

Pacific Journal of Mathematics

TRANSFORMATIONS OF FOURIER COEFFICIENTS

DANIEL RIDER

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Let A and B be function spaces on the unit circle and let F be a complex function defined in the plane. F is said to map A into B provided $\sum F(a_n) e^{in\theta}$ is the Fourier series of a function in B whenever $\sum a_n e^{in\theta}$ is the Fourier series of a function in A . For $1 \leq q < \infty$, let L^q denote the usual space of functions on the unit circle normed by

$$(1) \quad \|f\|_q = \left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(e^{i\theta})|^q d\theta \right\}^{1/q}.$$

Let $2 \leq q \leq \infty$ and p be given by $p^{-1} + q^{-1} = 1$.

It follows from the Hausdorff-Young theorem that if $b(z)$ is bounded near the origin, then

$$(2) \quad F(z) = c_1 z + c_2 \bar{z} + |z|^{2/p} b(z)$$

maps L^q into L^q .

In this paper it is shown that all functions mapping L^q into L^q have this form. In fact, all functions mapping the continuous functions into L^q have this form.

THEOREM 1. Let $2 \leq q \leq \infty$. The following are equivalent.

(i) F maps L^q into L^q .

(ii) F maps the continuous functions into L^q .

(iii) $F(z) = c_1 z + c_2 \bar{z} + |z|^{2/p} b(z)$ where $b(z)$ is bounded near the origin.

Rudin [2] proves that Theorem 1 is true provided F is an even function. Our proof consists primarily of applications of the method devised by Rudin.

\mathcal{C} will denote the continuous functions on the unit circle. The Fourier coefficients of $f \in L^1$ are given by

$$(3) \quad \hat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{i\theta}) e^{-in\theta} d\theta \quad (n = 0, \pm 1, \pm 2, \dots).$$

F maps A into B provided given $f \in A$ there is $g \in B$ such that $\hat{g} = F(\hat{f})$. This is written $g = F \circ f$.

2. Trigonometric polynomials with few coefficients. H. S. Shapiro in his Master's thesis [3], and, independently, Rudin [2], prove the existence of a sequence $\{\varepsilon_n\}$ with $\varepsilon_n = \pm 1$ such that

$$(4) \quad \left| \sum_{n=1}^N \varepsilon_n e^{in\theta} \right| < 5N^{1/2} \quad (0 \leq \theta \leq 2\pi; N = 1, 2, 3, \dots).$$

A similar construction yields

THEOREM 2. *Let r be a prime integer and $\alpha = \exp(2\pi i/r)$. There is a sequence $\{\varepsilon_r(n)\}$ with $\varepsilon_r(n)$ having for each n one of the values $1, \alpha, \dots, \alpha^{r-1}$ such that*

$$(5) \quad \left| \sum_{n=1}^N \varepsilon_r(n) e^{in\theta} \right| < r(1 + r^{1/2})N^{1/2} \quad (0 \leq \theta \leq 2\pi; N = 1, 2, 3, \dots).$$

Proof. Let A_0, A_1, \dots, A_{r-1} be complex numbers. A simple calculation based on the identity $\sum_{j=0}^{r-1} \alpha^{sj} = \begin{cases} r, & s = 0 \\ 0, & s = 1, 2, \dots, r-1 \end{cases}$ gives

$$(6) \quad \sum_{s=0}^{r-1} \left| \sum_{j=0}^{r-1} \alpha^{sj} A_j \right|^2 = r \sum_{j=0}^{r-1} |A_j|^2.$$

For $r = 2$, this is just the parallelogram law used in [2] and [3] to prove the theorem for the special case $r = 2$.

Let $P_0^0(x) = P_0^1(x) = \dots = P_0^{r-1}(x) = x$ and define polynomials P_k^s inductively by

$$(7) \quad P_{k+1}^s(x) = \sum_{j=0}^{r-1} x^{jr^k} \alpha^{sj} P_k^j(x) \quad (s = 0, 1, \dots, r-1).$$

P_k^s is a polynomial of degree r^k and it is easily seen by induction that each of its coefficients is a power of α and that P_k^0 is a partial sum of P_{k+1}^0 . The sequence $\varepsilon_r(n)$ is defined by letting $\varepsilon_r(n)$ be the n^{th} coefficient of P_k^0 when $r^k > n$.

