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# TRANSFORMATIONS OF FOURIER COEFFICIENTS

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**Let $A$ and $B$ be function spaces on the unit circle and let $F$ be a complex function defined in the plane. $F$ is said to map $A$ into $B$ provided $\sum F(a_n)\, e^{in\theta}$ is the Fourier series of a function in $B$ whenever $\sum a_n\, e^{in\theta}$ is the Fourier series of a function in $A$. For $1 \leqq q < \infty$, let $L^q$ denote the usual space of functions on the unit circle normed by**

$$\|f\|_q = \left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(e^{i\theta})|^q \, d\theta\right\}^{1/q} . \tag{1}$$

**Let $2 \leqq q \leqq \infty$ and $p$ be given by $p^{-1} + q^{-1} = 1$.**

**It follows from the Hausdorff-Young theorem that if $b(z)$ is bounded near the origin, then**

$$F(z) = c_1 z + c_2 \bar{z} + |z|^{2/p}\, b(z) \tag{2}$$

**maps $L^q$ into $L^q$.**

**In this paper it is shown that all functions mapping $L^q$ into $L^q$ have this form. In fact, all functions mapping the continuous functions into $L^q$ have this form.**

**THEOREM 1. Let $2 \leqq q \leqq \infty$. The following are equivalent.**

**(i) $F$ maps $L^q$ into $L^q$.**

**(ii) $F$ maps the continuous functions into $L^q$.**

**(iii) $F(z) = c_1 z + c_2 \bar{z} + |z|^{2/p}\, b(z)$ where $b(z)$ is bounded near the origin.**

Rudin [2] proves that Theorem 1 is true provided $F$ is an even function. Our proof consists primarily of applications of the method devised by Rudin.

$\mathscr{C}$ will denote the continuous functions on the unit circle. The Fourier coefficients of $f \in L^1$ are given by

$$\hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i\theta}) e^{-in\theta} d\theta \qquad (n = 0, \pm 1, \pm 2, \cdots) . \tag{3}$$

$F$ maps $A$ into $B$ provided given $f \in A$ there is $g \in B$ such that $\hat{g} = F(\hat{f})$. This is written $g = F \circ f$.

## 2. Trigonometric polynomials with few coefficients.

H. S. Shapiro in his Master's thesis [3], and, independently, Rudin [2], prove the existence of a sequence $\{\varepsilon_n\}$ with $\varepsilon_n = \pm 1$ such that

$$\left|\sum_{n=1}^{N} \varepsilon_n\, e^{in\theta}\right| < 5N^{1/2} \qquad (0 \leqq \theta \leqq 2\pi;\ N = 1, 2, 3, \cdots) . \tag{4}$$

A similar construction yields

THEOREM 2. *Let $r$ be a prime integer and $\alpha = \exp(2\pi i/r)$. There is a sequence $\{\varepsilon_r(n)\}$ with $\varepsilon_r(n)$ having for each $n$ one of the values $1, \alpha, \cdots, \alpha^{r-1}$ such that*

$$(5)\qquad \left|\sum_{n=1}^{N} \varepsilon_r(n)\, e^{in\theta}\right| < r(1 + r^{(1/2)})N^{1/2} \qquad (0 \leqq \theta \leqq 2\pi;\ N = 1, 2, 3, \cdots)\ .$$

*Proof.* Let $A_0, A_1, \cdots, A_{r-1}$ be complex numbers. A simple calculation based on the identity $\sum_{j=0}^{r-1} \alpha^{sj} = \begin{cases} r, & s = 0 \\ 0, & s = 1, 2, \cdots, r-1 \end{cases}$ gives

$$(6)\qquad \sum_{s=0}^{r-1} \left|\sum_{j=0}^{r-1} \alpha^{sj} A_j\right|^2 = r \sum_{j=0}^{r-1} |A_j|^2\ .$$

For $r = 2$, this is just the parallelogram law used in [2] and [3] to prove the theorem for the special case $r = 2$.

Let $P_0^0(x) = P_0^1(x) = \cdots = P_0^{r-1}(x) = x$ and define polynomials $P_k^s$ inductively by

$$(7)\qquad P_{k+1}^s(x) = \sum_{j=0}^{r-1} x^{jr^k} \alpha^{sj} P_k^j(x) \qquad (s = 0, 1, \cdots, r-1)\ .$$

$P_k^s$ is a polynomial of degree $r^k$ and it is easily seen by induction that each of its coefficients is a power of $\alpha$ and that $P_k^0$ is a partial sum of $P_{k+1}^0$. The sequence $\varepsilon_r(n)$ is defined by letting $\varepsilon_r(n)$ be the $n^{\text{th}}$ coefficient of $P_k^0$ when $r^k > n$.

