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AN EMBEDDING THEOREM FOR FUNCTION SPACES

COLIN W. CLARK

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Let G be an open set in E_n , and let $H_0^m(G)$ denote the Sobolev space obtained by completing $C_0^\infty(G)$ in the norm

$$\|u\|_m = \left\{ \int_G \sum_{|\alpha| \leq m} |D^\alpha u(x)|^2 dx \right\}^{1/2}.$$

We show that the embedding maps $H_0^{m+1}(G) \subset H_0^m(G)$ are completely continuous if G is "narrow at infinity" and satisfies an additional regularity condition. This generalizes the classical case of bounded sets G .

As an application, the resolvent operator R_λ , associated with a uniformly strongly elliptic differential operator A with zero boundary conditions is completely continuous in $\mathcal{L}_2(G)$ provided G satisfies the same conditions. This generalizes a theorem of A. M. Molcanov.

Let G be an open set in Euclidean n -space E_n . Following standard usage, we denote by $C_0^\infty(G)$ the space of infinitely differentiable complex valued functions having compact support in G . Let $H_0^m(G)$ denote the Sobolev space obtained by completing $C_0^\infty(G)$ relative to the norm

$$\|f\|_m = \left\{ \int_G \sum_{|\alpha| \leq m} |D^\alpha f(x)|^2 dx \right\}^{1/2}.$$

(See (3) below for notations.) It is an important and well-known result of functional analysis that each embedding

$$H_0^{m+1}(G) \subset H_0^m(G), \quad m = 0, 1, 2, \dots$$

is completely continuous provided G is a bounded set. In this paper we show that this assumption can be relaxed; it turns out that a certain condition on G called "narrowness at infinity" (see Definition 2), which is obviously necessary, is also sufficient for complete continuity of the embeddings, provided G also satisfies a certain regularity condition. This result could be anticipated on the basis of theorems of F. Rellich [4] and A. M. Molcanov [3] concerning discreteness of the spectrum for the Laplace operator (with zero boundary conditions) on G .

DEFINITION 1. For an arbitrary open set $G \subset E_n$, with boundary ∂G , define

$$(1) \quad \rho(G) = \sup_{x \in \bar{G}} \text{dist}(x, \partial G).$$

Clearly $\rho(G)$ is the supremum of the radii of spheres inscribable in G .

DEFINITION 2. The open set G is said to be "narrow at infinity" if

$$(2) \quad \lim_{R \rightarrow \infty} \rho(G_R) = 0, \quad \text{where } G_R = G \cap \{x : |x| > R\}.$$

Evidently G is narrow at infinity if and only if it does not contain infinitely many disjoint spherical balls of equal positive radius. Our main result concerns such sets G , but we also require the following regularity condition:

1. Corresponding to each $R \geq 0$ there exist positive numbers $d(R)$ and $\delta(R)$ satisfying
- (a) $d(R) + \delta(R) \rightarrow 0$ as $R \rightarrow \infty$
 - (b) $d(R)/\delta(R) \leq M < \infty$ for all R
 - (c) for each $x \in G_R$ there exists a point y such that $|x - y| < d(R)$ and $G \cap \{z : |z - y| < \delta(R)\} = \emptyset$.

Note that Condition 1 clearly implies that G is narrow at infinity. We use the following standard notations.

$$(3) \quad \begin{cases} D_i = \frac{\partial}{\partial x_i}, & i = 1, 2, \dots, n; \\ D^\alpha = D_1^{\alpha_1} \dots D_n^{\alpha_n} & \text{for } \alpha = (\alpha_1, \dots, \alpha_n); \\ |\alpha| = \sum \alpha_i. \end{cases}$$

The following theorem is a generalization of Poincaré's inequality, cf. Agmon [1]. Although the proof is similar to that of Agmon, we give it here for the sake of completeness.

THEOREM 1. Let G be an open set in E_n satisfying the Condition 1. Then there exists a constant c such that

$$(4) \quad \int_{G_R} |f(x)|^2 dx \leq c(d(R))^2 \int_G \sum_i |D_i f(x)|^2 dx$$

for all $f \in H_0^1(G)$. Moreover if G satisfies only Condition 1(c) for $R = 0$, then the inequality (4) is valid for $R = 0$.

Proof. Assume that G satisfies Condition 1. Let $R > 0$ be fixed, and write $d = d(R)$, $\delta = \delta(R)$. If $\alpha = (\alpha_1, \dots, \alpha_n)$ is an n -tuple of integers, let $Q_\alpha = \{x \in E_n : n^{-1/2}d\alpha_k \leq x_k \leq n^{-1/2}d(\alpha_k + 1), k = 1, \dots, n\}$. Then $E_n = \bigcup_\alpha Q_\alpha$.

