

# Pacific Journal of Mathematics

**AN INEQUALITY FOR THE DENSITY OF THE SUM OF SETS  
OF VECTORS IN  $n$ -DIMENSIONAL SPACE**

ALLEN ROY FREEDMAN

## AN INEQUALITY FOR THE DENSITY OF THE SUM OF SETS OF VECTORS IN $n$ -DIMENSIONAL SPACE

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**A Schnirelmann type density is defined for sets of "nonnegative" lattice points. If  $A, B$  and  $C = A + B$  are such sets with densities  $\alpha, \beta$  and  $\gamma$  respectively, then it is shown that  $\gamma \geq \beta/(1 - \alpha)$  provided  $\alpha + \beta < 1$ .**

1. Let  $n$  be a positive integer and let  $Q$  be the set of all vectors  $r = (\rho_1, \dots, \rho_n)$  where each  $\rho_i$  is a nonnegative integer and at least one  $\rho_i$  is positive. We define a partial order relation  $<$  on  $Q$  where  $r < s$  if and only if  $\rho_i \leq \sigma_i$  ( $i = 1, 2, \dots, n$ ) with strict inequality holding for at least one index. Denote by  $L(r)$  the set of all  $x$  in  $Q$  for which either  $x < r$  or  $x = r$ .

A nonempty finite subset  $F$  of  $Q$  is called fundamental if, whenever  $r \in F$ , then  $L(r) \subseteq F$ . For  $A, X \subseteq Q$  with  $X$  finite, let  $A(X)$  denote the number of vectors in  $A \cap X$ . Then the (Kvarda) density of  $A$  is

$$\alpha = \text{glb} \frac{A(F)}{Q(F)}$$

where  $F$  ranges over all fundamental subsets of  $Q$ .

Let  $B \subseteq Q$  and define  $A + B = \{a, b, a + b \mid a \in A, b \in B\}$  where addition of vectors is done coordinatewise. Let  $\beta$  and  $\gamma$  be the densities of  $B$  and  $C = A + B$  respectively. Kvarda [1] has proved the inequality  $\gamma = \alpha + \beta - \alpha\beta$  which for  $n = 1$  was first proved by Landau and Schnirelmann. In this paper we prove  $\gamma \geq \beta/(1 - \alpha)$  provided  $\alpha + \beta < 1$ . For  $n = 1$ , this has been proved by Schur [2].

### 2. Main results.

**LEMMA 1.** *Let  $\bar{C}$  denote the complement of  $C$  in  $Q$  and suppose  $\bar{C} \neq \emptyset$ . For a fundamental set  $F$  let  $F^*$  denote the set of maximal vectors of  $F$  with respect to the partial ordering  $<$ . Then*

$$\gamma = \text{glb} \frac{C(F)}{Q(F)}$$

where  $F$  ranges over all fundamental sets with  $F^* \subseteq \bar{C}$ .

*Proof.* Let  $\gamma'$  denote this glb. Clearly  $\gamma \leq \gamma'$ . Let  $G$  be any fundamental set. If  $C(G) = Q(G)$  then  $C(G)/Q(G) = 1 > \gamma'$ . If  $C(G) < Q(G)$  then  $\bar{C} \cap G \neq \emptyset$ . In this case let  $F$  be the union of all

sets  $L(g)$  where  $g \in \bar{C} \cap G$ . Evidently  $F$  is a fundamental set,  $F \subseteq G$ , and  $F^* \subseteq \bar{C}$ . Thus,

$$\frac{C(G)}{Q(G)} = \frac{C(F) + C(G - F)}{Q(F) + Q(G - F)} = \frac{C(F) + Q(G - F)}{Q(F) + Q(G - F)} \geq \frac{C(F)}{Q(F)} \geq \gamma',$$

and so  $\gamma \geq \gamma'$ .

LEMMA 2. *If  $F$  is a fundamental set with  $F^* \subseteq \bar{C}$ , then  $C(F) \geq \alpha C(F) + B(F)$ .*

*Proof.* Let  $g_1, g_2, \dots, g_k$  be the vectors of  $\bar{C} \cap F$ , indexed in such a way that

$$(1) \quad g_i < g_j \text{ implies } i < j.$$

Define  $H_1 = L(g_1)$  and  $H_{i+1} = L(g_{i+1}) - \bigcup_{j=1}^i H_j$ . Then

- (2) the  $H_i$  are disjoint,
- (3) the union of the  $H_i$  is  $F$ , and
- (4) for each  $i, g_i \in H_i$ .

