CONTRACTION SEMI-GROUPS IN A FUNCTION SPACE

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Using the concepts of a semi inner-product and a dissipative operator, it is proven that if \( X \) is a complex Banach space (under the supremum norm) of bounded complex valued functions on a set \( S \), \( p \) is a bounded positive function on \( S \) which is bounded away from zero, \( pX \subset X \), and \( A \) is the infinitesimal generator of a strongly continuous (class \( (C_0) \)) semi-group of contraction operators in \( X \), then \( pA \) is also the infinitesimal generator of such a semi-group.

The notion of a semi inner-product was introduced by G. Lumer in [3].

**Definition 1.** A semi inner-product for a complex (real) Banach space \( X \) is a function \( \langle \cdot, \cdot \rangle \) from \( X \times X \) into the complex (real) numbers which satisfies

\[
\langle ax + \beta y, z \rangle = a \langle x, z \rangle + \beta \langle y, z \rangle ,
\]

\[
|\langle x, z \rangle| \leq ||x|| \cdot ||z||,
\]

and

\[
\langle x, x \rangle = ||x||^2 .
\]

There is at least one semi inner-product for every Banach space \( X \), because we can define \( \langle x, y \rangle = f(x) \), where \( f \) is a bounded linear functional on \( X \) such that \( ||f|| = ||y|| \), and \( |f(y)| = ||y||^2 \) (see [4]).

By an operator in a Banach space \( X \), we mean a linear transformation (not necessarily bounded) from a subspace of \( X \) to a subspace of \( X \). The notion of a dissipative operator in a Banach space is treated by G. Lumer and R. S. Phillips in [4].

**Definition 2.** An operator \( A \) in a Banach space \( X \) is said to be dissipative (with respect to a given semi inner-product for \( X \)) if

\[
\text{re} \langle Ax, x \rangle \leq 0
\]

for all \( x \) in the domain of \( A \).

By a contraction semi-group in a Banach space \( X \) we mean a strongly continuous semi-group of contraction operators in \( X \) which is of class \( (C_0) \) (see [2]). A contraction operator in \( X \) is a bounded linear transformation \( T \) from \( X \) into \( X \) with \( ||T|| \leq 1 \). Lumer and Phillips have given the following characterization [4, Theorem 3.1] of the infinitesimal generator of a contraction semi-group.
THEOREM (Lumer and Phillips). Suppose $A$ is an operator in a Banach space $X$, the domain of $A$ is dense in $X$, and $[,]$ is a semi inner-product for $X$. Then $A$ is the infinitesimal generator of a contraction semi-group in $X$ if and only if $A$ is dissipative with respect to $[,]$, and the range of $I-A$ is all of $X$, where $I$ denotes the identity transformation on $X$.

THEOREM. Suppose $S$ is a set, $X$ is a complex Banach space (under the supremum norm) of bounded complex valued functions on $S$, $p$ is a bounded positive function on $S$ which is bounded away from zero, $pX \subset X$, and $A$ is the infinitesimal generator of a contraction semi-group in $X$. Then $pA$ is also the infinitesimal generator of a contraction semi-group in $X$.

Proof. Let $U$ denote the Banach algebra of all bounded complex valued functions on $S$, and let $S_1$ denote the set of all nonzero multiplicative linear functionals on $S$. It follows from [1, pp. 272–277], especially [1, Corollary 19, p. 276], that

(i) if $m$ is in $S_1$, and $q$ is a nonnegative function in $U$, then $m(q) \geq 0$, and
(ii) if $x$ is in $U$, then there is an $m$ in $S_1$ such that $|m(x)| = \|x\|$. For each $x$ in $X$, let $m_x$ denote an element $m$ of $S_1$ such that $|m(x)| = \|x\|$, and for each $x, y$ in $X$, let

$$[x, y] = m_y(x)[m_y(y)]^*,$$

where the $*$ denotes complex conjugation. Then $[,]$ is a semi inner-product for $X$; it is the only one to be used from this point on. A dissipative operator in $X$ will mean one which is dissipative with respect to this semi inner-product.

If $q$ is a bounded nonnegative function on $S$, and $qX \subset X$, then

$$\text{re}[qAx, x] = m_x(q) \text{re}[Ax, x] \leq 0,$$

for all $x$ in $\mathcal{D}(A)$, the domain of $A$, since $A$ is dissipative by [4, Theorem 3.1]. Therefore, $qA$ is dissipative. Also, the domain of $qA$ is $\mathcal{D}(A)$, which is dense in $X$ by [2, Theorem 12.3.1, p. 360]. If

$$\sup_{s \in S} |1 - q(s)| < 1/2,$$

then $\|I-q\|$, the operator norm of $I-q$, is less than $1/2$, so that $I-qA$ is invertible, since

$$I - qA = I - A + (I-q)A = (I + (I-q)AR(1, A))(I - A),$$

and
by [2, Theorem 12.3.1, p. 360]. Thus the range of \( I - qA \) is all of \( X \), and \( qA \) generates a contraction semi-group in \( X \) by [4, Theorem 3.1].

Since \( F(p)X \subset X \) for every polynomial \( F \), and \( p \) is bounded and nonnegative, it follows from the classical Weierstrass theorem that \( p^{(i/n)}X \subset X \) for every positive integer \( n \). Choose \( n \) so that

\[
\sup_{s \in S} |1 - [p(s)]^{(i/n)}| < 1/2 ,
\]
and let \( r = p^{(i/n)} \). This is possible because the range of \( p \) is contained in a closed and bounded interval of positive numbers. By what was shown in the previous paragraph, \( rA \) generates a contraction semi-group in \( X \). If \( 1 \leq j < n \), and \( r^jA \) generates a contraction semi-group in \( X \), then \( r^{j+1}A \) does also, for

\[
r^{j+1}A = r(r^jA) ,
\]
and we can substitute \( r \) for \( q \) and \( r^jA \) for \( A \) in the argument given in the previous paragraph.

REMARK. An argument similar to the one given will establish the theorem if \( X \) is taken to be a real Banach space (under the supremum norm) of bounded real valued functions on \( S \), and the rest of the hypothesis remains the same. Also, we could take \( A \) to be the generator of a class \((C_0)\) semi-group \([T(t); 0 \leq t < \infty]\) of operators in \( X \) such that for some \( \omega > 0 \),

\[
\| T(t) \| \leq e^{\omega t} \quad \text{for } t \geq 0 .
\]

If

\[
\tilde{T}(t) = e^{-\omega t}T(t) \quad \text{for } t \geq 0 ,
\]
then \([\tilde{T}(t)]\) is a contraction semi-group in \( X \) and has the generator \( \tilde{A} = A - \omega \).

If

\[
V(t) = e^{\omega t}p\tilde{V}(t) \quad \text{for } t \geq 0 ,
\]
where \([\tilde{V}(t); 0 \leq t < \infty]\) is the contraction semi-group generated by \( p\tilde{A} \), then \([V(t)]\) is a class \((C_0)\) semi-group of operators in \( X \),

\[
\| V(t) \| \leq e^{\omega t\|p\|} \quad \text{for } t \geq 0 ,
\]
and \([V(t)]\) is generated by \( pA \). The author wishes to express his thanks to the referee for his suggestions.
REFERENCES


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