

# Pacific Journal of Mathematics

## **RESTRICTED BIPARTITE PARTITIONS**

L. CARLITZ AND DAVID PAUL ROSELLE

## RESTRICTED BIPARTITE PARTITIONS

L. CARLITZ AND D. P. ROSELLE

Let  $\pi_k(n, m)$  denote the number of partitions

$$\begin{aligned} n &= n_1 + n_2 + \cdots + n_k \\ m &= m_1 + m_2 + \cdots + m_k \end{aligned}$$

subject to the conditions

$$\min(n_j, m_j) \geq \max(n_{j+1}, m_{j+1}) \quad (j = 1, 2, \dots, k-1).$$

Put

$$\xi^{(k)}(x, y) = \sum_{n, m=0}^{\infty} \pi_k(n, m) x^n y^m.$$

We show that

$$\begin{aligned} \xi^{(k)}(x, y) &= \prod_{j=1}^k \frac{1 - x^{2j-1} y^{2j-1}}{(1 - x^j y^j)(1 - x^j y^{j-1})(1 - x^{j-1} y^j)}, \\ \sum_{n, m=0}^{\infty} \pi(n, m; \lambda) x^n y^m &= 1 + (1 - \lambda) \sum_{k=1}^{\infty} \lambda^k \xi^{(k)}(x, y), \\ \sum_{n, m=0}^{\infty} \phi(n, m) x^n y^m &= \sum_{n=0}^{\infty} x^n y^n \xi^{(n)}(x^2, y^2), \end{aligned}$$

where  $\pi(n, m; \lambda)$  denotes the number of “weighted” partitions of  $(n, m)$  and  $\phi(n, m)$  is the number of partitions into odd parts ( $n_j, m_j$  all odd).

Consider partitions of the bipartite  $(n, m)$  of the type

$$(1.1) \quad \begin{aligned} n &= n_1 + n_2 + n_3 + \cdots \\ m &= m_1 + m_2 + m_3 + \cdots, \end{aligned}$$

where the  $n_j, m_j$  are nonnegative integers subject to the conditions

$$(1.2) \quad \min(n_j, m_j) \geq \max(n_{j+1}, m_{j+1}) \quad (j = 1, 2, 3, \dots).$$

For brevity we may write (1.2) in the form

$$(n_j, m_j) \geq (n_{j+1}, m_{j+1}) \quad (j = 1, 2, 3, \dots)$$

and say that the “parts” of the partition (1.1) decrease.

Let  $\pi(n, m)$  denote the number of partitions (1.1) that satisfy (1.2) and let  $\rho(n, m)$  denote the numbers of partitions (1.1) that satisfy

$$(1.3) \quad (n_j, m_j) > (n_{j+1}, m_{j+1}) \quad (j=1, 2, 3, \dots).$$

By the inequality (1.3) is understood

$$\min(n_j, m_j) > \max(n_{j+1}, m_{j+1}) \quad (j = 1, 2, 3, \dots).$$

The generating functions for  $\pi(n, m)$  and  $\rho(n, m)$  are given by [2]

$$(1.4) \quad \prod_{j=1}^{\infty} (1 - x^{2j}y^{2j})^{-1} (1 - x^jy^{j-1})^{-1} (1 - x^{j-1}y^j)^{-1},$$

$$(1.5) \quad \frac{1-xy}{(1-x)(1-y)} \sum_{n=0}^{\infty} (xy)^{n(n+1)/2} \prod_{j=1}^n \frac{1-x^{2j+1}y^{2j+1}}{(1-x^jy^j)(1-x^{j+1}y^j)(1-x^jy^{j+1})},$$

respectively.

For the case of unipartite (natural) numbers generating functions are known for partitions with parts restricted in various ways [3]. The notion of a part of the partition (1.1) implied by the conditions (1.2) suggests that these results can be extended to bipartite numbers. For example, we may think of  $\rho(n, m)$  as the number of partitions of  $(n, m)$  with unequal parts. We shall find generating functions for bipartite partitions with at most  $k$  parts, weighted parts, and odd parts.

