TWO THEOREMS ON METRIZABILITY OF MOORE SPACES

Ben Fitzpatrick, Jr. and Donald Reginald Traylor
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BEN FITZPATRICK, JR. AND D. R. TRAYLOR

One of the outstanding questions in point set topology is whether each normal Moore space is metrizable. The primary result of this paper is to reduce that question to the problem of deciding whether each normal Moore space is locally metrizable at some point. Also, the question of metrizability for normal, separable spaces is reduced to that for normal, separable, locally compact spaces.

The question as to whether every normal Moore space is metrizable has received considerable attention [1], [2]. In this note it is proved that if every normal Moore space is locally metrizable at some point, then every normal Moore space is metrizable. Also, the question of metrizability for normal, separable spaces is reduced to that for normal, separable locally compact spaces. This complements Jones' result (2, Theorem 5) that every normal separable Moore space is metrizable, provided $2^{\kappa_0} < 2^{\kappa_1}$. In the proof of the first theorem, a construction device similar to one described by Roy [4] is used.

By a development is meant a sequence of collections of regions satisfying Axiom 0 and the first three parts of Axiom 1 of [3].

THEOREM 1. If there is a normal Moore space which is not metrizable, there is one which is not locally metrizable at any point.

Proof. Suppose that $S^0$ is a nonmetrizable, normal Moore space, $M^0$ is a dense subset of $S^0$ which is of minimal cardinality, and $\{G^0_n\}$ is a development of $S^0$. (By taking $M^0$ to be of minimal cardinality, a property such as separability of $S^0$ is preserved.) There exist sequences $w = S^0, S^1, S^2, \ldots, w = M^0, M^1, M^2, \ldots$, and $v = \{G^0_n\}, \{G^1_n\}, \{G^2_n\}, \ldots$ such that for each integer $j > 0$

(i) $M^j$ is a dense subset of $S^j - S^{j-1}$ of minimal cardinality,
(ii) for each point $P$ of $M^j$ and each positive integer $k$,

$$S_{P,k}^{j+1} = S^0 \times (P) \times (k)$$

and $S^{j+1} = S^j + \sum_{P \in M^j} \sum_{k=1}^{\infty} S_{P,k}^{j+1}$,

(iii) the statement that $R^{j+1}$ is a region of $G^{j+1}$ means that either

(1) for some point $P$ of $M^j$ and some region $R^0$ of $G^j$ and some positive integer $i$, $R^{j+1}$ is $R^0 \times (P) \times (i)$, or

(2) for some point $P$ of $M^j$ and some region $R^j$ of $G^j_n$ containing $P$ and some positive integer $i \geq n$, 259
\[ R^{j+1} = R^j + \sum_{q \in \mathbb{N}} (S_{q,j}^{j+1} + S_{q,j+1}^{j+1} + \cdots). \]

It is clear that for each positive integer \( j \)

1. \( S^j \) is a Moore space with development \( \{G^j_n\} \),
2. \( S^{j+1} \) is a dense open set in \( S^{j+1} \),
3. the cardinality of \( M^{j+1} \) is the same as that of \( M^j \),
   (again, this preserves separability in case \( S^0 \) is separable)
4. if \( R \) is a region of \( G^j_n \) intersecting \( S^j \), then there is a region \( g \) of \( G^j_n \) such that \( R \cdot S^j \) is \( g \),
5. if \( R^j \) is a region of \( G^j_n \), there is a region \( R^{j+1} \) of \( G^{j+1}_n \) containing \( R^j \),
6. if each of \( i \) and \( j \) is a positive integer, \( P \) is a point in \( M^j \) and \( k \) is some positive integer greater than or equal to \( j \) such that \( R^{j+1} \) intersects \( S_{p,i}^j \), then \( S_{p,i}^j \) is a subset of \( R^{j+1} \).

