lambda$. Suppose $\theta \colon H_\lambda^n \cong H_\mu^n$. Since X^n is projective [3], there exists $\bar{\theta} \colon X^n \longrightarrow X^n$ such that $\theta\pi_\lambda = \pi_\mu\bar{\theta}$, where π_λ, π_μ are the natural projections of X^n onto H_λ^n, H_μ^n, respectively. There exist $x_1, \cdots, x_n \in eAe$, such that

$$\bar{\theta}\varepsilon_n(e) = \sum_{i=1}^{n} \varepsilon_i(x_i) .$$

Since $\pi_\lambda\varepsilon_n s(\lambda) = 0$, and $\theta\pi_\lambda = \pi_\mu\bar{\theta}$, we have $\pi_\mu\bar{\theta}\varepsilon_n s(\lambda) = 0$ and hence $\bar{\theta}\varepsilon_n s(\lambda) \in T_\mu^n$. Thus,

$$\sum_{i=1}^{n} \varepsilon_i(s(\lambda)x_i) = \bar{\theta}\varepsilon_n s(\lambda) \in T_\mu^n .$$

According to the definition of T_μ^n, there exist $a_0 = 0, a_1, \cdots, a_n \in A$, such that

$$s(\lambda)x_i = a_{i-1}v + a_i s(\mu) , \qquad i = 1, \cdots, n .$$

Using an induction and Lemma 1, we conclude that $x_1, \cdots, x_n \in eNe$, and hence

$$\theta\pi_\lambda\varepsilon_n(v) = \pi_\mu \sum_{i=1}^{n} \varepsilon_i(vx_i) = 0 .$$

This contradicts the assumption that θ is an isomorphism.

Next, suppose that H_λ^n decomposes. Let η be the idempotent endomorphism of H_λ^n associated with an indecomposable direct summand of H_λ^n such that $\eta\pi_\lambda\varepsilon_n(v) \neq 0$.

LEMMA 3. *The restriction of η to S^n_λ is a monomorphism.*

Proof. Since X^n is projective, η may be lifted to an endomorphism $\bar{\eta}$ of X^n. There exist $y_{ij} \in eAe$ such that

$$\bar{\eta}\varepsilon_j(e) = \sum_{i=1}^{n} \varepsilon_i(y_{ij}) , \qquad j = 1, \cdots, n .$$

From the definition of T^n_λ, we have that

$$\bar{\eta}(\varepsilon_{j-1}(s(\lambda)) + \varepsilon_j(v)) \in T^n_\lambda , \qquad j = 2, \cdots, n .$$

and $\bar{\eta}\varepsilon_n(s(\lambda)) \in T^n_\lambda$. Thus,

$$\bar{\eta}(\varepsilon_{j-1}(s(\lambda)) + \varepsilon_j(v)) = \sum_{i=1}^{n} \varepsilon_i(s(\lambda)y_{i,j-1} + vy_{ij}) \in T^n_\lambda ,$$

for $j = 2, \cdots, n$, and

$$\bar{\eta}\varepsilon_n s(\lambda) = \sum_{i=1}^{n} \varepsilon_i(s(\lambda)y_{in}) \in T^n_\lambda .$$

Hence, there exist $a_{ij} \in fAf$ such that

$$\begin{aligned} s_\lambda y_{1,j-1} + vy_{1j} &= a_{1,j-1}s_\lambda , \\ s_\lambda y_{i,j-1} + vy_{ij} &= a_{i,j-1}s_\lambda + a_{i-1,j-1}v , \\ s_\lambda y_{1n} &= a_{1n}s_\lambda , \end{aligned}$$

and

$$s_\lambda y_{in} = a_{in}s_\lambda + a_{i-1,n}v ,$$
$$\text{for } i, j = 2, 3, \cdots, n .$$

Since $fs_\lambda e = s_\lambda$ and $fve = v$, we may assume that $a_{ij} \in fAf$, $i, j = 1, 2, \cdots, n$. Applying Lemma 2, we obtain,

$$uy_{ij} = a_{ij}u ,$$

and

$$\begin{aligned} vy_{ij} &= a_{ij}v , &\quad i, j &= 1, 2, \cdots, n ; \\ vy_{ij} &= a_{i-1,j-1}v , &\quad i, j &= 2, 3, \cdots, n ; \end{aligned}$$