If $|x| = 1$, (6) and (7) yield

$$(8) \quad \sum_{s=0}^{r-1} |P_{k+1}^s(x)|^2 = r \sum_{j=0}^{r-1} |P_k^j(x)|^2.$$

Since $\sum_{s=0}^{r-1} |P_0^s(x)|^2 = r$, we have

$$(9) \quad \sum_{s=0}^{r-1} |P_k^s(e^{i\theta})|^2 = r^{k+1}.$$

Hence

$$(10) \quad |P_k^0(e^{i\theta})| \leq r^{1/2} r^{k/2}.$$

For $N = r^k$ this is stronger than (5). From it we can obtain (5) for all values of N by following the procedure of [2].

If we replace α by α^t ($t = 1, 2, \dots, r-1$) in (7) then we obtain a sequence $\{\varepsilon_{r,t}(n)\}$ such that

$$(11) \quad \varepsilon_{r,t}(n) = (\varepsilon_r(n))^t$$

and

$$(12) \quad \left| \sum_{n=1}^N \varepsilon_{r,t}(n) e^{in\theta} \right| < r(1 + r^{1/2})N^{1/2}.$$

Now let $\delta_r(n) = \sum_{i=1}^{r-1} \epsilon_{r,i}(n)$. Since $\epsilon_r(n)$ is an r th root of unity, it follows from (11) that $\delta_r(n) = r - 1$ or -1 . Thus (12) yields

THEOREM 3. *If r is a prime there is a sequence $\{\delta_r(n)\}$ with $\delta_r(n) = r - 1$ or -1 such that*

$$(13) \quad \left| \sum_{n=1}^N \delta_r(n) e^{in\theta} \right| < (r-1)r(1+r^{1/2})N^{1/2} \quad (0 \leq \theta \leq 2\pi; N = 1, 2, 3, \dots).$$

3. Proof of Theorem 1. To prove Theorem 1, we need only show that (ii) implies (iii). Furthermore, by [2, Theorem 4], we can assume that F is odd. For $q = 2$, Theorem 1 follows from [2, Theorem 4]. For if F maps \mathcal{E} into L^2 then $H(z) = |F(z)| + |F(-z)|$ is an even function mapping \mathcal{E} into L^2 so that $|F(z)/z|$ is bounded near the origin. In this section F will map \mathcal{E} into L^q ($q > 2; 1/p + 1/q = 1$).

The proof of the theorem relies primarily on the following lemma similar to [1; Lemma 3.2].

LEMMA 1. *Let F map \mathcal{E} into L^q . There are constants $\delta > 0$ and $M < \infty$ such that*

$$(14) \quad \|F \circ f\|_q \leq M$$

whenever $f \in \mathcal{E}$ and $\|f\|_\infty < \delta$.

Proof. It is sufficient to show that (14) holds for trigonometric polynomials.

For let $f \in \mathcal{E}$, $\|f\|_\infty < (1/3)\delta$, and define

$$(15) \quad K_m(e^{i\theta}) = \sum_{n=-2m}^{2m} \min\left(1, 2 - \frac{|n|}{m}\right) e^{in\theta} \quad (m = 1, 2, 3, \dots).$$

If $*$ denotes ordinary convolution then $f * K_m$ is a polynomial such that $\|f * K_m\|_\infty < \delta$. Hence $\|F \circ (f * K_m)\|_q \leq M$. But a subsequence of $\{F \circ (f * K_m)\}$ approaches $F \circ f$ weakly as elements of L^q . Hence $\|F \circ f\|_q \leq M$.