Now let $\varphi \in C_0^\infty(G)$ and let $x \in G_R \cap Q_\alpha$; let y satisfy 1(c). Note that $Q_\alpha \subset \{z : |z - y| < 2d\}$. Let $S = \{z : |z - y| < \delta\}$ and integrate $|\varphi|^2$ over $Q_\alpha - S$:

$$\begin{aligned} \int_{Q_{\alpha-s}} |\varphi|^2 dx &\leq \int_{\delta \leq |x-y| \leq 2\delta} |\varphi|^2 dx \\ &= \int_{\Sigma} \int_{\delta}^{2\delta} |\varphi(r, \sigma)|^2 r^{n-1} dr d\sigma, \end{aligned}$$

where Σ is the unit sphere centred at y . If $\delta \leq r \leq 2\delta$, we have by Schwarz's inequality

$$\begin{aligned} |\varphi(r, \sigma)|^2 r^{n-1} &= \left| \int_{\delta}^r \varphi_r(t, \sigma) dt \right|^2 r^{n-1} \\ &\leq (2\delta)^n \int_{\delta}^{2\delta} |\varphi_r(t, \sigma)|^2 dt \\ &\leq (2\delta)^n \delta^{1-n} \int_{\delta}^{2\delta} |\varphi_r(t, \sigma)|^2 t^{n-1} dt. \end{aligned}$$

Therefore, integrating over $\delta \leq |x - y| \leq 2\delta$, we obtain

$$\begin{aligned} \int_{Q_{\alpha-s}} |\varphi|^2 dx &\leq (2\delta)^{n+1} \delta^{1-n} \int_{\delta \leq |x-y| \leq 2\delta} \sum_i |D_i \varphi|^2 dx \\ &\leq (2\delta)^{n+1} \delta^{1-n} \int_{Q'_\alpha} \sum_i |D_i \varphi|^2 dx, \end{aligned}$$

where Q'_α is the union of all cubes Q_β which meet the set $\delta \leq |x - y| \leq 2\delta$. There is a number N , depending only on n , such that any $N + 1$ of the sets Q'_α have empty intersection. Summation of the above inequality over the set A of all indices α for which Q_α meets G_R therefore yields

$$\begin{aligned} \int_{G_R} |\varphi|^2 dx &\leq \int_{\cup_{\alpha \in A} (Q_{\alpha-s})} |\varphi|^2 dx \\ &\leq \sum_{\alpha \in A} (2\delta)^{n+1} \delta^{1-n} \int_{Q'_\alpha} \sum_i |D_i \varphi|^2 dx \\ &\leq N \cdot 2^{n+1} M^{n-1} (d(R))^2 \int_G \sum_i |D_i \varphi|^2 dx, \end{aligned}$$

where M is as in 1(b). This proves inequality (4) for $\varphi \in C_0^\infty(G)$; the extension to $H_0^1(G)$ is trivial.

The second assertion of the theorem is now obvious.

COROLLARY. *Let G be an open set in E_n , satisfying the condition 1(c) for $R = 0$, and consider the norm $\|\cdot\|_m$ defined in $H_0^m(G)$ by*

$$\|f\|_m^2 = \int_G \sum_{|\alpha|=m} |D^\alpha f(x)|^2 dx.$$

Then the norms $\|\cdot\|_m$ and $\|\cdot\|_m$ are equivalent in $H_0^m(G)$. On the other hand these norms are not equivalent for any open set G for which $\rho(G) = +\infty$.

Proof. Applying the second assertion of the theorem to the k -th order derivatives of $f \in H_0^m(G)$ ($k < m$), we get $|f|_k \leq \text{const. } |f|_{k+1}$ and hence $\|f\|_m^2 = \sum_0^m |f|_k^2 \leq \text{const. } |f|_m^2$. Since obviously $|f|_m \leq \|f\|_m$, this proves the first assertion. For the second assertion, note that G must contain spheres of arbitrarily large radius if $\rho(G) = \infty$. Thus for example $H_0^1(G)$ will contain suitable translates of the functions $g_\alpha(x) = g(\alpha^{-1}x)$ for arbitrarily large values of α , where $g(x) \neq 0$ is chosen as some function in $C_0^\infty(\{x : |x| < 1\})$. Since $|g_\alpha|_0 = \text{const. } \alpha |g_\alpha|_1$, an inequality of the form

$$\|g_\alpha\|_1^2 = |g_\alpha|_0^2 + |g_\alpha|_1^2 \leq \text{const. } |g_\alpha|_1^2$$

is precluded. This argument clearly extends to $H_0^m(G)$.