**2. Partitions with at most  $k$  parts.** We consider partitions of the type

$$(2.1) \quad \begin{aligned} n &= n_1 + n_2 + \dots + n_k \\ m &= m_1 + m_2 + \dots + m_k, \end{aligned}$$

where the  $n_j, m_j$  are nonnegative integers subject to the conditions

$$(2.2) \quad (n_j, m_j) \geq (n_{j+1}, m_{j+1}) \quad (j = 1, 2, \dots, k - 1).$$

Let  $\pi_k(n, m)$  denote the number of partitions (2.1) subject to the conditions (2.2) and let  $\pi_k(n, m | a, b)$  denote the numbers of these partitions that also satisfy

$$(2.3) \quad (a, b) \geq (n_1, m_1).$$

Note that  $\pi(n, m)$  defined in §1 satisfies

$$(2.4) \quad \pi(n, m) = \lim_{k \rightarrow \infty} \pi_k(n, m).$$

We define the rational function  $\xi_{ab}^{(k)}$  of  $x$  and  $y$  by the recurrence

$$(2.5) \quad \xi_{ab}^{(0)} = 1, \quad \xi_{ab}^{(k)} = \sum_{r,s=0}^{\min(a,b)} x^r y^s \xi_{rs}^{(k-1)} \quad (k \geq 1).$$

If we put

$$(2.6) \quad \xi^{(k)} = \xi_{\infty\infty}^{(k)},$$

then in the limit (2.5) becomes

$$(2.7) \quad \xi^{(k)} = \sum_{r,s=0}^{\infty} x^r y^s \xi_{rs}^{(k-1)} \quad (k \geq 1).$$

It is clear from (2.5) that  $\xi_{ab}^{(k)}$  is the generating function for  $\pi_k(n, m | a, b)$ . Thus it follows from (2.6) that  $\xi^{(k)}$  is the generating function for  $\pi_k(n, m)$ . Explicitly, we have

$$(2.8) \quad \xi_{ab}^{(k)} = \sum_{n,m=0}^{\infty} \pi_k(n, m | a, b) x^n y^m,$$

$$(2.9) \quad \xi^{(k)} = \sum_{n,m=0}^{\infty} \pi_k(n, m) x^n y^m.$$

We define the generating functions

$$(2.10) \quad F_k(u, v) = \sum_{r,s=0}^{\infty} u^r v^s \xi_{rs}^{(k-1)},$$

$$(2.11) \quad F_k^{(u)} = \sum_{n=0}^{\infty} u^n \xi_{nn}^{(k-1)},$$

so that

$$(2.12) \quad F_k(x, y) = \xi^{(k)}.$$

Using (2.10), (2.11) and

$$(2.13) \quad \xi_{rr}^{(k)} = \xi_{ab}^{(k)} \quad (r = \min(a, b)),$$

we get

$$\begin{aligned} F_k(u, v) &= \sum_{r \geq s} u^r v^s \xi_{ss}^{(k-1)} + \sum_{s \geq r} u^r v^s \xi_{rr}^{(k-1)} - \sum_{r=0}^{\infty} u^r v^r \xi_{rr}^{(k-1)} \\ &= \left( \frac{1}{1-u} + \frac{1}{1-v} - 1 \right) F_k(uv). \end{aligned}$$

It follows that

$$(2.14) \quad F_k(u, v) = \frac{1-uv}{(1-u)(1-v)} F_k(uv).$$

On the other hand, using (2.5), (2.11), and (2.13), we get

$$\begin{aligned} F_k(u) &= \sum_{n=0}^{\infty} u^n \sum_{r,s=0}^n x^r y^s \xi_{rs}^{(k-2)} \\ &= \frac{1}{1-u} \left( \sum_{r \geq s} u^r x^r y^s \xi_{ss}^{(k-1)} + \sum_{s \geq r} u^s y^s x^r \xi_{rr}^{(k-1)} - \sum_{r=0}^{\infty} (xyu)^r \xi_{rr}^{(k-1)} \right) \\ &= \frac{1}{1-u} \left( \frac{1}{1-ux} + \frac{1}{1-uy} - 1 \right) F_{k-1}(xyu), \end{aligned}$$

which implies

$$(2.15) \quad F_k(u) = \frac{1 - xyu^2}{(1 - u)(1 - xu)(1 - yu)} F_{k-1}(xyu) \quad (k \geq 1).$$