Thus if the lemma is false there is a sequence of polynomials $\{f_m\}$ with $\|f_m\|_\infty < 1/m^2$ and $\|F \circ f_m\|_q \rightarrow \infty$ as $m \rightarrow \infty$. Clearly we may assume that $\hat{f}_m(k) = 0$ if $k < 0$. Let N_m be the degree of f_m and choose integers n_m so that

$$(16) \quad n_m + 3N_m < n_{m+1} - N_{m+1}.$$

The series

$$(17) \quad f(e^{i\theta}) = \sum_{m=1}^{\infty} e^{in_m\theta} f_m(e^{i\theta})$$

converges uniformly to a continuous function. Let

$$H_m(e^{i\theta}) = e^{i(n_m+N_m)\theta} K_{N_m}(\theta).$$

The choice of $\{n_m\}$ implies that

$$(18) \quad (F \circ f) * H_m = e^{in_m\theta} (F \circ f_m).$$

Since $\|H_m\|_1 < 3$, it follows that

$$(19) \quad \|F \circ f_m\|_q < 3 \|F \circ f\|_q.$$

But this is impossible since $\|F \circ f\|_q \rightarrow \infty$.

LEMMA 2. $|F(z/2) - (1/2)F(z)| |z|^{-2/p}$ is bounded near the origin.

Proof. If the lemma is false there are numbers $z_m \neq 0$ ($m = 1, 2, 3, \dots$) such that $mz_m \rightarrow 0$ and

$$(20) \quad \left| F\left(\frac{z_m}{2}\right) - \frac{1}{2} F(z_m) \right| > m^3 |z_m|^{2/p}.$$

Let $N_m = [m^{-2}z_m^{-2}]$ and define

$$(21) \quad T_m(e^{i\theta}) = \frac{z_m}{2} \sum_{n=1}^{N_m} \delta_3(n) e^{in\theta}$$

where $\delta_3(n)$ is the sequence of Theorem 3 for $r = 3$. From Theorem 3 and the definition of N_m it follows that $\|T_m\|_\infty \rightarrow 0$ as $m \rightarrow \infty$. Hence, by Lemma 1, $\|F \circ T_m\|_q$ is bounded as $m \rightarrow \infty$.

Since F is an odd function

$$(22) \quad (F \circ T_m)(e^{i\theta}) = F(z_m) \sum_{1 \leq n \leq N_m, \delta_3(n)=2} e^{in\theta} - F\left(\frac{z_m}{2}\right) \sum_{1 \leq n \leq N_m, \delta_3(n)=-1} e^{in\theta}.$$

Thus

$$(23) \quad \begin{aligned} |F \circ T_m(e^{i\theta})| &\geq \frac{2}{3} \left| \frac{1}{2} F(z_m) - F\left(\frac{z_m}{2}\right) \right| \left| \sum_{n=1}^{N_m} e^{in\theta} \right| \\ &\quad - \frac{1}{3} \left| F(z_m) + F\left(\frac{z_m}{2}\right) \right| \left| \sum_{n=1}^{N_m} \delta_3(n) e^{in\theta} \right|. \end{aligned}$$

Now if F maps \mathcal{E} to L^q , $q > 2$, then, *a fortiori*, F maps \mathcal{E} to L^2 . Thus the truth of Theorem 1 for $q = 2$ implies that $|F(z)/z|$ is bounded near the origin. Thus, since $\|\sum_{n=1}^{N_m} e^{in\theta}\|_q \geq C_q N_m^{1/p}$, it follows that $|(1/2)F(z_m) - F(z_m/2)| \cdot N_m^{1/p}$ is bounded as $m \rightarrow \infty$. However this is a contradiction to (20).

LEMMA 3. $F(z) = F_1(z) + F_2(z)$ where

- (a) F_1 and F_2 map \mathcal{E} into L^q .
- (b) $|F_2(z)| |z|^{-2/p}$ is bounded near the origin.
- (c) $F_1(z/2) = (1/2)F_1(z)$ for all z .