and

$$y_{i-1,n} = 0 \pmod{eNe} , \qquad i = 2, 3, \cdots, n .$$

Suppose $i < j$. Then we have

$$vy_{ij} = a_{ij}v = vy_{i+1,j+1} = \cdots = vy_{i+n-j,n} = 0 .$$

Therefore, $y_{ij} = 0 \pmod{eNe}$. Suppose $i > j$. Then

$$vy_{ij} = a_{i-1,j-1}v = vy_{i-1,j-1} = \cdots = vy_{i-j+1,1} .$$

Also,

$$vy_{kk} = vy_{nn} , \qquad k = 1, 2, \cdots, n .$$

Since $\eta\pi_\lambda(\varepsilon_n(v)) = \pi_\lambda\varepsilon_n(vy_{nn}) \neq 0$, we have $y_{nn} \neq 0 \pmod{eNe}$. From these equations and the idempotence of η it follows that

$$y_{ij} = \begin{cases} e \pmod{eNe}, & \text{if } i = j . \\ 0 \pmod{eNe}, & \text{if } i < j . \\ y_{i-j+1,1} \pmod{eNe}, & \text{if } i > j . \end{cases}$$

Next assume that $x \in Y^n$ and $\eta\pi_\lambda(x) = 0$. Then $\bar{\eta}(x) \in T^n_\lambda$. There exist elements r_j of the socle of Ae such that $x = \sum_{j=1}^n \varepsilon_j r_j$, from which the equation

$$\bar{\eta}(x) = \sum_{i=1}^n \varepsilon_i\Big(r_i + \sum_{j=1}^{i-1} r_j y_{i-j+1,1}\Big)$$

follows. Since $\bar{\eta}(x) \in T^n_\lambda$, there exist $b_0 = 0, b_1, \cdots, b_n \in Ae$ such that

$$\sum_{j=1}^{i-1} r_j y_{i-j+1,1} + r_i = b_i s(\lambda) + b_{i-1}v , \qquad i = 2, \cdots, n .$$

Defining

$$\begin{aligned} \alpha_0 &= 0 , \\ \alpha_1 &= b_1 , \\ \alpha_k &= b_k - \sum_{j=1}^{k-1} \alpha_j a_{k-j+1,1} , \qquad k = 2, \cdots, n . \end{aligned}$$

it follows that

$$r_k = \alpha_k s(\lambda) + \alpha_{k-1} v , \qquad k = 1, \cdots, n .$$

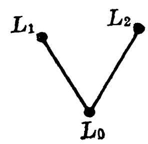
Thus, $x \in T^n_\lambda$ and $\pi_\lambda x = 0$. This proves Lemma 3.

From Lemma 3, we conclude that S^n_λ is contained in an idecomposable direct summand V_λ of H^n_λ. Calculation of $H^n_\lambda/S^n_\lambda \cong X^n/Y^n$ shows that every direct summand of H^n_λ not equal to V_λ is isomorphic to $Ae/S(Ae)$, $S(Ae)$ the socle of Ae. Thus, $V_\lambda \cong V_\mu$ if and only if $H^n_\lambda \cong H^n_\mu$ and hence $V_\lambda \not\cong V_\mu$ if $\lambda \neq \mu \in \Lambda$. This completes the proof of the theorem.

3. Commutative rings.

THEOREM. *If A is commutative, then A is of finite type if and only if the ideal lattice of A is distributive. Otherwise, A is of* *unbounded type, strongly so if the endomorphism ring of each simple A-module is infinite.*

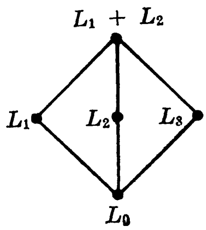
Proof. It is sufficient to show that, if the ideal lattice of A is distributive, A is generalized uni-serial (see [5]). Let e be a primitive idempotent in A and consider the lattice of submodules of Ae. Since A is commutative, these submodules are ideals in A. Suppose the lattice contains a vertex



where we assume, without loss of generality, that the lattice from (0) to L_0 is a chain. Then $L_0 = N^{k+1}e$ for some k, and $L_1 + L_2 \subseteqq N^k e$. Choose $\alpha_1 \in L_i - L_0$, $i = 1, 2$, and define

$$L_3 = Ae(\alpha_1 + \alpha_2) + L_0 .$$

The mapping $ae \longrightarrow ae(\alpha_1 + \alpha_2) + L_0$ induces an isomorphism $L_3/L_0 \cong Ae/Ne$ so that we have $L_0 \subset L_3 \subset L_1 + L_2$. Since $L_1 \cap L_2 = L_0$, it follows directly that $L_3 \cap L_1 = L_3 \cap L_2 = L_0$. Clearly $L_1 + L_2 = L_1 + L_3 = L_2 + L_3$. Hence the ideal lattice of A contains the projective root



which contradicts the assumption that the lattice is distributive. Thus, A is generalized uni-serial and of finite type.