It follows from (2.5), (2.11), and (2.15) that

$$(2.16) \quad F_k(u) = \frac{1}{1 - u} \prod_{j=0}^{k-2} \frac{1 - x^{2j+1}y^{2j+1}u^2}{(1 - x^{j+1}y^{j+1}u)(1 - x^jy^{j+1}u)(1 - x^{j+1}y^j u)}.$$

Thus, using (2.12) and (2.14), we have evidently proved

**THEOREM 1.** *If  $\xi^{(k)}$  is defined by (2.9) then*

$$(2.17) \quad \xi^{(k)} = \prod_{j=1}^k \frac{1 - x^{2j-1}y^{2j-1}}{(1 - x^jy^j)(1 - x^jy^{j-1})(1 - x^{j-1}y^j)}.$$

We may now write (1.5) in the form

$$(2.18) \quad \sum_{n=1}^{\infty} (xy)^{n(n-1)/2} (1 - x^n y^n) \xi^{(n)}(x, y),$$

which is analogous to the well-known identity

$$(2.19) \quad \prod_{n=1}^{\infty} (1 + x^n) = \sum_{n=1}^{\infty} x^{n(n-1)/2} \prod_{j=1}^{n-1} (1 - x^j)^{-1}.$$

**3. A  $q$ -identity.** If we put

$$(3.1) \quad \xi = \xi^{(\infty)}, \quad \xi_{ab} = \xi_{ab}^{(\infty)},$$

then it follows from (2.4) and (2.9) that  $\xi$  is the generating function for  $\pi(n, m)$ . Moreover, it is clear from (2.14) and (2.16) that

$$(3.2) \quad F(u, v) = \sum_{r,s=0}^{\infty} u^r v^s \xi_{rs} = \frac{1 - uv}{(1 - u)(1 - v)} F(uv),$$

$$(3.3) \quad \begin{aligned} F(u) &= \sum_{n=0}^{\infty} u^n \xi_{nn} \\ &= e(u, xy) e(xu, xy) e(yu, xy) \prod_{j=0}^{\infty} (1 - x^{2j+1}y^{2j+1}u^2), \end{aligned}$$

where

$$(3.4) \quad \begin{aligned} e(t) &= e(t, q) = \prod_0^{\infty} (1 - q^n t)^{-1} = \prod_0^{\infty} \frac{t^n}{(q)_n}, \\ (q)_n &= (1 - q)(1 - q^2) \cdots (1 - q^n). \end{aligned}$$

We define the polynomial

$$(3.5) \quad H_n(x) = H_n(x, q) = \sum_{r=0}^n \begin{bmatrix} n \\ r \end{bmatrix} x^r,$$

where

$$\left[ \begin{matrix} n \\ r \end{matrix} \right] = \frac{(q)_n}{(q)_r(q)_{n-r}} .$$

It has been shown [1] that

$$(3.6) \quad \sum_0^\infty \frac{H_k(x)H_k(y)}{(q)_k} t^k = \frac{e(t) e(xt) e(yt) e(xyt)}{e(xyt^2)} .$$

Using (3.3), (3.4), and (3.6), we then have

$$\sum_0^\infty u^n \xi_{nn} = \sum_0^\infty \frac{H_k(x)H_k(y)}{(xy)_k} u^k \sum_0^\infty (-1)^r \frac{x^r y^r u^r}{(xy)_r} .$$

Comparing coefficients of  $u^n$ , we get

$$(3.7) \quad \xi_{nn} = \frac{1}{(xy)_n} \sum_{k=0}^n (-1)^{n-k} \left[ \begin{matrix} n \\ k \end{matrix} \right] x^{n-k} y^{n-k} H_k(x)H_k(y) .$$

Note that  $xy = q$  in the right member of (3.7).

