Pacific Journal of Mathematics

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Vol. 19, No. 2 June 1966

PACIFIC JOURNAL OF MATHEMATICS
Vol. 19, No. 2, 1966

RESTRICTED BIPARTITE PARTITIONS

L. CARLITZ AND D. P. ROSELLE

Let $\pi_k(n, m)$ denote the number of partitions

$$n = n_1 + n_2 + \cdots + n_k$$
$$m = m_1 + m_2 + \cdots + m_k$$

subject to the conditions

$$\min(n_j, m_j) \geqq \max(n_{j+1}, m_{j+1}) \qquad (j = 1, 2, \cdots, k-1)\,.$$

Put

$$\xi^{(k)}(x, y) = \sum_{n,m=0}^{\infty} \pi_k(n, m) x^n y^m\,.$$

We show that

$$\xi^{(k)}(x, y) = \prod_{j=1}^{k} \frac{1 - x^{2j-1}y^{2j-1}}{(1 - x^j y^j)(1 - x^j y^{j-1})(1 - x^{j-1}y^j)}\,,$$

$$\sum_{n,m=0}^{\infty} \pi(n, m; \lambda) x^n y^m = 1 + (1 - \lambda) \sum_{k=1}^{\infty} \lambda^k \xi^{(k)}(x, y)\,,$$

$$\sum_{n\ m=0}^{\infty} \phi(n, m) x^n y^m = \sum_{n=0}^{\infty} x^n y^n \xi^{(n)}(x^2, y^2)\,,$$

where $\pi(n, m; \lambda)$ denotes the number of "weighted" partitions of (n, m) and $\phi(n, m)$ is the number of partitions into odd parts (n_j, m_j all odd).

Consider partitions of the bipartite (n, m) of the type

$$n = n_1 + n_2 + n_3 + \cdots \tag{1.1}$$
$$m = m_1 + m_2 + m_3 + \cdots\,,$$

where the n_j, m_j are nonnegative integers subject to the conditions

$$\min(n_j, m_j) \geqq \max(n_{j+1}, m_{j+1}) \qquad (j = 1, 2, 3, \cdots)\,. \tag{1.2}$$

For brevity we may write (1.2) in the form

$$(n_j, m_j) \geqq (n_{j+1}, m_{j+1}) \qquad (j = 1, 2, 3, \cdots)$$

and say that the "parts" of the partition (1.1) decrease.

Let $\pi(n, m)$ denote the number of partitions (1.1) that satisfy (1.2) and let $\rho(n, m)$ denote the numbers of partitions (1.1) that satisfy

$$(n_j, m_j) > (n_{j+1}, m_{j+1}) \qquad (j=1, 2, 3, \cdots)\,. \tag{1.3}$$

By the inequality (1.3) is understood

$$\min(n_j, m_j) > \max(n_{j+1}, m_{j+1}) \qquad (j = 1, 2, 3, \cdots) .$$

The generating functions for $\pi(n, m)$ and $\rho(n, m)$ are given by [2]

$$(1.4) \qquad \prod_{j=1}^{\infty} (1 - x^{2j}y^{2j})^{-1} (1 - x^j y^{j-1})^{-1} (1 - x^{j-1}y^j)^{-1} ,$$

$$(1.5) \qquad \frac{1-xy}{(1-x)(1-y)} \sum_{n=0}^{\infty} (xy)^{n(n+1)/2} \prod_{j=1}^{n} \frac{1-x^{2j+1}y^{2j+1}}{(1-x^j y^j)(1-x^{j+1}y^j)(1-x^j y^{j+1})} ,$$

respectively.

For the case of unipartite (natural) numbers generating functions are known for partitions with parts restricted in various ways [3]. The notion of a part of the partition (1.1) implied by the conditions (1.2) suggests that these results can be extended to bipartite numbers. For example, we may think of $\rho(n, m)$ as the number of partitions of (n, m) with unequal parts. We shall find generating functions for bipartite partitions with at most k parts, weighted parts, and odd parts.

