CONTRACTION SEMI-GROUPS IN A FUNCTION SPACE

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Using the concepts of a semi inner-product and a dissipative operator, it is proven that if $X$ is a complex Banach space (under the supremum norm) of bounded complex valued functions on a set $S$, $p$ is a bounded positive function on $S$ which is bounded away from zero, $pX \subset X$, and $A$ is the infinitesimal generator of a strongly continuous (class $(C_0)$) semi-group of contraction operators in $X$, then $pA$ is also the infinitesimal generator of such a semi-group.

The notion of a semi inner-product was introduced by G. Lumer in [3].

**Definition 1.** A semi inner-product for a complex (real) Banach space $X$ is a function $[\cdot, \cdot]$ from $X \times X$ into the complex (real) numbers which satisfies

$$
[\alpha x + \beta y, z] = \alpha [x, z] + \beta [y, z],
$$

$$
|| [x, z] || \leq || x || \cdot || z ||,
$$

and

$$
[x, x] = || x ||^2.
$$

There is at least one semi inner-product for every Banach space $X$, because we can define $[x, y] = f(x)$, where $f$ is a bounded linear functional on $X$ such that $||f|| = ||y||$, and $||f(y)|| = ||y||^2$ (see [4]).

By an operator in a Banach space $X$, we mean a linear transformation (not necessarily bounded) from a subspace of $X$ to a subspace of $X$. The notion of a dissipative operator in a Banach space is treated by G. Lumer and R. S. Phillips in [4].

**Definition 2.** An operator $A$ in a Banach space $X$ is said to be dissipative (with respect to a given semi inner-product for $X$) if

$$
\text{re} [Ax, x] \leq 0
$$

for all $x$ in the domain of $A$.

By a contraction semi-group in a Banach space $X$ we mean a strongly continuous semi-group of contraction operators in $X$ which is of class $(C_0)$ (see [2]). A contraction operator in $X$ is a bounded linear transformation $T$ from $X$ into $X$ with $||T|| \leq 1$. Lumer and Phillips have given the following characterization [4, Theorem 3.1] of the infinitesimal generator of a contraction semi-group.
THEOREM (Lumer and Phillips). Suppose \( A \) is an operator in a Banach space \( X \), the domain of \( A \) is dense in \( X \), and \([\cdot, \cdot]\) is a semi inner-product for \( X \). Then \( A \) is the infinitesimal generator of a contraction semi-group in \( X \) if and only if \( A \) is dissipative with respect to \([\cdot, \cdot]\), and the range of \( I-A \) is all of \( X \), where \( I \) denotes the identity transformation on \( X \).

THEOREM. Suppose \( S \) is a set, \( X \) is a complex Banach space (under the supremum norm) of bounded complex valued functions on \( S \), \( p \) is a bounded positive function on \( S \) which is bounded away from zero, \( pX \subset X \), and \( A \) is the infinitesimal generator of a contraction semi-group in \( X \). Then \( pA \) is also the infinitesimal generator of a contraction semi-group in \( X \).

Proof. Let \( U \) denote the Banach algebra of all bounded complex valued functions on \( S \), and let \( S \) denote the set of all nonzero multiplicative linear functionals on \( S \). It follows from [1, pp. 272–277], especially [1, Corollary 19, p. 276], that

(i) if \( m \) is in \( S \), and \( q \) is a nonnegative function in \( U \), then \( m(q) \geq 0 \), and

(ii) if \( x \) is in \( U \), then there is an \( m \) in \( S \) such that \( |m(x)| = ||x|| \). For each \( x \) in \( X \), let \( m_x \) denote an element \( m \) of \( S \), such that \( |m(x)| = ||x|| \), and for each \( x, y \) in \( X \), let

\[
[x, y] = m_y(x)[m_y(y)]^* ,
\]

where the * denotes complex conjugation. Then \([\cdot, \cdot]\) is a semi inner-product for \( X \); it is the only one to be used from this point on. A dissipative operator in \( X \) will mean one which is dissipative with respect to this semi inner-product.

If \( q \) is a bounded nonnegative function on \( S \), and \( qX \subset X \), then

\[
\text{re} [qAx, x] = m_x(q) \text{ re} [Ax, x] \leq 0 ,
\]

for all \( x \) in \( \mathcal{D}(A) \), the domain of \( A \), since \( A \) is dissipative by [4, Theorem 3.1]. Therefore, \( qA \) is dissipative. Also, the domain of \( qA \) is \( \mathcal{D}(A) \), which is dense in \( X \) by [2, Theorem 12.3.1, p. 360]. If

\[
\sup_{s \in S} |1 - q(s)| < 1/2 ,
\]

then \( ||I - q|| \), the operator norm of \( I - q \), is less than 1/2, so that \( I - qA \) is invertible, since

\[
I - qA = I - A + (I - q)A = \{I + (I - q)AR(1, A)\}(I - A) ,
\]

and
by [2, Theorem 12.3.1, p. 360]. Thus the range of $I - qA$ is all of $X$, and $qA$ generates a contraction semi-group in $X$ by [4, Theorem 3.1].

Since $F(p)X \subset X$ for every polynomial $F$, and $p$ is bounded and nonnegative, it follows from the classical Weierstrass theorem that $p^{(j/n)}X \subset X$ for every positive integer $n$. Choose $n$ so that

$$\sup_{s \in S} |1 - [p(s)]^{(1/n)}| < 1/2,$$

and let $r = p^{(j/n)}$. This is possible because the range of $p$ is contained in a closed and bounded interval of positive numbers. By what was shown in the previous paragraph, $rA$ generates a contraction semi-group in $X$. If $1 \leq j < n$, and $r^jA$ generates a contraction semi-group in $X$, then $r^{j+1}A$ does also, for

$$r^{j+1}A = r(r^jA),$$

and we can substitute $r$ for $q$ and $r^jA$ for $A$ in the argument given in the previous paragraph.

**Remark.** An argument similar to the one given will establish the theorem if $X$ is taken to be a real Banach space (under the supremum norm) of bounded real valued functions on $S$, and the rest of the hypothesis remains the same. Also, we could take $A$ to be the generator of a class $(C_0)$ semi-group $[T(t); 0 \leq t < \infty]$ of operators in $X$ such that for some $\omega > 0$,

$$\|T(t)\| \leq e^{\omega t} \quad \text{for } t \geq 0.$$

If

$$\tilde{T}(t) = e^{-\omega t}T(t) \quad \text{for } t \geq 0,$$

then $[\tilde{T}(t)]$ is a contraction semi-group in $X$ and has the generator $\tilde{A} = A - \omega$.

If

$$V(t) = e^{\omega t}p\tilde{V}(t) \quad \text{for } t \geq 0,$$

where $[\tilde{V}(t); 0 \leq t < \infty]$ is the contraction semi-group generated by $p\tilde{A}$, then $[V(t)]$ is a class $(C_0)$ semi-group of operators in $X$,

$$\|V(t)\| \leq e^{\omega t\|p\|} \quad \text{for } t \geq 0,$$

and $[V(t)]$ is generated by $pA$. The author wishes to express his thanks to the referee for his suggestions.
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Received October 19, 1965, and in revised form December 2, 1965. Presented to the Society, November 12, 1965, under the title *Contraction semi-groups in a function algebra.*

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