We next introduce some useful notation. If R is a positive real number, set

$$B_R^n = \{x \in E_n : |x| < R\};$$

$$G'_R = G \cap B_R^n \text{ if } G \text{ is an open set in } E_n.$$

DEFINITION 3. Let G be an open set in E_n and let $R > 0$. Denote by $C_0^\infty(G, R)$ the space of all C^∞ functions on E_n whose support is a compact subset of $G \cap \bar{B}_R^n$. We define $H^m(G, R)$ to be the completion of $C_0^\infty(G, R)$ with respect to the norm $\|\cdot\|_m$.

DEFINITION 4. We say that a sequence $\{x_n\}$ in a Hilbert space H is *compact* if every subsequence of $\{x_n\}$ has a subsequence converging in H .

Thus a linear operator $T: H_1 \rightarrow H_2$ (H_2 a separable Hilbert space) is completely continuous if and only if it maps bounded sequences into compact sequences.

THEOREM 2. *If G is an arbitrary open set in E_n then the embeddings*

$$H^{m+1}(G, R) \subset H^m(G, R), \quad m = 0, 1, 2, \dots$$

are completely continuous.

Proof. This follows easily from the complete continuity of the embeddings $H^{m+1}(B_R^n) \subset H^m(B_R^n) = H^m(E_n, R)$ [2, Ch. XIV]. For let $f \in H^m(G, R)$ and let $\{f_k\}$ be a sequence in $C_0^\infty(G, R)$ with $\|f_k - f\|_m \rightarrow 0$. Extending f_k to be zero outside its support, we get $f_k \rightarrow \hat{f}$ in $H^m(B_R^n)$ where \hat{f} is obtained by extending f to be zero in $B_R^n - \bar{G}'_R$. Now if $\{\varphi_j\}$ is a bounded sequence in $H^{m+1}(G, R)$, then $\{\hat{\varphi}_j\}$ is bounded in

$H^{m+1}(B_R^n)$ and hence compact in $H^m(B_R^n)$, and therefore $\{\varphi_j\}$ itself is compact in $H^m(G, R)$.

The following criterion for compactness is well-known.

LEMMA. Let $\{f_k\}$ be a bounded sequence in $\mathcal{L}_2(G)$, where $G \subset E_n$. Suppose that

- (a) $\{f_k|G'\}$ is compact for every bounded subset G' of G , and
- (b) given $\varepsilon > 0$, there exists $R > 0$ such that for all k ,

$$\int_{a_R} |f_k(x)|^2 dx < \varepsilon .$$

Then $\{f_k\}$ is compact in $\mathcal{L}_2(G)$.

THEOREM 3. Let G be an open set in E_n , satisfying the Condition 1. Then G is narrow at infinity and each of the embedding maps

$$H_0^{m+1}(G) \subset H_0^m(G) , \quad m = 0, 1, 2, \dots$$

is completely continuous. On the other hand if $G \subset E_n$ is not narrow at infinity, then the indicated embeddings are not completely continuous.

Proof. First, if G is not narrow at infinity, it must contain an infinite denumerable family $\{U_j\}$ of nonintersecting spherical balls of equal positive radius. Let f_1 be an arbitrary nonzero function in $C_0^\infty(U_1)$, and let f_j be constructed for $j = 2, 3, \dots$ by translating f_1 to have support contained in U_j . Then we have

$$(f_j, f_k)_m = c_m \delta_{k,j}$$

where $(\ , \)_m$ is the natural inner product in $H_0^m(G)$ and c_m is a nonzero constant depending only on m and f_1 . Consequently none of the embeddings can be completely continuous.

To prove that if G satisfies Condition 1 then the embeddings are completely continuous, it suffices by the standard inductive argument to consider the case $m = 0$. Thus suppose $\{f_k\}$ is a sequence in $H_0^1(G)$ with $\|f_k\|_1 \leq 1$. If G' is a bounded subset of G , then $G' \subset G'_R$ for some R , and by Theorem 2 the sequence $\{f_k|G'_R\}$ is compact in $\mathcal{L}_2(G'_R)$, and *a fortiori* $\{f_k|G'\}$ is compact in $\mathcal{L}_2(G')$. Thus (a) of the Lemma is satisfied; to verify (b) we merely have to apply the inequality (4) to f_k :

$$\int_{a_R} |f_k(x)|^2 dx \leq c(d(R))^2 \|f_k\|_1^2 \leq c(d(R))^2 .$$

By hypothesis the right hand side approaches zero as $R \rightarrow \infty$.