If $|x| = 1$, (6) and (7) yield

$$(8)\qquad \sum_{s=0}^{r-1} |P_{k+1}^s(x)|^2 = r \sum_{j=0}^{r-1} |P_k^s(x)|^2\ .$$

Since $\sum_{s=0}^{r-1} |P_0^s(x)|^2 = r$, we have

$$(9)\qquad \sum_{s=0}^{r-1} |P_k^s(e^{i\theta})|^2 = r^{k+1}\ .$$

Hence

$$(10)\qquad |P_k^0(e^{i\theta})| \leqq r^{1/2} r^{k/2}\ .$$

For $N = r^k$ this is stronger than (5). From it we can obtain (5) for all values of $N$ by following the procedure of [2].

If we replace $\alpha$ by $\alpha^t$ $(t = 1, 2, \cdots, r-1)$ in (7) then we obtain a sequence $\{\varepsilon_{r,t}(n)\}$ such that

$$(11)\qquad \varepsilon_{r,t}(n) = (\varepsilon_r(n))^t$$

and

$$(12)\qquad \left|\sum_{n=1}^{N} \varepsilon_{r,t}(n) e^{in\theta}\right| < r(1 + r^{1/2})N^{1/2}\ .$$

Now let $\delta_r(n) = \sum_{t=1}^{r-1} \varepsilon_{r,t}(n)$. Since $\varepsilon_r(n)$ is an $r$th root of unity, it follows from (11) that $\delta_r(n) = r - 1$ or $-1$. Thus (12) yields

THEOREM 3. *If $r$ is a prime there is a sequence $\{\delta_r(n)\}$ with $\delta_r(n) = r - 1$ or $-1$ such that*

$$\left|\sum_{n=1}^{N} \delta_r(n)e^{in\theta}\right| < (r-1)r(1 + r^{1/2})N^{1/2} \quad (0 \leqq \theta \leqq 2\pi;\ N = 1, 2, 3, \cdots) . \tag{13}$$

**3. Proof of Theorem 1.** To prove Theorem 1, we need only show that (ii) implies (iii). Furthermore, by [2, Theorem 4], we can assume that $F$ is odd. For $q = 2$, Theorem 1 follows from [2, Theorem 4]. For if $F$ maps $\mathscr{C}$ into $L^2$ then $H(z) = |F(z)| + |F(-z)|$ is an even function mapping $\mathscr{C}$ into $L^2$ so that $|F(z)/z|$ is bounded near the origin. In this section $F$ will map $\mathscr{C}$ into $L^q$ $(q > 2;\ 1/p + 1/q = 1)$.

The proof of the theorem relies primarily on the following lemma similar to [1; Lemma 3.2].

LEMMA 1. *Let $F$ map $\mathscr{C}$ into $L^q$. There are constants $\delta > 0$ and $M < \infty$ such that*

$$\|F \circ f\|_q \leqq M \tag{14}$$

*whenever $f \in \mathscr{C}$ and $\|f\|_\infty < \delta$.*

*Proof.* It is sufficient to show that (14) holds for trigonometric polynomials.

For let $f \in \mathscr{C}$, $\|f\|_\infty < (1/3)\delta$, and define

$$K_m(e^{i\theta}) = \sum_{n=-2m}^{2m} \min\left(1, 2 - \frac{|n|}{m}\right)e^{in\theta} \quad (m = 1, 2, 3, \cdots) . \tag{15}$$

If $*$ denotes ordinary convolution then $f * K_m$ is a polynomial such that $\|f * K_m\|_\infty < \delta$. Hence $\|F \circ (f * K_m)\|_q \leqq M$. But a subsequence of $\{F \circ (f * K_m)\}$ approaches $F \circ f$ weakly as elements of $L^q$. Hence $\|F \circ f\|_q \leqq M$.

Thus if the lemma is false there is a sequence of polynomials $\{f_m\}$ with $\|f_m\|_\infty < 1/m^2$ and $\|F \circ f_m\|_q \to \infty$ as $m \to \infty$. Clearly we may assume that $\hat{f}_m(k) = 0$ if $k < 0$. Let $N_m$ be the degree of $f_m$ and choose integers $n_m$ so that

$$n_m + 3N_m < n_{m+1} - N_{m+1} . \tag{16}$$

The series

$$f(e^{i\theta}) = \sum_{m=1}^{\infty} e^{in_m\theta} f_m(e^{i\theta}) \tag{17}$$

converges uniformly to a continuous function. Let

$$H_m(e^{i\theta}) = e^{i(n_m+N_m)\theta}K_{N_m}(\theta) .$$

The choice of $\{n_m\}$ implies that

$$(F\circ f)*H_m = e^{in_m\theta}(F\circ f_m) . \tag{18}$$

Since $\| H_m \|_1 < 3$, it follows that

$$\| F\circ f_m \|_q < 3 \| F\circ f \|_q . \tag{19}$$

But this is impossible since $\| F\circ f \|_q \to \infty$.