Moreover, each \( S^j \) is a normal Moore space. Since \( S^0 \) is normal, assume that \( S^{j-1} \) is normal and consider \( S^j \). If \( H \) and \( K \) are mutually exclusive, closed subsets of \( S^j \), there exist domains \( D_{H}^{j-1} \) and \( D_{K}^{j-1} \), in \( S^{j-1} \), containing \( H \cdot S^{j-1} \) and \( K \cdot S^{j-1} \) respectively. Since \( S^j \) is a Moore space, it follows that if \( P \) is in \( H \cdot S^{j-1} \), there is an integer \( nP \) such that if \( R \) is a region of \( G^i_{nP} \) containing \( P \) then \( R \cdot S^{j-1} \) is a subset of \( D_{H}^{j-1} \) and \( \bar{R} \) does not intersect \( K \). If \( P \) is in \( H \cdot S^{j-1} \), denote by \( R_P \) some region of \( G^i_{nP} \) containing \( P \) and similarly for \( Q \) a point of \( K \cdot S^{j-1} \). If \( D_H = \sum_{P \in H} R_P \) and \( D_K = \sum_{Q \in K} R_Q \), then it is obvious that \( D_H \) and \( D_K \) are mutually exclusive domains in \( S^j \) containing \( H \cdot S^{j-1} \) and \( K \cdot S^{j-1} \) respectively. Now no point of \( K \) is a limit point of \( D_H \) and no point of \( H \) is a limit point of \( D_K \). For suppose that \( Q \) belongs to \( K \) and is not in \( S^{j-1} \). Then there is a point \( Y \) of \( M^{j-1} \) and a positive integer \( i \) such that \( Q \) is a point of \( S^0 \times (Y) \times (i) \). But each region of \( G^i_{j+1} \) containing \( Q \) is a subset of \( S^0 \times (Y) \) and \( i \) and no region of \( G^i_{j+1} \) containing a point of \( S^{j-1} \) intersects \( S^0 \times (Y) \times (i) \). It is clear then that no region of \( G^i_{j+1} \) contains \( Q \) and intersects a region \( R_P \) of \( D_H \), else \( R_P \) contains \( Q \) and this is impossible. If \( i \) is a positive integer and \( Y \) is a point of \( M^{j-1} \) such that neither \( D_H \) nor \( D_K \) intersects \( S^0 \times (Y) \times (i) \), then \( S^0 \times (Y) \times (i) \) is normal and there exist mutually exclusive domains \( D_{H \cdot Y, i} \) and \( D_{K \cdot Y, i} \) containing \( H \cdot (S^0 \times (Y) \times (i)) \) and \( K \cdot (S^0 \times (Y) \times (i)) \) respectively. Then \( D_H = D_{H}^i + \sum D_{H \cdot Y, i} \) and \( D_K = D_{K}^i + \sum D_{K \cdot Y, i} \) are mutually exclusive domains, in \( S^j \), containing \( H \) and \( K \) respectively. Thus \( S^j \) is normal.

Denote by \( S^w \) the space in which \( P \) is a point if and only if \( P \) is a point of some \( S^j \) and suppose that \( H^w_n \) is defined as follows:

\( R^w \) belongs to \( H^w_n \) if and only if it is true that there are an integer \( k \), an integer \( m \geq n \), and a sequence \( R^k_m, R^{k+1}_m, R^{k+2}_m, \ldots \) such that
(1) $R^k_m$ is a region of $G^k_m$,

(2) $S^{k+j-1}_m \cap R^{k+j}_m = R^{k+j-1}_m$, $j = 1, 2, 3, \ldots$

(3) Subject to condition (2), $R^{k+j-1}_m$ is the maximal element in $G^{k+j-1}_m$ containing $R^{k+j}_m$, for $j = 0, 1, 2, \ldots$. (That is, the positive integer $i$ of part (2) of the definition (iii) given in the proof of Theorem 1 is chosen to be minimal for all possible points $P$ of $M^i \cdot R^{k+j}$),

(4) $R^{k+i}_m \subset R^{k+i+1}_m \subset R^{k+i+2}_m \subset \cdots$,

(5) $R^{k+i}_m \subset S^{k+i}_m$ and $R^{k+i}_m \not\subset S^{k+i-1}_m$,

(6) $R^w = \sum_{i=0}^{\infty} R^{k+i}_m$. For each positive integer $t$, denote by $G^w_t$ the collection to which $\beta$ belongs if and only if there is an integer $s \geq t$ and an element $R^w$ of $H^w_s$ such that $R$ is $R^w$. Note that if $R^w = \sum_{i=0}^{\infty} R^{k+i}_m$ is a region then the boundary of $R^w$ is the boundary of $R^w_m$ and is a subset of $S^k$. It is clear that for each positive integer $n$, $G^w_n$ is a covering of the space and $G^w_{n+1}$ is a subcollection of $G^w_n$.