Functions in $H_0^m(G)$ vanish in some sense on ∂G . This property is essential for the embedding theorem in the case of unbounded sets G , as is indicated in the following theorem. Here $H^m(G)$ is the Hilbert space of functions which together with their (distribution) derivatives of all orders $\leq m$ are in $\mathcal{L}_2(G)$.

THEOREM 4. *Let G be an open set in E_n , contained in a cylinder of finite $n - 1$ dimensional cross-section. If G has infinite n dimensional volume, then the embedding $H^1(G) \subset \mathcal{L}_2(G)$ is not completely continuous.*

Proof. Assume that the x_1 -axis is the centre of the cylinder containing G , and let C denote the $n - 1$ dimensional volume of the section of the cylinder by the hyperplane $x_1 = 0$. We may also suppose that $\mu_n(G \cap \{x : x_1 > 0\}) = \infty$; then for fixed a , $\mu_n(G \cap \{x : a \leq x_1 \leq b\})$ is a continuous increasing function of $b \geq a$, with range the half-line $[0, \infty)$.

For $x \in E_n$ define the function $f_1(x)$ as follows.

$$f_1(x) = \begin{cases} x_1 & \text{if } 0 \leq x_1 \leq 1 \\ 1 & \text{if } 1 \leq x_1 \leq b_1 \\ 1 + b_1 - x_1 & \text{if } b_1 \leq x_1 \leq b_1 + 1 \\ 0 & \text{otherwise,} \end{cases}$$

where $\mu_n(G \cap \{x : 1 \leq x_1 \leq b_1\}) = 1$. Similarly define $f_2(x)$ to have support in the strip $b_1 + 1 \leq x_1 \leq b_2 + 1$, where $\mu_n(G \cap \{x : b_1 + 1 \leq x_1 \leq b_2\}) = 1$, and so on. Then $f_k \perp f_j$ ($j \neq k$) and

$$1 \leq \|f_k\|_0^2 \leq 1 + 2C .$$

Moreover

$$\begin{aligned} \|f_k\|_1^2 &= \|f_k\|_0^2 + \int_G \sum_i |D_i f_k(x)|^2 dx \\ &\leq \|f_k\|_0^2 + 2C \leq 1 + 4C . \end{aligned}$$

Thus the sequence $\{f_k\}$ is bounded in $H^1(G)$ but not compact in $\mathcal{L}_2(G)$, so that the embedding $H^1(G) \subset H^0(G) = \mathcal{L}_2(G)$ is not completely continuous.

As an application of Theorem 3, consider a given differential operator $a(x, D)$ of order $2m$:

$$a(x, D) = \sum_{|\alpha| \leq 2m} a_\alpha(x) D^\alpha .$$

We assume that the coefficients are infinitely differentiable, bounded complex functions on an open set G in E_n . Let $a(x, D)$ be *uniformly strongly elliptic* in the following sense:

$$(-1)^m \operatorname{Re} (a_0(x, \xi)) \geq \text{const.} \cdot |\xi|^{2m}, \quad x \in G, \xi \in E_n,$$

where $a_0(x, \xi)$ is the characteristic form,

$$a_0(x, \xi) = \sum_{|\alpha|=2m} a_\alpha(x) \xi^\alpha.$$

Under certain additional conditions on the coefficients $a_\alpha(x)$ and on the set G , it is known that the following inequalities are valid (cf. [1]).

$$(5) \quad |(a(x, D)\varphi, \psi)| \leq \text{const.} \cdot \|\varphi\|_m \|\psi\|_m, \quad \varphi, \psi \in C_0^\infty(G);$$

and ‘‘Gårding’s inequality’’

$$(6) \quad \operatorname{Re} (a(x, D)\varphi, \varphi) \geq c_1 \|\varphi\|_m^2 - c_2 \|\varphi\|_0^2, \quad \varphi \in C_0^\infty(G),$$

where $c_1 > 0$ and c_2 are constants. For the purpose of the following theorem we use these inequalities as hypotheses. Theorem 5 was obtained in the case of the Laplacian operator in a smoothly bounded domain G by A. M. Molcanov [3].