LEMMA 2. *$|F(z/2) - (1/2)F(z)|\,|z|^{-2/p}$ is bounded near the origin.*

*Proof.* If the lemma is false there are numbers $z_m \neq 0$ ($m = 1, 2, 3, \cdots$) such that $mz_m \to 0$ and

$$\left| F\left(\frac{z_m}{2}\right) - \frac{1}{2}F(z_m)\right| > m^3 |z_m|^{2/p} . \tag{20}$$

Let $N_m = [m^{-2}z_m^{-2}]$ and define

$$T_m(e^{i\theta}) = \frac{z_m}{2}\sum_{n=1}^{N_m} \delta_3(n)e^{in\theta} \tag{21}$$

where $\delta_3(n)$ is the sequence of Theorem 3 for $r = 3$. From Theorem 3 and the definition of $N_m$ it follows that $\| T_m \|_\infty \to 0$ as $m \to \infty$. Hence, by Lemma 1, $\| F\circ T_m \|_q$ is bounded as $m \to \infty$.

Since $F$ is an odd function

$$(F\circ T_m)(e^{i\theta}) = F(z_m)\sum_{1\leqq n\leqq N_m,\delta_3(n)=2} e^{in\theta} - F\left(\frac{z_m}{2}\right)\sum_{1\leqq n\leqq N_m,\delta_3(n)=-1} e^{in\theta} . \tag{22}$$

Thus

$$\begin{aligned} |F\circ T_m(e^{i\theta})| \geqq \frac{2}{3}\left|\frac{1}{2}F(z_m) - F\left(\frac{z_m}{2}\right)\right| \left|\sum_{n=1}^{N_m} e^{in\theta}\right| \\ - \frac{1}{3}\left|F(z_m) + F\left(\frac{z_m}{2}\right)\right| \left|\sum_{n=1}^{N_m}\delta_3(n)e^{in\theta}\right| . \end{aligned} \tag{23}$$

Now if $F$ maps $\mathscr{C}$ to $L^q$, $q > 2$, then, *a fortiori*, $F$ maps $\mathscr{C}$ to $L^2$. Thus the truth of Theorem 1 for $q = 2$ implies that $|F(z)/z|$ is bounded near the origin. Thus, since $\| \sum_{n=1}^{N_m} e^{in\theta} \|_q \geqq C_q N_m^{1/p}$, it follows that $|(1/2)F(z_m) - F(z_m/2)|\cdot N_m^{1/p}$ is bounded as $m \to \infty$. However this is a contradiction to (20).

LEMMA 3. *$F(z) = F_1(z) + F_2(z)$ where*

(a) *$F_1$ and $F_2$ map $\mathscr{C}$ into $L^q$.*
(b) *$|F_2(z)|\,|z|^{-2/p}$ is bounded near the origin.*
(c) *$F_1(z/2) = (1/2)F_1(z)$ for all $z$.*

REMARK. $F_2$ is the "small" part of $F$. Lemmas 5 and 6 show that because of (a) and (c)

$$F_1(z) = c_1 z + c_2 \bar{z} \ .$$

*Proof.* By Lemma 2 there are finite positive constants $B$ and $C$ such that for $|z| \leqq B$

$$\begin{aligned} \left|F\left(\frac{z}{2^k}\right) - \frac{1}{2^k}F(z)\right| &\leqq \sum_{j=0}^{k-1} \frac{1}{2^j}\left|F\left(\frac{z}{2^{k-j}}\right) - \frac{1}{2}F\left(\frac{z}{2^{k-j-1}}\right)\right| \\ &\leqq C\sum_{j=0}^{k-1}\frac{1}{2^j}\left|\frac{z}{2^{k-j-1}}\right|^{2/p} \\ &\leqq C'\frac{|z|^{2/p}}{2^k} \qquad (k = 1, 2, 3, \cdots) \ . \end{aligned} \tag{24}$$

Define

$$F_1(z) = \lim_{n\to\infty} 2^n F\left(\frac{z}{2^n}\right) . \tag{25}$$

This limit exists. For if $n > j$ and we apply (24) to $z/2^j$ with $k = n - j$ and multiply by $2^n$ then

$$\left|2^n F\left(\frac{z}{2^n}\right) - 2^j F\left(\frac{z}{2^j}\right)\right| \leqq 2^j C'\left|\frac{z}{2^j}\right|^{2/p} . \tag{26}$$

Since $p < 2$, the right side of (26) $\longrightarrow 0$ as $j$ and $n \longrightarrow \infty$.