REMARK. F_2 is the “small” part of F . Lemmas 5 and 6 show that because of (a) and (c)

$$F_1(z) = c_1 z + c_2 \bar{z} .$$

Proof. By Lemma 2 there are finite positive constants B and C such that for $|z| \leq B$

$$\begin{aligned} \left| F\left(\frac{z}{2^k}\right) - \frac{1}{2^k} F(z) \right| &\leq \sum_{j=0}^{k-1} \frac{1}{2^j} \left| F\left(\frac{z}{2^{k-j}}\right) - \frac{1}{2} F\left(\frac{z}{2^{k-j-1}}\right) \right| \\ (24) \qquad \qquad \qquad &\leq C \sum_{j=0}^{k-1} \frac{1}{2^j} \left| \frac{z}{2^{k-j-1}} \right|^{2/p} \\ &\leq C' \frac{|z|^{2/p}}{2^k} \qquad \qquad \qquad (k = 1, 2, 3, \dots) . \end{aligned}$$

Define

$$(25) \qquad \qquad \qquad F_1(z) = \lim_{n \rightarrow \infty} 2^n F\left(\frac{z}{2^n}\right) .$$

This limit exists. For if $n > j$ and we apply (24) to $z/2^j$ with $k = n - j$ and multiply by 2^n then

$$(26) \qquad \qquad \qquad \left| 2^n F\left(\frac{z}{2^n}\right) - 2^j F\left(\frac{z}{2^j}\right) \right| \leq 2^j C' \left| \frac{z}{2^j} \right|^{2/p} .$$

Since $p < 2$, the right side of (26) $\rightarrow 0$ as j and $n \rightarrow \infty$.

It is clear from the definition of F_1 that (c) holds. $F_2(z) = F(z) - F_1(z)$ and (b) is a result of (24). F_2 maps \mathcal{E} into L^q because of (b). Thus F_1 does also. Note that F_1 is odd (since F is).

LEMMA 4. F_1 is continuous.

Proof. It is sufficient to show it is continuous at 1. If not, there is a sequence $z_m \rightarrow 1$ such that $F_1(z_m) \not\rightarrow F_1(1)$. The z_m can be chosen so that

$$(27) \qquad \qquad \qquad |1 - z_m| < 2^{-m} .$$

Let $N_m = [2^{2m} \cdot m^{-2}]$ and define

$$(28) \qquad T_m(e^{i\theta}) = 2^{-m} \sum_{n=1}^{N_m} \left\{ \varepsilon_2(n) + \frac{(1 - z_m)}{2} (1 - \varepsilon_2(n)) \right\} e^{in\theta}$$

where $\{\varepsilon_2(n)\}$ is the sequence of Theorem 2.

Theorem 2, (27) and the choice of N_m imply that $\|T_m\|_\infty = O(1/m)$ so that, by Lemma 1, $\|F \circ T_m\|_q$ is bounded as $m \rightarrow \infty$. But then since $F_1(z/2) = (1/2)F_1(z)$,

$$\begin{aligned}
 |F_1 \circ T_m(e^{i\theta})| &= 2^{-m} \left| F_1(1) \sum_{1 \leq n \leq N_m, \varepsilon_2(n)=1} e^{in\theta} - F_1(z_m) \sum_{1 \leq n \leq N_m, \varepsilon_2(n)=-1} e^{in\theta} \right| \\
 (29) \qquad &\geq 2^{-m-1} |F_1(1) - F_1(z_m)| \left| \sum_{n=1}^{N_m} e^{in\theta} \right| \\
 &\quad - 2^{-m-1} |F_1(1) + F_1(z_m)| \left| \sum_{n=1}^{N_m} \varepsilon_2(n) e^{in\theta} \right|.
 \end{aligned}$$

As in Lemma 2 this implies that $|F_1(1) - F_1(z_m)| N_m^{1/p} \cdot 2^{-m}$ is bounded as $m \rightarrow \infty$, which is impossible unless $F_1(z_m) \rightarrow F_1(1)$. Hence F_1 is continuous.