4. Lattices with vertex of order four. In this section we assume that the center of the endomorphism ring of each simple A-module is infinite.

THEOREM. *If the ideal lattice of A contains a vertex V of order greater than three such that for some primitive idempotent $e \in A$, the image Ve of V is a vertex of order greater than three in the submodule lattice of Ae, then A is of strongly unbounded module type.*

Proof. There exists an ideal $B_0 \subsetneqq A$ with distinct covers B_1, B_2, B_3, B_4 such that $B_i e \supset B_0 e$, $i = 1, 2, 3, 4$. Since A/B_0 modules are A-modules we can assume that $B_0 = 0$. Because of the theorem of §1, we assume that the ideal lattice of A is distributive and hence that

$$\sum_{i=1}^{4} B_i = \bigoplus \sum_{i=1}^{4} B_i .$$

There exist primitive idempotents $f_i \in A$ such that $f_i B_i e \neq 0$, $i = 1, 2, 3, 4$. Let $\Lambda \subset eAe$ be a set of representatives for the nonzero cosets of the center of eAe/eNe. Choose $u_i = f_i u_i e \neq 0 \in B_i e$, $i = 1, 2, 3, 4$. For $\lambda \in \Lambda$ we have $Af_i/Nf_i \cong Au_i \cong Au_i\lambda$, $i = 1, 2, 3, 4$. For each positive integer n define

$$X^n = \bigoplus \sum_{i=1}^{2n} \varepsilon_i(Ae)$$

and denote the socle of X^n by Y^n. Define

$$T_\lambda^n = \Big\{\sum_{i=1}^{n} \varepsilon_i(a_i u_1 + c_i u_3 + d_i u_4 \lambda + d_{i-1} u_4) + \varepsilon_{i+n}(b_i u_2 + c_i u_3 + d_i u_4): \\ d_0 = 0,\ a_i, b_i, c_i, d_i \in A,\ i = 1, \cdots, n.\Big\},$$

$$H_\lambda^n = X^n / T_\lambda^n ,$$

and

$$S_\lambda^n = Y^n / T_\lambda^n .$$

Since the composition length of T_λ^n is equal to $4n$ and the composition length of Y^n is greater than or equal to $8n$, the composition length of S_λ^n increases without bound as n increases.

Let $\lambda \neq \mu$ be elements of Λ. We next prove that H_λ^n and H_μ^n are not isomorphic. Suppose θ is an isomorphism from H_λ^n onto H_μ^n. Since X^n is projective, θ can be lifted to a endomorphism $\bar\theta$ of X^n. There exist $x_1, \cdots, x_{2n}, y_1, \cdots, y_{2n}$ in eAe such that

$$\bar\theta\varepsilon_{2n}(e) = \sum_{i=1}^{2n} \varepsilon_i(x_i)$$

and

$$\bar\theta\varepsilon_n(e) = \sum_{i=1}^{2n} \varepsilon_i(y_i) .$$

Since, $\theta\pi_\lambda\varepsilon_n(u_4) \neq 0$, we have

$$\pi_\mu\Big(\sum_{i=1}^{2n} \varepsilon_i(u_4 y_i)\Big) = \theta\pi_\lambda\varepsilon_n(u_4) \neq 0 .$$

Thus, since $u_4 \in M$, there exists $k, 1 \leqq k \leqq 2n$, such that

$$y_k \notin eNe .$$

Since $u_1 y_i \in Au_2 + Au_3 + Au_4$ for $i > n$, we have $u_1 y_i = 0$ for $i > n$, and hence, since eAe/eNe is a division ring, $y_i \in eNe$, for $i > n$. Similarly, $\bar{\theta}\varepsilon_{2n}(u_2) \in T^n_\mu$ implies $x_i \in eNe$, for $i \leqq n$. It follows that

$$\bar{\theta}(\varepsilon_n u_3 + \varepsilon_{2n} u_3) = \sum_{i=1}^{n} \varepsilon_i(u_3 y_i) + \sum_{i=n+1}^{2n} \varepsilon_i(u_3 x_i) \in T^n_\mu .$$

Therefore, $u_3 y_i = u_3 x_{i+n}$ for $i = 1, \cdots, n$, and hence,

$$y_i = x_{i+n}, \ (\operatorname{mod} eNe) \qquad i = 1, \cdots, n .$$

From this we obtain

$$\bar{\theta}(\varepsilon_n(u_4\lambda) + \varepsilon_{2n}(u_4)) = \sum_{i=1}^{n} \varepsilon_i(u_4 \lambda y_i) + \sum_{i=n+1}^{2n} \varepsilon_i(u_4 y_{i-n}) \in T^n_\mu .$$