It is clear from (3.7) that

$$(3.8) \quad P_n(x, y) = (xy)_n \xi_{nn}$$

is a polynomial in  $x, y$  with integral coefficients which satisfies

$$\begin{aligned} P_n(x, y) &= P_n(y, x) , \\ P_n(x, 0) &= \frac{1 - x^{n+1}}{1 - x} , \\ x^n P_n\left(x, \frac{1}{x}\right) &= (x^2 + x + 1)^n . \end{aligned}$$

Also it follows from (2.15) that  $P_n(x, y)$  satisfies the recurrence

$$(3.9) \quad \begin{aligned} P_n - (1 + x + y)P_{n-1} + [n - 1] (x + y + xy + x^{n-1}y^{n-1})P_{n-2} \\ - xy[n - 1][n - 2]P_{n-3} = 0 , \end{aligned}$$

where  $[j] = 1 - x^j y^j$ .

4. **Weighted partitions.** We define  $\pi(n, m; \lambda)$ , the number of weighted partitions of the bipartite  $(n, m)$ , by the relation

$$(4.1) \quad \pi(n, m; \lambda) = \sum_{k=0}^\infty \lambda^k \sum 1 ,$$

where the inner sum is extended over all partitions of the form (2.1) subject to the conditions (2.2) and the additional condition  $\max(n_k, m_k) > 0$ ; that is, over all partitions with exactly  $k$  parts. It follows from the definition of  $\pi_k(n, m)$  that we may write (4.1) in the form

$$(4.2) \quad \pi(n, m; \lambda) = \sum_{k=0}^{\infty} \lambda^k (\pi_k(n, m) - \pi_{k-1}(n, m)) .$$

It should be remarked that the sum in (4.2) is finite, the upper bound for  $k$  being  $\max(n, m)$ .

Multiplying both members of (4.2) by  $x^n y^m$  and summing over  $n, m$  it follows from (2.9) and (2.17) that we have established

**THEOREM 2.** *We have*

$$(4.3) \quad \sum_{n, m=0}^{\infty} \pi(n, m; \lambda) x^n y^m = 1 + (1 - \lambda) \sum_{k=1}^{\infty} \lambda^k \xi^{(k)}(x, y) .$$

Note that (4.3) is a direct analogue of the well-known identity

$$(4.4) \quad \prod_{n=1}^{\infty} (1 - \lambda x^n)^{-1} = \sum_{n=0}^{\infty} \lambda^n x^n \sum_{j=1}^n (1 - x^j)^{-1} .$$

We remark that (4.3) may be proved in a different manner. If we put

$$(4.5) \quad \xi_{ab}(\lambda) = 1 + \lambda \sum_{r, s=0}^{\min(a, b)} x^r y^s \xi_{rs} ,$$

where the prime denotes that we sum over all  $r, s$  in the indicated range except  $r = s = 0$ , then it follows from (4.1) that

$$(4.6) \quad \xi(\lambda) = \xi_{\infty\infty}(\lambda)$$

is the generating function for  $\pi(n, m; \lambda)$ . We may then evaluate  $\xi(\lambda)$  by the methods of §2.

**5. Partitions into odd parts.** We shall say that the  $j$ -th part of the partition (1.1) is odd if each of  $n_j, m_j$  is odd.