It is clear from the definition of $F_1$ that (c) holds. $F_2(z) = F(z) - F_1(z)$ and (b) is a result of (24). $F_2$ maps $\mathscr{C}$ into $L^q$ because of (b). Thus $F_1$ does also. Note that $F_1$ is odd (since $F$ is).

LEMMA 4. *$F_1$ is continuous.*

*Proof.* It is sufficient to show it is continuous at 1. If not, there is a sequence $z_m \longrightarrow 1$ such that $F_1(z_m) \longrightarrow F_1(1)$. The $z_m$ can be chosen so that

$$|1 - z_m| < 2^{-m} . \tag{27}$$

Let $N_m = [2^{2m}\cdot m^{-2}]$ and define

$$T_m(e^{i\theta}) = 2^{-m}\sum_{n=1}^{N_m}\left\{\varepsilon_2(n) + \frac{(1 - z_m)}{2}\,(1 - \varepsilon_2(n))\right\}e^{in\theta} \tag{28}$$

where $\{\varepsilon_2(n)\}$ is the sequence of Theorem 2.

Theorem 2, (27) and the choice of $N_m$ imply that $\| T_m \|_\infty = 0(1/m)$ so that, by Lemma 1, $\| F \circ T_m \|_q$ is bounded as $m \to \infty$. But then since $F_1(z/2) = (1/2)F_1(z)$,

$$
\begin{aligned}
| F_1 \circ T_m(e^{i\theta}) | &= 2^{-m} \left| F_1(1) \sum_{1 \leqq n \leqq N_m, \varepsilon_2(n)=1} e^{in\theta} - F_1(z_m) \sum_{1 \leqq n \leqq N_m, \varepsilon_2(n)=-1} e^{in\theta} \right| \\
&\geqq 2^{-m-1} | F_1(1) - F_1(z_m) | \left| \sum_{n=1}^{N_m} e^{in\theta} \right| \\
&\quad - 2^{-m-1} | F_1(1) + F_1(z_m) | \left| \sum_{n=1}^{N_m} \varepsilon_2(n) e^{in\theta} \right| .
\end{aligned} \tag{29}
$$

As in Lemma 2 this implies that $| F_1(1) - F_1(z_m) | N_m^{1/p} \cdot 2^{-m}$ is bounded as $m \to \infty$, which is impossible unless $F_1(z_m) \to F_1(1)$. Hence $F_1$ is continuous.

LEMMA 5. *There are continuous functions $C_1$ and $C_2$ on $(0, \infty)$ such that*

$$F_1(xe^{i\theta}) = C_1(x)e^{i\theta} + C_2(x)e^{-i\theta} \qquad (0 < x < \infty) .$$

*Proof.* We will show that if $r$ is an integer $(r \neq 0, 1)$ and $z$ a complex number then

$$\sum_{j=1}^{r} F_1\left(z \exp \frac{2\pi i j}{r}\right) = 0 . \tag{30}$$

Now consider $F_1(xe^{i\theta}) = G_x(e^{i\theta})$ for a fixed $x$. $G_x$ is a continuous function of $\theta$ by Lemma 4. It follows from (30) that for each integer $r \neq 0, 1$.

$$\sum_{j=1}^{r} G_x\left(\exp i\left(\theta + \frac{2\pi j}{r}\right)\right) = 0 \qquad (0 \leqq \theta \leqq 2\pi) . \tag{31}$$

By considering the Fourier coefficients of $G_x$ it is easily seen that $G_x(e^{i\theta}) = C_1(x)e^{i\theta} + C_2(x)e^{-i\theta}$. $C_1$ and $C_2$ are continuous because of Lemma 4.

To prove (30) it is sufficient to assume that $z = 1$. It is also sufficient to assume $r$ is prime. For if $r = pq$ where $p$ is a prime then (30) can be written

$$\sum_{s=1}^{q} \sum_{j=1}^{p} F_1\left(z \exp \frac{2\pi i(jq + s)}{pq}\right) . \tag{32}$$

If (30) holds for primes then each summand of the outer sum of (32) is zero.