To establish that $S^w$ is a Moore space, it will suffice to prove that the third part of Axiom 1 is satisfied. To this end, suppose that each of $A$ and $B$ is a point of $S^w$ and $R$ is a region containing $A$ and $B$.

Case 1. There are a positive integer $j$, a point $P$ of $M^j$, and a positive integer $k$ such that each of $A$ and $B$ belongs to $S^0 \times (P) \times (k)$. But $S^{j+1}$ is a Moore space so there is an integer $n$ such that no region of $G^{j+1}_n$ contains both $A$ and $B$, but the closure of each region of $G^{j+1}_n$ is a subset of $S^k$. It follows as in Case 1 that no region of $G^w$ contains $A$ does not contain $B$, but the closure of it is a subset of $R$.

Case 2. The conditions of Case 1 are not satisfied. Denote by $j$ the least integer such that for some point $P$ of $M^j$ and for some integer $k$, $A$ is a point of $S^0 \times (P) \times (k)$, and denote by $t$ the least integer such that for some point $Q$ of $M^t$ and for some integer $e$, $B$ is a point of $S^0 \times (Q) \times (e)$. If $t$ is $j$, it is clear that there is a positive integer $n$ such that the closure of no region of $G^{j+1}_n$ intersects both $S^0 \times (Q) \times (e)$ and $S^0 \times (P) \times (k)$. If $t$ is not $j$, assume $t$ is less than $j$. Then there must be a finite sequence of positive integers, $n_1, n_2, \ldots, n_{j-t}$, and a finite sequence of points $P_1, P_2, \ldots, P_{j-t}$ such that for each $i, P_i$ is a point of $S^{t+i-1} \cap P_{i+1}$ is a point of $S^0 \times (P_i) \times (n_i)$, and $B$ is in $S^0 \times (P_{j-t}) \times (n_{j-t})$. (This essentially traces the construction of $S^w$ as regards $A$ and $B$). But again, if $A$ is not $P_1$, there is a positive integer $n$ such that the closure of no region of $G^w_n$ intersects $A$ and $P_1$, and the closure of each region of $G^w_n$ containing $A$ is a subset of $R \cdot S^t$. It follows as in Case 1 that no region of $G^w_n$ contains
both $A$ and $B$, but the closure of each region of $G^w$ containing $A$ is a subset of $R$. If $A$ is $P_{1}$, no region of $G^w_{n_{1}+1}$ contains both $A$ and $B$. Thus $S^w$ is a Moore space.

To prove that $S^w$ is normal, suppose that $H$ and $K$ are mutually exclusive, closed point sets. Then $H \cdot S^0$ and $K \cdot S^0$ are mutually exclusive, closed point sets, if each exists. In $S^0$ there are mutually exclusive domains $D_{H \cdot S^0}$ and $D_{K \cdot S^0}$ containing $H \cdot S^0$ and $K \cdot S^0$, respectively. For each point $P$ of $H \cdot S^0$, denote by $R_P$ a region in $S^w$ containing $P$ such that $R_P \cdot S^0$ is a subset of $D_{H \cdot S^0}$ and $\bar{R}_P \cdot K$ does not exist, and similarly for $P$ a point of $K \cdot S^0$. Denote by $D_0$ the domain to which $x$ belongs if and only if there is a point $P$ of $H \cdot S^0$ such that $x$ is point of $R_P$, and denote by $D'_0$ the domain to which $x$ belongs if and only if there is a point $P$ of $K \cdot S^0$ such that $x$ is a point of $R_P$. Then $D_0$ does not intersect $K$ and $D'_0$ does not intersect $H$. For suppose that $Q$ is a limit point of $D_0$ which is in $K$. Then $Q$ is in some $S^j$ where $j$ is positive. Indeed, if $R_m$ is a region of $G^j$ containing $Q$, then there is a point $P$ of $H \cdot S^0$ such that $R_m$ intersects $R_P$. But since $j$ is positive, this means that $R_P$ contains $R_m$ and this is a contradiction to $Q$ belonging to $K$.