Hence, using the definition of T^n_μ there exist $d_1, \cdots, d_n \in A$ such that

$$u_4 \lambda y_1 = d_1 u_4 \mu ,$$
$$u_4 \lambda y_j = d_j u_4 \mu + d_{j-1} u_4 , \qquad j = 2, \cdots, n ,$$

and

$$u_4 y_j = d_j u_4 , \qquad j = 1, \cdots, n .$$

Replacing $d_j u_4$ by $u_4 y_j$ in these equations, we have

$$u_4 \lambda y_1 = u_4 y_1 \mu$$

and

$$u_4 \lambda y_j = u_4 y_j \mu + u_4 y_{j-1} , \qquad j = 2, \cdots, n .$$

Since $u_4 \in M$, a simple induction shows that

$$y_i \in eNe , \qquad i = 1, \cdots, n .$$

We conclude that H^n_λ and H^n_μ are not isomorphic.

Next, suppose that H^n_λ decomposes and let η be an idempotent endomorphism of H^n_λ such that $\eta\pi_\lambda(\varepsilon_n(u_3)) \neq 0$. Since X is projective, η can be lifted to an endomorphism $\bar{\eta}$ of X^n. There exist $y_{ij} \in eAe$ such that $\bar{\eta}(\varepsilon_j(e)) = \sum_{i=1}^{2n} \varepsilon_i(y_{ij})$. If $j \leqq n$, we have

$$\bar{\eta}(\varepsilon_j u_1) = \sum_{i=1}^{2n} \varepsilon_i(u_1 y_{ij}) \in T^n_\lambda$$

and hence

$$y_{ij} = 0 , \quad (\operatorname{mod} eNe) \qquad 1 \leqq i \leqq n, n + 1 \leqq j \leqq 2n .$$

For $j \leqq n$, we have,

$$\bar{\eta}(\varepsilon_j(u_3) + \varepsilon_{j+n}(u_3)) = \sum_{i=1}^{n} \varepsilon_i(u_3 y_{ij}) + \sum_{i=n+1}^{2n} \varepsilon_i(u_3 y_{i,j+n}) \in T_\lambda^n .$$

Thus, by the definition of T_λ^n,

$$y_{ij} = y_{i+n,j+n} \pmod{eNe} , \qquad 1 \leqq i, j \leqq n .$$

We infer that

$$\bar{\eta}(\varepsilon_n(u_4\lambda) + \varepsilon_{2n}(u_4)) = \sum_{i=1}^{n} \varepsilon_i(u_4\lambda y_{in}) + \sum_{i=n+1}^{2n} \varepsilon_i(u_4 y_{i-n,n}) \in T_\lambda^n .$$

Hence, there exist $d_0, \cdots, d_n \in A$, $d_0 = 0$, such that

$$u_4\lambda y_{jn} = d_j u_4 \lambda + d_{j-1} u_4 ,$$

and

$$u_4 y_{jn} = d_j u_4 , \qquad j = 1, \cdots, n .$$

Replacing $d_j u_4$ by $u_4 y_{jn}$, we have

$$u_4\lambda y_{1n} = u_4 y_{1n} \lambda ,$$

and

$$u_4\lambda y_{jn} = u_4 y_{jn}\lambda + u_4 y_{j-1,n} , \qquad j = 2, \cdots, n .$$

Hence, for $i < n$ we obtain $y_{in} = 0 \pmod{eNe}$. And, since η is idempotent and eAe/eNe is a division ring, $y_{nn} = e \pmod{eNe}$. Now suppose $k < n$. Then

$$\begin{aligned} &\bar{\eta}(\varepsilon_k(u_4\lambda) + \varepsilon_{k+1}(u_4) + \varepsilon_{k+n}(u_4)) \\ &\quad = \sum_{i=1}^{n} \varepsilon_i(u_4\lambda y_{ik} + u_4 y_{i,k+1}) + \sum_{i=n+1}^{2n} \varepsilon_i(u_4 y_{i-n,k}) \in T_\lambda^n . \end{aligned}$$

