

# Pacific Journal of Mathematics

**CONTRACTION SEMI-GROUPS IN A FUNCTION SPACE**

JAMES ROBERT DORROH

# CONTRACTION SEMI-GROUPS IN A FUNCTION SPACE

J. R. DORROH

Using the concepts of a semi inner-product and a dissipative operator, it is proven that if  $X$  is a complex Banach space (under the supremum norm) of bounded complex valued functions on a set  $S$ ,  $p$  is a bounded positive function on  $S$  which is bounded away from zero,  $pX \subset X$ , and  $A$  is the infinitesimal generator of a strongly continuous (class  $(C_0)$ ) semi-group of contraction operators in  $X$ , then  $pA$  is also the infinitesimal generator of such a semi-group.

The notion of a *semi inner-product* was introduced by G. Lumer in [3].

DEFINITION 1. A *semi inner-product* for a complex (real) Banach space  $X$  is a function  $[\cdot, \cdot]$  from  $X \times X$  into the complex (real) numbers which satisfies

$$[\alpha x + \beta y, z] = \alpha[x, z] + \beta[y, z],$$

$$|[x, z]| \leq \|x\| \cdot \|z\|,$$

and

$$[x, x] = \|x\|^2.$$

There is at least one semi inner-product for every Banach space  $X$ , because we can define  $[x, y] = f(x)y$ , where  $f$  is a bounded linear functional on  $X$  such that  $\|f\| = \|y\|$ , and  $|f(y)| = \|y\|^2$  (see [4]).

By an operator in a Banach space  $X$ , we mean a linear transformation (not necessarily bounded) from a subspace of  $X$  to a subspace of  $X$ . The notion of a *dissipative operator* in a Banach space is treated by G. Lumer and R. S. Phillips in [4].

DEFINITION 2. An operator  $A$  in a Banach space  $X$  is said to be *dissipative* (with respect to a given semi inner-product for  $X$ ) if

$$\operatorname{re}[Ax, x] \leq 0$$

for all  $x$  in the domain of  $A$ .

By a contraction semi-group in a Banach space  $X$  we mean a strongly continuous semi-group of contraction operators in  $X$  which is of class  $(C_0)$  (see [2]). A contraction operator in  $X$  is a bounded linear transformation  $T$  from  $X$  into  $X$  with  $\|T\| \leq 1$ . Lumer and Phillips have given the following characterization [4, Theorem 3.1] of the infinitesimal generator of a contraction semi-group.

**THEOREM (Lumer and Phillips).** *Suppose  $A$  is an operator in a Banach space  $X$ , the domain of  $A$  is dense in  $X$ , and  $[\cdot, \cdot]$  is a semi inner-product for  $X$ . Then  $A$  is the infinitesimal generator of a contraction semi-group in  $X$  if and only if  $A$  is dissipative with respect to  $[\cdot, \cdot]$ , and the range of  $I - A$  is all of  $X$ , where  $I$  denotes the identity transformation on  $X$ .*

**THEOREM.** *Suppose  $S$  is a set,  $X$  is a complex Banach space (under the supremum norm) of bounded complex valued functions on  $S$ ,  $p$  is a bounded positive function on  $S$  which is bounded away from zero,  $pX \subset X$ , and  $A$  is the infinitesimal generator of a contraction semi-group in  $X$ . Then  $pA$  is also the infinitesimal generator of a contraction semi-group in  $X$ .*

*Proof.* Let  $U$  denote the Banach algebra of all bounded complex valued functions on  $S$ , and let  $S_1$  denote the set of all nonzero multiplicative linear functionals on  $S$ . It follows from [1, pp. 272-277], especially [1, Corollary 19, p. 276], that

(i) if  $m$  is in  $S_1$ , and  $q$  is a nonnegative function in  $U$ , then  $m(q) \geq 0$ , and

(ii) if  $x$  is in  $U$ , then there is an  $m$  in  $S_1$  such that  $|m(x)| = \|x\|$ . For each  $x$  in  $X$ , let  $m_x$  denote an element  $m$  of  $S_1$  such that  $|m(x)| = \|x\|$ , and for each  $x, y$  in  $X$ , let

$$[x, y] = m_y(x)[m_y(y)]^*,$$

where the  $*$  denotes complex conjugation. Then  $[\cdot, \cdot]$  is a semi inner-product for  $X$ ; it is the only one to be used from this point on. A dissipative operator in  $X$  will mean one which is dissipative with respect to this semi inner-product.