Similarly, if $H_1$ is $H \cdot S^1$ and $K_1$ is $K \cdot S^1$, there exist mutually exclusive domains $D_1$ and $D'_1$ in $S^w$ such that $D_1$ contains $H_1$, $D'_1$ contains $K_1$, $D_1$ does not intersect $(D_0 + D'_0)$, and $D'_1$ does not intersect $(D_0 + D'_0)$. In general, there exist sequences $\alpha = D_0, D_1, D_2, \ldots$ and $\beta = D'_1, D'_2, D'_3, \ldots$ such for each $n$, $D_n$ contains $H \cdot S^n$, $D'_n$ contains $K \cdot S^n$, $D_n$ does not intersect $D'_n$, $\bar{D}_{n+1}$ does not intersect $\left(\sum_{i=0}^{n} D_i + \sum_{i=0}^{n} D'_i\right)$, and $D'_n$ does not intersects $\left(\sum_{i=0}^{n} D_i + \sum_{i=0}^{n} D'_i\right)$. It follows that $D_H = \sum_{n=0}^{\infty} D_n$ and $D_K = \sum_{n=0}^{\infty} D'_n$ are mutually exclusive domains such that $D_H$ contains $H$ and $D_K$ contains $K$. Thus $S^w$ is normal.

That $S^w$ is not locally metrizable at any point follows from the property that each region in the development of $S^w$ contains a non-metrizable, normal Moore space.

**Corollary 1.** If there is a normal, separable Moore space which is not metrizable, then there is one which is not locally metrizable at any point.

**Proof.** The cardinality of each $M^j$ of Theorem 1 is $\aleph_0$ and $\sum_{j=1}^{\infty} M^j$ is countable.

**Corollary 2.** There exists a Moore space which is not locally metrizable at any point.

**Proof.** Remove the condition of normality from the hypothesis of Theorem 1.
THEOREM 2. If there is a normal, separable, nonmetrizable Moore space, then there is one which is also locally compact.

Proof. Suppose \((S, \Omega)\) is normal, separable Moore space which is not metrizable. There exist [2], in \(S\), an uncountable point set \(M\) with no limit point and a countable subset \(K\) of \(S \cdot M\) such that every point of \(M\) is a limit point of \(K\). Denote the subspace \(K + M\) with the relative topology by \((S_1, \Omega_1)\). This space is normal, separable, nonmetrizable and is a Moore space.

Enumerate \(K\): \(A_1, A_2, A_3, \ldots\). For each point \(x\) of \(M\) denote by \(\{n_i(x)\}_{i=1}^{\infty}\) an increasing sequence of positive integers such that \(\lim_{n \to \infty} A_{n_i}(x) = x\), according to the topology \(\Omega_1\). Now consider the space \((S_1, \Omega_2)\), where \(\Omega_2\) is the topology induced by the following definition of region:

The point set \(R\) is a region if and only if either (1) for some point \(P\) of \(K\), \(R\) is the degenerate set whose only element is \(P\), or (2) some point \(x\) of \(M\) and some positive integer \(i\), \(R\) is the set to which \(P\) belongs if and only if \(P = x\) or \(P = A_{n_j}(x)\) for some \(j\) greater than or equal to \(i\). Such a region is denoted by \(R(x, i)\). Note that if the point \(x\) of \(M\) is a limit point of the subset \(K'\) of \(K\) according to the topology \(\Omega_2\), then it is according to \(\Omega_1\).

Now \((S_1, \Omega_2)\) is clearly locally compact, separable, but not completely separable, and thus not metrizable. Also, it is a Moore space, for let \(G_i'\) be the collection to which the region \(R\) belongs if and only if \(R\) is a degenerate region, or, for some point \(x\) of \(M\) and some positive integer \(i \geq n\), \(R = R(x, i)\). Then \(G_1', G_2', G_3', \ldots\) gives a development for \((S_1, \Omega_2)\). It remains to be shown that \((S_1, \Omega_2)\) is normal. To this end, suppose that \(I\) and \(J\) are mutually exclusive closed point sets in \(S_1\) (according to \(\Omega_2\)). There exist mutually exclusive domains \(U\) and \(V\) (in \(\Omega_1\)) containing \(I \cdot M\) and \(J \cdot M\) respectively. Then \(U\) and \(V\) are open according to \(\Omega_2\), and \((U - U \cdot J + I \cdot K)\) and \((V - V \cdot I + J \cdot K)\) are mutually exclusive domains in \(\Omega_2\) containing \(I\) and \(J\) respectively. Thus \((S_1, \Omega_2)\) is normal.

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