LEMMA 5. *There are continuous functions C_1 and C_2 on $(0, \infty)$ such that*

$$F_1(xe^{i\theta}) = C_1(x)e^{i\theta} + C_2(x)e^{-i\theta} \qquad (0 < x < \infty).$$

Proof. We will show that if r is an integer ($r \neq 0, 1$) and z a complex number then

$$(30) \qquad \sum_{j=1}^r F_1\left(z \exp \frac{2\pi i j}{r}\right) = 0.$$

Now consider $F_1(xe^{i\theta}) = G_x(e^{i\theta})$ for a fixed x . G_x is a continuous function of θ by Lemma 4. It follows from (30) that for each integer $r \neq 0, 1$,

$$(31) \qquad \sum_{j=1}^r G_x\left(\exp i\left(\theta + \frac{2\pi j}{r}\right)\right) = 0 \qquad (0 \leq \theta \leq 2\pi).$$

By considering the Fourier coefficients of G_x it is easily seen that $G_x(e^{i\theta}) = C_1(x)e^{i\theta} + C_2(x)e^{-i\theta}$. C_1 and C_2 are continuous because of Lemma 4.

To prove (30) it is sufficient to assume that $z = 1$. It is also sufficient to assume r is prime. For if $r = pq$ where p is a prime then (30) can be written

$$(32) \qquad \sum_{s=1}^q \sum_{j=1}^p F_1\left(z \exp \frac{2\pi i(jq + s)}{pq}\right).$$

If (30) holds for primes then each summand of the outer sum of (32) is zero.

Let $N_m = [2^{2m}m^{-2}]$ and define

$$(33) \quad T_m^t(e^{i\theta}) = 2^{-m} \sum_{n=1}^{N_m} \{\varepsilon_r(n)\}^t e^{in\theta} \quad (t = 1, 2, \dots, r - 1),$$

where $\{\varepsilon_r(n)\}$ is the sequence of Theorem 2. $\|T_m^t\|_\infty = O(1/m)$ so that if $\beta = \sum_{j=1}^r F_1(\exp 2\pi i j/r)$ and

$$(34) \quad H_m(e^{i\theta}) = \sum_{t=1}^{r-1} \left\{ F_1 \circ T_m^t + \left\{ \frac{\beta}{r} - F_1(1) \right\} T_m^t \right\}$$

then, by Lemma 1, $\|H_m\|_q$ is bounded as $m \rightarrow \infty$. Now since $F_1(z/2) = (1/2)F(z)$

$$(35) \quad |H_m(e^{i\theta})| = 2^{-m} \left| \sum_{t=1}^{r-1} \left\{ \sum_{n=1}^{N_m} F_1\{\{\varepsilon_r(n)\}^t\} e^{in\theta} + \left\{ \frac{\beta}{r} - F_1(1) \right\} \sum_{n=1}^{N_m} \{\varepsilon_r(n)\}^t e^{in\theta} \right\} \right|.$$

Suppose $\varepsilon_r(n) = 1$. The coefficient of $e^{in\theta}$ in (35) is then

$$(36) \quad (r - 1)F_1(1) + (r - 1) \left\{ \frac{\beta}{r} - F_1(1) \right\} = \left(1 - \frac{1}{r} \right) \beta.$$

Suppose $\varepsilon_r(n) \neq 1$, so that $\varepsilon_r(n)$ is a primitive r^{th} root of unity. Then $\sum_{t=1}^{r-1} F_1\{\{\varepsilon_r(n)\}^t\} = \beta - F_1(1)$ and $\sum_{t=1}^{r-1} \{\varepsilon_r(n)\}^t = -1$ so that the coefficient of $e^{in\theta}$ is

$$(37) \quad \beta - F_1(1) - \left\{ \frac{\beta}{r} - F_1(1) \right\} = \left(1 - \frac{1}{r} \right) \beta.$$

Hence

$$(38) \quad |H_m(e^{i\theta})| = 2^{-m} \left(1 - \frac{1}{r} \right) |\beta| \left| \sum_{n=1}^{N_m} e^{in\theta} \right|$$

so that

$$(39) \quad \|H_m\|_q \geq \left(1 - \frac{1}{r} \right) \frac{|\beta|}{2^m} C_q N_m^{1/p} \quad (m = 1, 2, \dots).$$

But this is impossible unless $\beta = 0$. That is

$$(40) \quad \sum_{j=1}^r F_1 \left(\exp \frac{2\pi i j}{r} \right) = 0$$

which was to be proved

LEMMA 6. $C_j(x) = xC_j(1) \quad (0 < x < \infty; j = 1, 2)$.