2. Partitions with at most k parts. We consider partitions of the type

$$(2.1) \qquad \begin{aligned} n &= n_1 + n_2 + \cdots + n_k \\ m &= m_1 + m_2 + \cdots + m_k , \end{aligned}$$

where the n_j, m_j are nonnegative integers subject to the conditions

$$(2.2) \qquad (n_j, m_j) \geqq (n_{j+1}, m_{j+1}) \qquad (j = 1, 2, \cdots, k - 1) .$$

Let $\pi_k(n, m)$ denote the number of partitions (2.1) subject to the conditions (2.2) and let $\pi_k(n, m \mid a, b)$ denote the numbers of these partitions that also satisfy

$$(2.3) \qquad (a, b) \geqq (n_1, m_1) .$$

Note that $\pi(n, m)$ defined in § 1 satisfies

$$(2.4) \qquad \pi(n, m) = \lim_{k=\infty} \pi_k(n, m) .$$

We define the rational function $\xi_{ab}^{(k)}$ of x and y by the recurrence

$$(2.5) \qquad \xi_{ab}^{(0)} = 1, \qquad \xi_{ab}^{(k)} = \sum_{r,s=0}^{\min(a,b)} x^r y^s \xi_{rs}^{(k-1)} \qquad (k \geqq 1) .$$

If we put

$$(2.6) \qquad \xi^{(k)} = \xi_{\infty\infty}^{(k)} ,$$

then in the limit (2.5) becomes

(2.7) $$\xi^{(k)} = \sum_{r\ s=0}^{\infty} x^r y^s \xi_{rs}^{(k-1)} \qquad (k \geqq 1)\ .$$

It is clear from (2.5) that $\xi_{ab}^{(k)}$ is the generating function for $\pi_k(n, m \mid a, b)$. Thus it follows from (2.6) that $\xi^{(k)}$ is the generating function for $\pi_k(n, m)$. Explicitly, we have

(2.8) $$\xi_{ab}^{(k)} = \sum_{n,m=0}^{\infty} \pi_k(n, m \mid a, b) x^n y^m\ ,$$

(2.9) $$\xi^{(k)} = \sum_{n,m=0}^{\infty} \pi_k(n, m) x^n y^m\ .$$

We define the generating functions

(2.10) $$F_k(u, v) = \sum_{r,s=0}^{\infty} u^r v^s \xi_{rs}^{(k-1)}\ ,$$

(2.11) $$F_k^{(u)} = \sum_{n=0}^{\infty} u^n \xi_{nn}^{(k-1)}\ ,$$

so that

(2.12) $$F_k(x, y) = \xi^{(k)}\ .$$

Using (2.10), (2.11) and

(2.13) $$\xi_{rr}^{(k)} = \xi_{ab}^{(k)} \qquad (r = \min(a, b))\ ,$$

we get

$$\begin{aligned} F_k(u, v) &= \sum_{r\geqq s} u^r v^s \xi_{ss}^{(k-1)} + \sum_{s\geqq r} u^r v^s \xi_{rr}^{(k-1)} - \sum_{r=0}^{\infty} u^r v^r \xi_{rr}^{(k-1)} \\ &= \Big(\frac{1}{1-u} + \frac{1}{1-v} - 1\Big) F_k(uv)\ . \end{aligned}$$

It follows that

(2.14) $$F_k(u, v) = \frac{1-uv}{(1-u)(1-v)} F_k(uv)\ .$$

On the other hand, using (2.5), (2.11), and (2.13), we get

$$\begin{aligned} F_k(u) &= \sum_{n=0}^{\infty} u^n \sum_{r,s=0}^{n} x^r y^s \xi_{rs}^{(k-2)} \\ &= \frac{1}{1-u}\Big(\sum_{r\geqq s} u^r x^r y^s\, \xi_{ss}^{(k-1)} + \sum_{s\geqq r} u^s y^s x^r \xi_{rr}^{(k-1)} - \sum_{r=0}^{\infty} (xyu)^r \xi_{rr}^{(k-1)}\Big) \\ &= \frac{1}{1-u}\Big(\frac{1}{1-ux} + \frac{1}{1-uy} - 1\Big) F_{k-1}(xyu)\ , \end{aligned}$$

which implies

$$F_k(u) = \frac{1 - xyu^2}{(1-u)(1-xu)(1-yu)} F_{k-1}(xyu) \qquad (k \geqq 1) . \tag{2.15}$$

