A THEOREM ON ONE-TO-ONE MAPPINGS

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Let $X$ be a locally connected generalized continuum with the property that the complement of each compact set has only one nonconditionally compact component. The author proves the following theorem. If $f$ is a one-to-one mapping of $X$ onto Euclidean 2-space, then $f$ is a homeomorphism.

An example of K. Whyburn implies that if $f$ is a one-to-one mapping of $X$ onto Euclidean $n$-space ($n \geq 3$), then $X$ can have many nice properties yet $f$ need not be a homeomorphism. However the complement of a compact set in the domain space of his example may have more than one non-conditionally compact component.

It is interesting to note that a characterization of closed 2-cells in the plane is obtained in the course of proving the theorem.

Positive results in connection with the following problem would be useful in classifying mappings from a Euclidean space into itself. "What properties must a topological space $X$ have before one can conclude that every one-to-one mapping $f$ of $X$ into a Euclidean space $E^n$ of dimension $n$ is a homeomorphism?" A very general theorem of this type was supposedly obtained in [2]. However, several counterexamples have been obtained which show the main theorems of [2] to be false. One of these is an example of K. Whyburn [6], which implies that if $n \geq 3$, $X$ may have many nice properties, yet $f$ need not be a homeomorphism. We prove that if the Euclidean space has dimension two, the mapping $f$ is onto, and $X$ has appropriate properties, then $f$ is indeed a homeomorphism. It is interesting to note that we assign a property to the space $X$ which is not a property of the domain space of the example in [6].

2. Notation. A mapping is a continuous function. A generalized continuum is a connected, locally compact, separable metric space. The cyclic element theory used is that of reference [4]. A set $A$ in a topological space is conditionally compact if its closure is a compact set. A dendrite is a compact locally connected generalized continuum containing no simple closed curve. A topological line is a homeomorphic image of the real line. A topological ray is a homeomorphic image of a ray in the real line.

3. Theorem and proof.

THEOREM. Let $X$ be a locally connected generalized continuum
with the property that the complement of each compact set has only one nonconditionally compact component. If $f(X) = E^2$ is a one-to-one mapping, then $f$ is a homeomorphism.

Proof. The proof consists of proving a series of five statements concerning the structure of $X$ if $f$ is not a homeomorphism. Then in (vi) with the aid of a theorem of G. T. Whyburn [5], a contradiction is obtained.

(i) $X$ contains simple closed curves.

Proof of (i). The space $X$ has the representation, $X = \bigcup_{i=1}^{\infty} A_i$, where each $A_i$ is a locally connected continuum. If $X$ has no simple closed curve, then no $A_i$ contains a simple closed curve. Thus each $A_i$ is a dendrite and therefore has dimension one. Using the sum theorem for dimension, we obtain $\dim \bigcup_{i=1}^{\infty} f(A_i) \leq 1$, but $\bigcup_{i=1}^{\infty} f(A_i) = E^2$ and $\dim E^2 = 2$. Clearly then each such $X$ must contain simple closed curves.

(ii) Every simple closed curve $J$ in $X$ separates $X$ and is the boundary of an open two cell which is an open subset of $X$.

Proof of (ii). For a simple closed curve $J$ in $X, f(J)$ is a simple closed curve. Since $f(J)$ separates $E^2$, its inverse image $J$ separates $X$. The complement of $J, X - J$, can have at most countably many components, $C_i, i = 0, 1, 2, \ldots$, and only one of these, say $C_0$, is not conditionally compact. Each $f(C_i), i \neq 0$, is closed in $E^2 - f(J)$ and each $f(C_i), i = 0, 1, 2, \ldots$ is either in the bounded component $W$ or the unbounded component $M$ of $E^2 - f(J)$. The set $f(C_0)$ is not contained in $W$ for this would imply that $M$ is the countable union of pairwise disjoint bounded closed (in $M$) sets $f(C_{nk}), k = 1, 2, \ldots$. No arcwise connected space has such a representation hence $f(C_0) \subset M$. Applying the same theorem to $W$ shows there is one and only one $C_i, i \neq 0$, for which $f(C_i) \subset W$ and hence $f(C_i) = W$. It easily follows that $f(F_r C_i) = f(J)$ and therefore $F_r C_i = J$.

(iii) Each compact nondegenerate cyclic element of $X$ is topologically a closed 2-cell.

Proof of (iii). Let $C$ be a compact nondegenerate cyclic element of $X$ and note by (ii) that every simple closed curve in $C$ is the boundary of an open 2-cell of $C$. Since $f/C$ is a homeomorphism we can assume that $C$ is a subset of $E^2$.