Hence, there exist $d_1^k, d_2^k, \cdots, d_n^k \in A$, $d_0^k = 0$, such that

$$u_4\lambda y_{ik} + u_4 y_{i,k+1} = d_j^k u_4 \lambda + d_{j-1}^k u_4 ,$$

and

$$u_4 y_{jk} = d_j^k u_4 , \qquad j = 1, \cdots, n .$$

Replacing $d_j^k u_4$ by $u_4 y_{jk}$ we obtain $u_4 y_{1,k+1} = 0$, and $u_4 y_{j,k+1} = u_4 y_{j-1,k}$, $j = 2, \cdots, n$, $k = 1, \cdots, n-1$. It follows from these equations that $y_{1k} = 0, \pmod{eNe}$ for $k = 2, \cdots, n$, and $y_{jk} = y_{j+1,k+1} \pmod{eNe}$, $j, k = 1, \cdots, n-1$. If $i < j \leqq n$, then

$$y_{ij} = y_{i-1,j-1} = \cdots = y_{1,j-i+1} = 0 \pmod{eNe} .$$

And, if $n \geqq i \geqq j$,

$$y_{ij} = y_{i-1,j-1} = \cdots = y_{i-j+1,1} \,(\mathrm{mod}\, eNe) \,.$$

These results imply

$$y_{ij} = \begin{cases} 0 \,(\mathrm{mod}\, eNe) \,, & \text{if } i < j, \text{ or } j \leqq n < i \,, \\ e \,(\mathrm{mod}\, eNe) \,, & \text{if } i = j \,, \\ y_{i-j+1,1} \,(\mathrm{mod}\, eNe) \,, & \text{if } j < i \leqq n, \text{ or } n < j < i \,. \end{cases}$$

We shall now show that the restriction of η to S_λ^n is a monomorphism and that $\eta(S_\lambda^n) = S_\lambda^n$. Suppose that $x \in Y^n$ is such that $\pi_\lambda(s)$ is an element of the kernel of η,

$$x = \sum_{i=1}^{2n} \varepsilon_i(x_i) \,.$$

We have $\eta\pi_\lambda(x) = \pi_\lambda\bar{\eta}(x) = 0$, and so

$$\begin{aligned} &\bar{\eta}(x) \in T_\lambda^n \,. \\ &\bar{\eta}(x) = \sum_{j=1}^{2n} \bar{\eta}\varepsilon_j(x_j) \\ &\quad = \sum_{j=1}^{2n}\sum_{i=1}^{2n} \varepsilon_i(x_j y_{ij}) \\ &\quad = \sum_{j=1}^{n}\sum_{i=1}^{n} \varepsilon_i(x_j y_{i-j+1,1}) + \sum_{j=n+1}^{2n}\sum_{i=j}^{2n} \varepsilon_i(x_j y_{i-j+1,1}) \,. \end{aligned}$$

Thus, there exist $a_1, b_i, c_i, d_i, i = 1, \cdots, n$ in A, $d_0 = 0$ such that

$$\sum_{j=1}^{i} x_j y_{ij} = a_i u_1 + c_i u_3 + d_i u_4 \lambda + d_{i-1} u_4 \,,$$

and

$$\sum_{j=1}^{i} x_{n+j} y_{ij} = b_i u_2 + c_i u_3 + d_i u_4 \,, \qquad \text{for } i = 1, 2, \cdots, n \,.$$

Using the definition of T_λ^n, it follows that

$$x_j = \alpha_j u_1 + \gamma_j u_3 + \delta_j u_4 \lambda + \delta_{j-1} u_4 \,,$$

and

$$x_{j+n} = \beta_j u_2 + \gamma_j u_3 + \delta_j u_4 \,, \qquad j = 1, \cdots, n-1 \,,$$

where $\alpha_1 = a_1$, $\beta_1 = b_1$, $\gamma_1 = c_1$, $\delta_0 = 0$, $\delta_1 = d_1$, and

$$\begin{aligned} \alpha_k u_1 &= a_k u_1 - \alpha_{k-1} u_1 y_{21} - \cdots - \alpha_1 u_1 y_{k1} \\ \beta_k u_2 &= b_k u_2 - \beta_{k-1} u_2 y_{21} - \cdots - \beta_1 u_2 y_{k1} \\ \gamma_k u_3 &= c_k u_3 - \gamma_{k-1} u_3 y_{21} - \cdots - \gamma_1 u_3 y_{k1} \\ \delta_k u_1 &= d_k u_4 - \delta_{k-1} u_4 y_{21} - \cdots - \delta_1 u_4 y_{k1} \,, \qquad \text{for } k > 1 \,. \end{aligned}$$

Hence, $\pi_\lambda(x) = 0$, and the restriction of η to S_λ^n is a monomorphism.

The proof can now be completed as in § 1.

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Received July 30, 1965. This paper consists of part of the author's doctoral thesis.

THE BOEING COMPANY AND
THE UNIVERSITY OF WASHINGTON

Printed in Japan by International Academic Printing Co., Ltd., Tokyo Japan

Pacific Journal of Mathematics
Vol. 19, No. 1 May, 1966