It follows from (2.5), (2.11), and (2.15) that

$$F_k(u) = \frac{1}{1-u} \prod_{j=0}^{k-2} \frac{1 - x^{2j+1}y^{2j+1}u^2}{(1 - x^{j+1}y^{j+1}u)(1 - x^j y^{j+1}u)(1 - x^{j+1}y^j u)} . \tag{2.16}$$

Thus, using (2.12) and (2.14), we have evidently proved

THEOREM 1. *If $\xi^{(k)}$ is defined by* (2.9) *then*

$$\xi^{(k)} = \prod_{j=1}^{k} \frac{1 - x^{2j-1}y^{2j-1}}{(1 - x^j y^j)(1 - x^j y^{j-1})(1 - x^{j-1}y^j)} . \tag{2.17}$$

We may now write (1.5) in the form

$$\sum_{n=1}^{\infty} (xy)^{n(n-1)/2}(1 - x^n y^n)\xi^{(n)}(x, y) , \tag{2.18}$$

which is analogous to the well-known identity

$$\prod_{n=1}^{\infty} (1 + x^n) = \sum_{n=1}^{\infty} x^{n(n-1)/2} \prod_{j=1}^{n-1} (1 - x^j)^{-1} . \tag{2.19}$$

3. **A q-identity.** If we put

$$\xi = \xi^{(\infty)} , \quad \xi_{ab} = \xi_{ab}^{(\infty)} , \tag{3.1}$$

then it follows from (2.4) and (2.9) that ξ is the generating function for $\pi(n, m)$. Moreover, it is clear from (2.14) and (2.16) that

$$F(u, v) = \sum_{r,s=0}^{\infty} u^r v^s \xi_{rs} = \frac{1 - uv}{(1-u)(1-v)} F(uv) , \tag{3.2}$$

$$\begin{aligned} F(u) &= \sum_{n=0}^{\infty} u^n \xi_{nn} \\ &= e(u, xy)\, e(xu, xy)\, e(yu, xy) \prod_{j=0}^{\infty} (1 - x^{2j+1}y^{2j+1}u^2) , \end{aligned} \tag{3.3}$$

where

$$\begin{aligned} e(t) = e(t, q) &= \prod_{0}^{\infty} (1 - q^n t)^{-1} = \prod_{0}^{\infty} \frac{t^n}{(q)_n} , \\ (q)_n &= (1-q)(1-q^2) \cdots (1-q^n) . \end{aligned} \tag{3.4}$$

We define the polynomial

$$H_n(x) = H_n(x, q) = \sum_{r=0}^{n} \begin{bmatrix} n \\ r \end{bmatrix} x^r , \tag{3.5}$$

where

$$\begin{bmatrix} n \\ r \end{bmatrix} = \frac{(q)_n}{(q)_r(q)_{n-r}} .$$

It has been shown [1] that

$$\sum_0^\infty \frac{H_k(x)H_k(y)}{(q)_k} t^k = \frac{e(t)\, e(xt)\, e(yt)\, e(xyt)}{e(xyt^2)} . \tag{3.6}$$

Using (3.3), (3.4), and (3.6), we then have

$$\sum_0^\infty u^n \xi_{nn} = \sum_0^\infty \frac{H_k(x)H_k(y)}{(xy)_k} u^k \sum_0^\infty (-1)^r \frac{x^r y^r u^r}{(xy)_r} .$$

Comparing coefficients of u^n, we get

$$\xi_{nn} = \frac{1}{(xy)_n} \sum_{k=0}^n (-1)^{n-k} \begin{bmatrix} n \\ k \end{bmatrix} x^{n-k} y^{n-k} H_k(x) H_k(y) . \tag{3.7}$$

Note that $xy = q$ in the right member of (3.7).