Now (2) follows immediately by definition, and (3) from the fact that since  $F^* \subseteq \bar{C}$ , we have for each  $x \in F$ , that  $x \in L(g_i)$  for some  $i$ . To prove (4) notice that  $g_i \notin H_i$  implies  $g_i \in \bigcup_{j=1}^{i-1} H_j$ , which in turn implies  $g_i \in L(g_{j_0})$  for some  $j_0 < i$ , contrary to (1).

For each  $i$  let  $tH_i$  be the set of all vectors  $g_i - x$  where  $x$  ranges over  $H_i - \{g_i\}$ . Then

- (5)  $tH_i$  is either empty or is a fundamental set, and
- (6)  $Q(tH_i) = Q(H_i) - 1$ .

To show (5) let  $z$  be an arbitrary vector in  $tH_i$  and let  $y \in L(z)$ . We have  $g_i - z \leq g_i - y < g_i$ . Thus  $g_i - y \in L(g_i) - \{g_i\}$  and, since  $g_i - z \in H_i$ , we have  $g_i - y \in H_i - \{g_i\}$ . Hence  $g_i - (g_i - y) = y \in tH_i$  and so  $L(z) \subseteq tH_i$ . Equation (6) is immediate.

Now, for each  $a \in A \cap tH_i$ , there exists a unique  $x \in H_i - \{g_i\}$  such that  $a = g_i - x$ . Thus  $x \in \bar{B}$ . Also, by (4), we have  $g_i \in \bar{B} \cap H_i$  and so

$$\begin{aligned} \bar{B}(H_i) &\geq A(tH_i) + 1 \\ &\geq \alpha Q(tH_i) + 1 \quad (\text{from (5) and the definition of } \alpha) \\ &= \alpha(Q(H_i) - 1) + 1 \quad (\text{from (6)}). \end{aligned}$$

Summing over  $i$ , using (2) and (3), we obtain

$$\begin{aligned} \bar{B}(F) &\geq \alpha(Q(F) - k) + k \\ &= \alpha C(F) + \bar{C}(F) \end{aligned}$$

that is,

$$C(F) \geq \alpha C(F) + B(F).$$

**THEOREM.** *If  $\alpha + \beta < 1$  then  $\gamma \geq \beta/(1 - \alpha)$ .*

*Proof.* Since  $\beta < 1 - \alpha$  and  $\alpha < 1$ , then  $\beta/(1 - \alpha) < 1$ . Hence if  $\gamma = 1$ , the theorem follows. If  $\gamma < 1$ , then  $\bar{C} \neq \emptyset$  and for any fundamental set  $F$  with  $F^* \subseteq \bar{C}$  we have by Lemma 2

$$C(F) \geq \alpha C(F) + B(F).$$

Hence,

$$\frac{C(F)}{Q(F)} \geq \alpha \frac{C(F)}{Q(F)} + \frac{B(F)}{Q(F)} \geq \alpha\gamma + \beta.$$

By Lemma 1  $\gamma \geq \alpha\gamma + \beta$  that is,  $\gamma \geq \beta/(1 - \alpha)$ .

**3. Remark.** A result of Kvarda [1] states that if  $\alpha + \beta \geq 1$  then  $\gamma = 1$ . This result and the above theorem can be used to prove quickly that if  $\alpha > 0$  then  $A$  is a basis for  $Q$ , that is,  $nA = Q$  for some  $n$ , where  $iA = (i - 1)A + A$  for  $i \geq 2$ . Thus let  $\gamma_i$  denote the density of  $iA$  and assume that  $nA \neq Q$  for all  $n$ . Then, for all  $k$ ,  $\gamma_k + \alpha < 1$ , and so

$$\gamma_{k+1} \geq \frac{\gamma_k}{1 - \alpha} \geq \frac{\gamma_{k-1}}{(1 - \alpha)^2} \geq \cdots \geq \frac{\gamma_1}{(1 - \alpha)^k} = \frac{\alpha}{(1 - \alpha)^k}$$

But, for  $k$  sufficiently large,  $(\alpha/(1 - \alpha)^k) \geq 1$ , a contradiction.

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2. I. Schur, *Über den Begriff der Dichte in der additiven Zahlentheorie*, S. B. Preuss. Akad. Wiss. Phys. Math. Kl. (1936), 269-297.

Received June 12, 1965. This paper is part of the author's Ph. D. thesis, written at Oregon State University under the direction of Professor Robert Stalley.



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The *Pacific Journal of Mathematics* is published monthly. Effective with Volume 16 the price per volume (3 numbers) is \$8.00; single issues, \$3.00. Special price for current issues to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: \$4.00 per volume; single issues \$1.50. Back numbers are available.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley 8, California.

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), No. 6, 2-chome, Fujimi-cho, Chiyoda-ku, Tokyo, Japan.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

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