If  $q$  is a bounded nonnegative function on  $S$ , and  $qX \subset X$ , then

$$\operatorname{re} [qAx, x] = m_x(q) \operatorname{re} [Ax, x] \leq 0,$$

for all  $x$  in  $\mathfrak{D}(A)$ , the domain of  $A$ , since  $A$  is dissipative by [4, Theorem 3.1]. Therefore,  $qA$  is dissipative. Also, the domain of  $qA$  is  $\mathfrak{D}(A)$ , which is dense in  $X$  by [2, Theorem 12.3.1, p. 360]. If

$$\sup_{s \in S} |1 - q(s)| < 1/2,$$

then  $\|I - q\|$ , the operator norm of  $I - q$ , is less than  $1/2$ , so that  $I - qA$  is invertible, since

$$I - qA = I - A + (I - q)A = \{I + (I - q)AR(1, A)\}(I - A),$$

and

$$\|AR(1, A)\| = \|R(1, A) - I\| \leq 2$$

by [2, Theorem 12.3.1, p. 360]. Thus the range of  $I - qA$  is all of  $X$ , and  $qA$  generates a contraction semi-group in  $X$  by [4, Theorem 3.1].

Since  $F(p)X \subset X$  for every polynomial  $F$ , and  $p$  is bounded and nonnegative, it follows from the classical Weierstrass theorem that  $p^{(1/n)}X \subset X$  for every positive integer  $n$ . Choose  $n$  so that

$$\sup_{s \in S} |1 - [p(s)]^{(1/n)}| < 1/2,$$

and let  $r = p^{(1/n)}$ . This is possible because the range of  $p$  is contained in a closed and bounded interval of positive numbers. By what was shown in the previous paragraph,  $rA$  generates a contraction semi-group in  $X$ . If  $1 \leq j < n$ , and  $r^jA$  generates a contraction semi-group in  $X$ , then  $r^{j+1}A$  does also, for

$$r^{j+1}A = r(r^jA),$$

and we can substitute  $r$  for  $q$  and  $r^jA$  for  $A$  in the argument given in the previous paragraph.

**REMARK.** An argument similar to the one given will establish the theorem if  $X$  is taken to be a real Banach space (under the supremum norm) of bounded real valued functions on  $S$ , and the rest of the hypothesis remains the same. Also, we could take  $A$  to be the generator of a class  $(C_0)$  semi-group  $[T(t); 0 \leq t < \infty]$  of operators in  $X$  such that for some  $\omega > 0$ ,

$$\|T(t)\| \leq e^{\omega t} \quad \text{for } t \geq 0.$$

If

$$\tilde{T}(t) = e^{-\omega t}T(t) \quad \text{for } t \geq 0,$$

then  $[\tilde{T}(t)]$  is a contraction semi-group in  $X$  and has the generator  $\tilde{A} = A - \omega$ .

If

$$V(t) = e^{\omega t p} \tilde{V}(t) \quad \text{for } t \geq 0,$$

where  $[\tilde{V}(t); 0 \leq t < \infty]$  is the contraction semi-group generated by  $p\tilde{A}$ , then  $[V(t)]$  is a class  $(C_0)$  semi-group of operators in  $X$ ,

$$\|V(t)\| \leq e^{\omega t \|p\|} \quad \text{for } t \geq 0,$$

and  $[V(t)]$  is generated by  $pA$ . The author wishes to express his thanks to the referee for his suggestions.

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