Proof. Fix x and φ . Let r be a prime, $N_m = [2^{2m}m^{-2}]$, and define

$$(41) \quad T_m(e^{i\theta}) = \frac{x e^{i\varphi}}{(r-1)2^m} \sum_{n=1}^{N_m} \delta_r(n) e^{in\theta}$$

where $\{\delta_r(n)\}$ is the sequence of Theorem 3.

Since F_1 is odd and $F_1(z) = 2F_1(z/2)$ we can write

$$(42) \quad \begin{aligned} F_1 \circ T_m(e^{i\theta}) &= \frac{1}{2^m r} \left\{ F_1(x e^{i\varphi}) - (r-1) F_1\left(\frac{x e^{i\varphi}}{r-1}\right) \right\} \sum_{n=1}^{N_m} e^{in\theta} \\ &+ \frac{1}{2^m r} \left\{ F_1(x e^{i\varphi}) + F_1\left(\frac{x e^{i\varphi}}{r-1}\right) \right\} \sum_{n=1}^{N_m} \delta_r(n) e^{in\theta}. \end{aligned}$$

As in the proofs of Lemma 2 and 4, $\|F_1 \circ T_m\|_q$ and $2^{-m} \|\sum \delta_r(n) e^{in\theta}\|_q$ are bounded. Hence $2^{-m} N_m^{1/p} |F_1(x e^{i\varphi}) - (r-1) F_1(x e^{i\varphi}/(r-1))|$ is bounded. But $2^{-m} N_m^{1/p}$ is unbounded so that

$$(43) \quad F_1(x e^{i\varphi}) - (r-1) F_1\left(\frac{x e^{i\varphi}}{r-1}\right) = 0 \quad (0 < x < \infty; 0 \leq \varphi \leq 2\pi).$$

By Lemma 5, (43) can be written

$$(44) \quad \begin{aligned} &\left\{ C_1(x) - (r-1) C_1\left(\frac{x}{r-1}\right) \right\} e^{i\varphi} \\ &+ \left\{ C_2(x) - (r-1) C_2\left(\frac{x}{r-1}\right) \right\} e^{-i\varphi} = 0. \end{aligned}$$

Clearly this possible only if

$$(45) \quad C_j(x) = (r-1) C_j\left(\frac{x}{r-1}\right) \quad (0 < x < \infty; j = 1, 2).$$

Thus, if r and q are primes and n is an integer,

$$(46) \quad C_j\left(\left(\frac{r-1}{q-1}\right)^n\right) = \left(\frac{r-1}{q-1}\right)^n C_j(1) \quad (j = 1, 2).$$

Now $\{(r-1/q-1)^n; r, q, \text{ primes; } n \text{ an integer}\}$ is dense in the positive real numbers. This is true since given $\varepsilon > 0$ there are infinitely many pairs of consecutive primes q_n, q_{n+1} such that $q_{n+1} < (1 + \varepsilon)q_n$.

Since C_j is continuous (46) then implies $C_j(x) = x C_j(1)$ for all x .

The proof of Theorem 1 follows from Lemmas 3, 5, and 6.

4. The general case. We remark here that Theorem 1 holds if we consider any compact Abelian group G . If Γ , the dual group of G , has elements of arbitrarily large order then it is possible to construct polynomials as in § 2 and the proof proceeds as in § 3. When, Γ , and hence G , has an exponent the construction of the polynomials is slightly different (it depends on the structure of Γ) but the remainder of the proof is similar.

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