THEOREM 5. *Let G be an open set in E_n , satisfying the hypotheses of Theorem 3. Let $a(x, D)$ be a uniformly strongly elliptic differential operator with coefficients defined in G , and suppose that the inequalities (5) and (6) are satisfied. Define the operator T in $\mathcal{L}_2(G)$ by*

$$\begin{aligned} \mathcal{D}(T) &= H_0^m(G) \cap \{f \in \mathcal{L}_2(G) : a(x, D)f \in \mathcal{L}_2(G)\} \\ Tf &= a(x, D)f, \quad f \in \mathcal{D}(T). \end{aligned}$$

Then T is a closed linear operator; the spectrum $\sigma(T)$ is discrete and has no finite limit points; for $\lambda \notin \sigma(T)$, the resolvent operator $R_\lambda(T) = (\lambda I - T)^{-1}$ is completely continuous.

Proof. We have worded the theorem to agree with Corollary 14.6.11 of [2]; in fact the proof is the same. At the suggestion of the referee, however, we include an outline here.

If λ is a given complex number with $\operatorname{Re} \lambda > c_2$, we have by (5) and (6)

$$(7) \quad |((a + \lambda)\varphi, \psi)| \leq k_1 \|\varphi\|_m \|\psi\|_m, \quad \varphi, \psi \in C_0^\infty(G);$$

$$(8) \quad \operatorname{Re} ((a + \lambda)\varphi, \varphi) \geq k_2 \|\varphi\|_m^2, \quad \varphi \in C_0^\infty(G).$$

Hence $((a + \lambda)\varphi, \psi)$ can be extended to a continuous bilinear form $B[\varphi, \psi]$ on $H_0^m(G)$, satisfying (7) and (8). By the Lax-Milgram lemma (cf. [1], p. 98), to each $\varphi \in H_0^m(G)$ there corresponds an element $A\varphi \in H_0^m(G)$ such that

$$(9) \quad B[A\varphi, \psi] = (\varphi, \psi)_m, \quad \text{for all } \psi \in H_0^m(G).$$

Moreover $A: H_0^m(G) \rightarrow H_0^m(G)$ is bounded, one-to-one, and hence onto. By the open mapping theorem, A^{-1} is also bounded.

Next, if T is the operator defined in the theorem, we will show that

$$(10) \quad ((T + \lambda I)\varphi, \psi) = (A^{-1}\varphi, \psi)_m, \quad \varphi \in \mathcal{D}(T), \psi \in H_0^m(G).$$

This relation is evident for $\varphi, \psi \in C_0^\infty(G)$. If $\varphi \in H_0^m(G)$, $\psi \in C_0^\infty(G)$, and if $\varphi_n \in C_0^\infty(G) \rightarrow \varphi$ in the norm of $H_0^m(G)$, then $\varphi_n \rightarrow \varphi$ in the sense of distributions on G , so that $((a + \lambda)\varphi_n, \psi) \rightarrow ((a + \lambda)\varphi, \psi)$, and therefore

$$((a + \lambda)\varphi, \psi) = (A^{-1}\varphi, \psi)_m, \quad \varphi \in H_0^m(G), \psi \in C_0^\infty(G).$$

This implies (10) immediately.

By (8), (9), and (10) we have for $\varphi \in \mathcal{D}(T)$

$$(11) \quad \begin{aligned} \|(T + \lambda I)\varphi\|_0 \|\varphi\|_m &\geq |((T + \lambda I)\varphi, \varphi)_0| \\ &= |(A^{-1}\varphi, \varphi)_m| = |B[\varphi, \varphi]| \geq k_2 \|\varphi\|_m^2. \end{aligned}$$

Hence $(T + \lambda I)^{-1}$ exists and is bounded on $\text{Range}(T + \lambda I)$. Another simple argument shows that $\text{Range}(T + \lambda I) = \mathcal{L}_2(G)$. We therefore conclude that T is closed and $\lambda \in \rho(T)$, the resolvent set of T .

By (11) we have

$$\|(T + \lambda I)^{-1}\varphi\|_m \leq k_2^{-1} \|\varphi\|_0, \quad \varphi \in \mathcal{L}_2(G).$$

Thus $(T + \lambda I)^{-1}$ maps a bounded set in $\mathcal{L}_2(G)$ into a bounded set in $H_0^m(G)$, which, according to Theorem 3, is precompact in $\mathcal{L}_2(G)$. Therefore $(T + \lambda I)^{-1}$ is a completely continuous operator in $\mathcal{L}_2(G)$.

The remaining assertions of the theorem follow from the Riesz-Schauder theory of completely continuous operators.

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