It is clear from (3.7) that

$$P_n(x, y) = (xy)_n \xi_{nn} \tag{3.8}$$

is a polynomial in x, y with integral coefficients which satisfies

$$P_n(x, y) = P_n(y, x) ,$$

$$P_n(x, 0) = \frac{1 - x^{n+1}}{1 - x} ,$$

$$x^n P_n\left(x, \frac{1}{x}\right) = (x^2 + x + 1)^n .$$

Also it follows from (2.15) that $P_n(x, y)$ satisfies the recurrence

$$\begin{aligned} P_n - (1 + x + y)P_{n-1} + [n-1]\,(x + y + xy + x^{n-1}y^{n-1})P_{n-2} \\ - xy[n-1]\,[n-2]P_{n-3} = 0 , \end{aligned} \tag{3.9}$$

where $[j] = 1 - x^j y^j$.

4. Weighted partitions. We define $\pi(n, m; \lambda)$, the number of weighted partitions of the bipartite (n, m), by the relation

$$\pi(n, m; \lambda) = \sum_{k=0}^\infty \lambda^k \sum 1 , \tag{4.1}$$

where the inner sum is extended over all partitions of the form (2.1) subject to the conditions (2.2) and the additional condition $\max(n_k, m_k) > 0$; that is, over all partitions with exactly k parts. It follows from the definition of $\pi_k(n, m)$ that we may write (4.1) in the form

$$(4.2)\qquad \pi(n, m; \lambda) = \sum_{k=0}^{\infty} \lambda^k(\pi_k(n, m) - \pi_{k-1}(n, m)) .$$

It should be remarked that the sum in (4.2) is finite, the upper bound for k being $\max(n, m)$.

Multiplying both members of (4.2) by $x^n y^m$ and summing over n, m it follows from (2.9) and (2.17) that we have established

THEOREM 2. *We have*

$$(4.3)\qquad \sum_{n,m=0}^{\infty} \pi(n, m; \lambda) x^n y^m = 1 + (1 - \lambda) \sum_{k=1}^{\infty} \lambda^k \xi^{(k)}(x, y) .$$

Note that (4.3) is a direct analogue of the well-known identity

$$(4.4)\qquad \prod_{n=1}^{\infty} (1 - \lambda x^n)^{-1} = \sum_{n=0}^{\infty} \lambda^n x^n \sum_{j=1}^{n} (1 - x^j)^{-1} .$$

We remark that (4.3) may be proved in a different manner. If we put

$$(4.5)\qquad \xi_{ab}(\lambda) = 1 + \lambda \sum_{r,s=0}^{\min(a,b)}{}' x^r y^s \xi_{rs} ,$$

where the prime denotes that we sum over all r, s in the indicated range except $r = s = 0$, then it follows from (4.1) that

$$(4.6)\qquad \xi(\lambda) = \xi_{\infty\infty}(\lambda)$$

is the generating function for $\pi(n, m; \lambda)$. We may then evaluate $\xi(\lambda)$ by the methods of § 2.

5. Partitions into odd parts. We shall say that the j-th part of the partition (1.1) is odd if each of n_j, m_j is odd.

Let $\psi(n, m)$ denote the number of partitions of the form (1.1) with parts odd and subject to the conditions (1.2). Let $\psi(n, m \mid a, b)$ denote the number of these partitions that satisfy the additional condition

$$(5.1)\qquad (2a + 1, 2b + 1) \geqq (n_1, m_1) .$$

We define the rational function $\beta_{2a+1,2b+1}$ of x, y by the relation

$$(5.2)\qquad \beta_{2a+1,2b+1} = 1 + \sum_{r,s=0}^{\min(a\ b)} x^{2r+1} y^{2s+1} \beta_{2r+1,2s+1} ,$$

so that

$$(5.3)\qquad \beta_{2r+1,2r+1} = \beta_{2a+1,2b+1} \qquad (r = \min(a, b)) .$$

If we put

$$\beta = \beta_{\infty\infty} , \tag{5.4}$$

then in the limit (5.2) becomes

$$\beta = 1 + \sum_{r,s=0}^{\infty} x^{2r+1} y^{2s+1} \beta_{2r+1,2s+1} . \tag{5.5}$$