Let  $\psi(n, m)$  denote the number of partitions of the form (1.1) with parts odd and subject to the conditions (1.2). Let  $\psi(n, m | a, b)$  denote the number of these partitions that satisfy the additional condition

$$(5.1) \quad (2a + 1, 2b + 1) \geq (n_1, m_1) .$$

We define the rational function  $\beta_{2a+1, 2b+1}$  of  $x, y$  by the relation

$$(5.2) \quad \beta_{2a+1, 2b+1} = 1 + \sum_{r, s=0}^{\min(a, b)} x^{2r+1} y^{2s+1} \beta_{2r+1, 2s+1} ,$$

so that

$$(5.3) \quad \beta_{2r+1, 2r+1} = \beta_{2a+1, 2b+1} \quad (r = \min(a, b)) .$$

If we put

$$(5.4) \quad \beta = \beta_{\infty\infty} ,$$

then in the limit (5.2) becomes

$$(5.5) \quad \beta = 1 + \sum_{r,s=0}^{\infty} x^{2r+1}y^{2s+1}\beta_{2r+1,2s+1} .$$

It follows from (5.2) that

$$(5.6) \quad \beta_{2a+1,2b+1} = \sum_{n,m=0}^{\infty} \psi(n, m | a, b)x^n y^m .$$

Thus, using (5.5), we get

$$(5.7) \quad \beta = \sum_{n,m=0}^{\infty} \psi(n, m)x^n y^m .$$

We define the generating functions

$$(5.8) \quad H(u, v) = \sum_{r,s=0}^{\infty} u^r v^s \beta_{2r+1,2s+1} ,$$

$$(5.9) \quad H(u) = \sum_{n=0}^{\infty} u^n \beta_{2n+1,2n+1} ,$$

so that

$$(5.10) \quad \beta = 1 + xy H(x^2, y^2) .$$

Using (5.3), (5.8) and (5.9), we have

$$(5.11) \quad H(u, v) = \frac{1 - uv}{(1 - u)(1 - v)} H(uv) .$$

The proof of (5.11) is exactly like that of (2.14).

On the other hand, it follows from (5.2), (5.3), and (5.9) that

$$\begin{aligned} H(u) &= \sum_{n=0}^{\infty} u^n \left( 1 + \sum_{r,s=0}^n x^{2r+1}y^{2s+1}\beta_{2r+1,2s+1} \right) \\ &= \frac{1}{1 - u} + \frac{xy}{1 - u} \sum_{r,s=0}^{\infty} x^{2r}y^{2s}u^{\max(r,s)}\beta_{2r+1,2s+1} \\ &= \frac{1}{1 - u} + \frac{xy}{1 - u} \left( \frac{1}{1 - x^2u} + \frac{1}{1 - y^2u} - 1 \right) H(x^2y^2u) , \end{aligned}$$

which implies

$$(5.12) \quad H(u) = \frac{1}{1 - u} \left( 1 + \frac{1 - x^2y^2u^2}{(1 - x^2u)(1 - y^2u)} H(x^2y^2u) \right) .$$

Repeated applications of (5.12) yield



$$(5.13) \quad H(u) = \frac{1}{1-u} \sum_{n=0}^{\infty} x^n y^n \prod_{j=1}^n \frac{1 - x^{4j+2} y^{4j+2} u^2}{(1 - x^{2j+2} y^{2j+2} u) (1 - x^{2j} y^{2j+2} u) (1 - x^{2j+2} y^{2j} u)} .$$

Thus, using (5.10), (5.11), and (2.17), we may state

**THEOREM 3.** *If  $\psi(n, m)$  denotes the number of partitions of  $(n, m)$  with odd parts, then*

$$(5.14) \quad \sum_{n, m=0}^{\infty} \psi(n, m) x^n y^m = \sum_{n=0}^{\infty} x^n y^n \xi^{(n)}(x^2, y^2) ,$$

where  $\xi^{(n)}(x, y)$  is defined by (2.17).

The fact that (2.18) and (5.14) are analogous to well-known identities for unipartite numbers leads one to conjecture that  $\rho(n, m) = \psi(n, m)$ . There are, however, counterexamples to this conjecture. For example, it is easily verified that

$$\rho(5, 4) = 6 \neq 4 = \psi(5, 4) .$$

It would be of interest to know whether generally

$$\rho(n, m) \geq \psi(n, m) .$$

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