Let $H$ be the set of points of $C$ that are interior to an open 2-cell
of $C$. By cyclic connectedness of $C$, $H$ is dense in $C$. To show $H$ is connected let $a$ and $b$ be distinct points of $H$ and let $J_1$ and $J_2$ be disjoint simple closed curves in $C$ that bound nonintersecting open 2-cells $C_1$ and $C_2$ containing $a$ and $b$ respectively. By cyclic connectedness of $C$ there exist mutually exclusive arcs $1_1$ and $1_2$ in $C$ with $1_1 \cap (C_1 \cup J_1) = x_{11}$, $1_1 \cap (C_2 \cup J_2) = x_{12}$, $1_2 \cap (C_1 \cup J_1) = x_{21}$, and $1_2 \cap (C_2 \cup J_2) = x_{22}$. The set $1_1 \cup (x_{11}x_{21}) \cup 1_2 \cup (x_{12}x_{22})$, where $(x_{11}x_{21}), (x_{12}x_{22})$ are arcs on $J_1$ and $J_2$ respectively, is a simple closed curve in $C$. The proper choice of arcs $(x_{11}x_{21})$ and $(x_{12}x_{22})$ on $J_1$ and $J_2$, respectively, gives a simple closed curve $J_0$ in $C$ that bounds an open 2-cell $C_0$ which contains both $a$ and $b$.

We use the Zoretti Theorem, p. 109, [4], to prove $C-H$ is connected. Suppose $C-H$ is not connected and $K$ is one of its compact components. By Zoretti's Theorem there is a simple closed curve $J_3$ in $E^2$ enclosing $K$ and not enclosing $C-H$ and is such that $J_3 \cap (C-H) = \emptyset$. The set $J_3 \cap C = J_3 \cap H$ is not empty and is both open and closed in $J_3$. Hence $J_3 \subset H$ and this implies $K \subset H$ which is false.

Let $x$ and $y$ be distinct points of $C-H$. By the cyclic connectedness of $C$ and the connectedness of $H$ there is a simple arc $(xy)$ in $C$ with $(xy) \cap (C-H) = x \cup y$. Suppose this arc does not separate $C$ and let $z \in (xy)$, $z \neq x, z \neq y$. Since $z \in H$ there is a closed 2-cell $C_4$ in $H$ with boundary $J_4$ such that $z$ is interior to $C_4$ and $(xy)$ separates $C_4$ into two connected sets $A$ and $B$. Let $a \in A$ and $b \in B$ and suppose $(ab)$ is a simple arc in $C-(xy)$.

In $C_4$ determine an arc $azb$ such that $(ab)$ union $azb$ is a simple closed curve $J_5$. The curve $J_5$ is the boundary of a closed 2-cell $C_5$ in $C$. The 2-cell $C_5$ contains points of $A$ and $B$ and hence points of one of the subarcs $(xz)$ or $(zy)$ of $(xy)$ other than $z$. Since $J_5$ meets $(xy)$ only in the point $z$, at least one of $x$ or $y$ is interior to $C_5$. This contrary to the choice of $x$ and $y$. Therefore, each such arc spanning $C-H$ in $C$ separates $C$. Furthermore, $H-(xy)$ has only two components and hence $C-(xy)$ has only two components since $H$ is dense in $C$. Also, each component of $C-(xy)$ contains points of $C-H$, otherwise there would exist a bounded open subset of the plane with a simple arc as its frontier. Thus each pair of points $x, y$ of $C-H$ separates $C-H$ and therefore $C-H$ is a simple closed curve $J$. Clearly $H$ is the open two cell of $C$ bounded by $J$.

In order to make repeated use of a theorem in [5] we set up the following notation. Let $f(X) = Y$ be a one-to-one continuous mapping of one locally compact separable metric space onto another. Let $S$ be the set of points in $X$ at which $f$ is a local homeomorphism and let $T$ be its complement. From a result in [3] the set $S$ is open, $T$ is closed, $f(S)$ is an open dense set in $Y$. The sets $S$ and $T$ will be used in the remaining parts of the proof. The following is a theorem
of G. T. Whyburn [5].

**Theorem A.** Let $X$ be a locally compact arcwise connected separable metric space, let $Y$ be a locally connected generalized continuum. If $f(X) = Y$ is a one-to-one continuous function which is not a homeomorphism, then there exists a topological ray $R$ in $X$ with $f(R)$ a simple closed curve in $Y$. Moreover, if $r$ is the initial point of $R$, there is a subray $R'$ of $R$ such that $f(R' \cup r)$ is a simple arc and $R' \subset S$.

(iv) There is only one noncompact cyclic element in $X$.