It follows from (5.2) that

$$\beta_{2a+1,2b+1} = \sum_{n,m=0}^{\infty} \psi(n, m \mid a, b) x^n y^m . \tag{5.6}$$

Thus, using (5.5), we get

$$\beta = \sum_{n,m=0}^{\infty} \psi(n, m) x^n y^m . \tag{5.7}$$

We define the generating functions

$$H(u, v) = \sum_{r,s=0}^{\infty} u^r v^s \beta_{2r+1,2s+1} , \tag{5.8}$$

$$H(u) = \sum_{n=0}^{\infty} u^n \beta_{2n+1,2n+1} , \tag{5.9}$$

so that

$$\beta = 1 + xy\, H(x^2, y^2) . \tag{5.10}$$

Using (5.3), (5.8) and (5.9), we have

$$H(u, v) = \frac{1 - uv}{(1 - u)(1 - v)} H(uv) . \tag{5.11}$$

The proof of (5.11) is exactly like that of (2.14).

On the other hand, it follows from (5.2), (5.3), and (5.9) that

$$\begin{aligned} H(u) &= \sum_{n=0}^{\infty} u^n \Big(1 + \sum_{r,s=0}^{n} x^{2r+1} y^{2s+1} \beta_{2r+1,2s+1}\Big) \\ &= \frac{1}{1-u} + \frac{xy}{1-u} \sum_{r,s=0}^{\infty} x^{2r} y^{2s} u^{\max(r,s)} \beta_{2r+1,2s+1} \\ &= \frac{1}{1-u} + \frac{xy}{1-u}\Big(\frac{1}{1 - x^2 u} + \frac{1}{1 - y^2 u} - 1\Big) H(x^2 y^2 u) , \end{aligned}$$

which implies

$$H(u) = \frac{1}{1-u}\Big(1 + \frac{1 - x^2 y^2 u^2}{(1 - x^2 u)(1 - y^2 u)} H(x^2 y^2 u)\Big). \tag{5.12}$$

Repeated applications of (5.12) yield

(5.13) $H(u) =$

$$\frac{1}{1-u}\sum_{n=0}^{\infty} x^n y^n \prod_{j=1}^{n} \frac{1 - x^{4j+2}y^{4j+2}u^2}{(1 - x^{2j+2}y^{2j+2}u)(1 - x^{2j}y^{2j+2}u)(1 - x^{2j+2}y^{2j}u)} .$$

Thus, using (5.10), (5.11), and (2.17), we may state

THEOREM 3. *If $\psi(n, m)$ denotes the number of partitions of (n, m) with odd parts, then*

$$\sum_{n,m=0}^{\infty} \psi(n, m)x^n y^m = \sum_{n=0}^{\infty} x^n y^n \xi^{(n)}(x^2, y^2) , \tag{5.14}$$

where $\xi^{(n)}(x, y)$ is defined by (2.17).

The fact that (2.18) and (5.14) are analogous to well-known identities for unipartite numbers leads one to conjecture that $\rho(n, m) = \psi(n, m)$. There are, however, counterexamples to this conjecture. For example, it is easily verified that

$$\rho(5, 4) = 6 \neq 4 = \psi(5, 4) .$$

It would be of interest to know whether generally

$$\rho(n, m) \geqq \psi(n, m) .$$

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Received March 20, 1965.

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Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley 8, California.

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), No. 6, 2-chome, Fujimi-cho, Chiyoda-ku, Tokyo, Japan.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

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Pacific Journal of Mathematics

Vol. 19, No. 2 June, 1966