*Proof of (iv).* If there were two or more noncompact cyclic elements then one could find a compact set (namely a point) whose complement would necessarily have two or more nonconditionally compact components. This is contrary to part of the hypotheses on $X$.

If all the cyclic elements were compact then by (iii), all the true cyclic elements would be closed 2-cells. Thus $S$ would be the union of open 2-cells. By Theorem A there is a ray $R'$ and a point $r$ not a $R'$ such that $f(R' \cup r)$ is a simple arc and $R' \subset S$. Thus $R'$ must be a closed subset of $X$ which is entirely in an open 2-cell and this is not possible.

(v) Let $M$ be the noncompact cyclic element of $X$ and let $B = M \cap T$. The set $B$ is a topological line.

*Proof of (v).* As in the proof of (iii), the set $M-B$ is connected. For two distinct points $a$ and $b$ of $B$ there is a simple arc $[ab]$ in $M$ with $[ab] \cap B = a \cup b$. Using the techniques of the proof of (iii), it follows that the arc $[ab]$ separates $M$ into two connected sets. The closure of the conditionally compact component $D$ is cyclically connected and every simple closed curve in $\bar{D}$ bounds an open 2-cell of $\bar{D}$. Thus by (iii), $\bar{D}$ is a closed 2-cell and this implies that there is a simple arc $(ab)$ which is entirely in $B$. Moreover, the set $(ab)-\{a \cup b\}$ is an open subset of $B$. If $c$ is any other point of $B$ not on $(ab)$, then there is a simple arc joining $c$ to $a$ and the first point (ordered from $c$ to $a$) in which it meets $(ab)$ can only be $a$ or $b$. Thus, either $a$ or $b$ is in an open one cell which is an open subset of $B$. It follows that every point of $B$ with the possible exception of at most two points is in an open one cell which is an open subset of $B$. That is, $B$ is a simple arc, a topological ray or a topological line. The set $B$ cannot have a point $d$ which is not interior to a one cell of $B$ for this would imply that $M$ is not locally compact at $d$. 
(vi) Completion of proof.

The structure of the space $X$ is now clear in the sense that certain properties can be assigned to the true cyclic elements. Also, each component of the complement of the noncompact cyclic element $M$ is conditionally compact and has only a single point of $B$ as its frontier.

There is a ray $R_0$ in $X$ with initial point $r_0$ such that $f(R_0)$ is a simple closed curve bounding a closed 2-cell $C_0$. From the proof of Theorem A and the structure of $X$ we can assume $R_0$ meets $B$ in only one point $x_0$. The set $f^{-1}(C_0) = N_0$ is closed, connected, locally connected, and contains one of the two components of $M-R_0$. Let $y \in B \cap N_0$ and let $(x_0, y)$ be the simple arc in $B$. Let $K$ be $(x_0, y)$ union the conditionally compact components of $X-(x_0, y)$. The set $K$ is compact and connected. The set $f(K \cap T)$ is compact in $C_0$ so that $(f(T) \cap C_0)-f(K \cap T)$ is an open subset of $f(T) \cap C_0$. Thus, in applying the proof of Theorem A to the map $f\vert_{N_0} : N_0 \rightarrow C_0$ we can use the points of $(f(T) \cap C_0)-f(K \cap T)$ to get a ray $R_1$ with the property that $R_1 \subset N_0-K$. Assume the initial point of $R_1$ is $r_1$, $R_1 \cap B = x_1, C_1$ is the closed 2-cell bounded by $f(R_1)$, and $N_1 = f^{-1}(C_1)$. The set $N_1$ is connected, locally connected, and $N_1 \cap K = \emptyset$. In fact, the arc $(x_0, x_1)$ in $B$ maps onto an arc in the closed annular region determined by $f(R_0)$ and $f(R_1)$. Also implied is that a sequence of rays $R_0, R_1, R_2, \cdots$ can be obtained such that $\lim \sup R_n \cap T = \emptyset$. We can also suppose the rays were chosen so that a monotone sequence of locally connected generalized continua, $N_0 \supset N_1 \supset N_2 \supset \cdots$ with corresponding closed 2-cell images $C_0 \supset C_1 \supset C_2 \supset \cdots$ is obtained. For each $i$, $i = 0, 1, 2, \cdots$, the set $C_i \cap f(T)$ is nonempty and compact. Thus, $L = \bigcap_{i=0}^{\infty} [C_i \cap f(T)]$ is not empty and for $y \in L$ there exists an $x \in (T \cap N_i), i = 0, 1, 2, \cdots$ such that $f(x) = y$. However, by the construction of the $N_i$, $\bigcap_{i=0}^{\infty} (N_i \cap T) = \emptyset$.

REFERENCES


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