Let $N_m = [2^{2m} m^{-2}]$ and define

(33) $$T_m^t(e^{i\theta}) = 2^{-m} \sum_{n=1}^{N_m} \{\varepsilon_r(n)\}^t e^{in\theta} \qquad (t = 1, 2, \cdots, r-1),$$

where $\{\varepsilon_r(n)\}$ is the sequence of Theorem 2. $\| T_m^t \|_\infty = 0(1/m)$ so that if $\beta = \sum_{j=1}^r F_1(\exp 2\pi i j/r)$ and

(34) $$H_m(e^{i\theta}) = \sum_{t=1}^{r-1} \left\{ F_1 \circ T_m^t + \left\{ \frac{\beta}{r} - F_1(1) \right\} T_m^t \right\}$$

then, by Lemma 1, $\| H_m \|_q$ is bounded as $m \to \infty$. Now since $F_1(z/2) = (1/2)F(z)$

(35) $$| H_m(e^{i\theta}) | = 2^{-m} \left| \sum_{t=1}^{r-1} \left\{ \sum_{n=1}^{N_m} F_1\{(\varepsilon_r(n))^t\} e^{in\theta} + \left\{ \frac{\beta}{r} - F_1(1) \right\} \sum_{n=1}^{N_m} \{\varepsilon_r(n)\}^t e^{in\theta} \right\} \right| .$$

Suppose $\varepsilon_r(n) = 1$. The coefficient of $e^{in\theta}$ in (35) is then

(36) $$(r-1)F_1(1) + (r-1)\left\{ \frac{\beta}{r} - F_1(1) \right\} = \left(1 - \frac{1}{r}\right)\beta .$$

Suppose $\varepsilon_r(n) \neq 1$, so that $\varepsilon_r(n)$ is a primitive $r^{\text{th}}$ root of unity. Then $\sum_{t=1}^{r-1} F_1\{(\varepsilon_r(n))^t\} = \beta - F_1(1)$ and $\sum_{t=1}^{r-1} (\varepsilon_r(n))^t = -1$ so that the coefficient of $e^{in\theta}$ is

(37) $$\beta - F_1(1) - \left\{ \frac{\beta}{r} - F_1(1) \right\} = \left(1 - \frac{1}{r}\right)\beta .$$

Hence

(38) $$| H_m(e^{i\theta}) | = 2^{-m}\left(1 - \frac{1}{r}\right) |\beta| \left| \sum_{n=1}^{N_m} e^{in\theta} \right|$$

so that

(39) $$\| H_m \|_q \geqq \left(1 - \frac{1}{r}\right) \frac{|\beta|}{2^m} C_q N_m^{1/p} \qquad (m = 1, 2, \cdots) .$$

But this is impossible unless $\beta = 0$. That is

(40) $$\sum_{j=1}^{r} F_1\left(\exp \frac{2\pi i j}{r}\right) = 0$$

which was to be proved

LEMMA 6. $C_j(x) = xC_j(1) \qquad (0 < x < \infty;\ j = 1, 2).$

*Proof.* Fix $x$ and $\varphi$. Let $r$ be a prime, $N_m = [2^{2m} m^{-2}]$, and define

$$T_m(e^{i\theta}) = \frac{xe^{i\varphi}}{(r-1)2^m} \sum_{n=1}^{N_m} \delta_r(n) e^{in\theta} \tag{41}$$

where $\{\delta_r(n)\}$ is the sequence of Theorem 3.

Since $F_1$ is odd and $F_1(z) = 2F_1(z/2)$ we can write

$$\begin{aligned} F_1 \circ T_m(e^{i\theta}) &= \frac{1}{2^m r}\left\{F_1(xe^{i\varphi}) - (r-1)F_1\left(\frac{xe^{i\varphi}}{r-1}\right)\right\} \sum_{n=1}^{N_m} e^{in\theta} \\ &+ \frac{1}{2^m r}\left\{F_1(xe^{i\varphi}) + F_1\left(\frac{xe^{i\varphi}}{r-1}\right)\right\} \sum_{n=1}^{N_m} \delta_r(n) e^{in\theta} . \end{aligned} \tag{42}$$

As in the proofs of Lemma 2 and 4, $\| F_1 \circ T_m \|_q$ and $2^{-m} \| \sum \delta_r(n) e^{in\theta} \|_q$ are bounded. Hence $2^{-m} N_m^{1/p} | F_1(xe^{i\varphi}) - (r-1) F_1(xe^{i\varphi}/r-1) |$ is bounded. But $2^{-m} N_m^{1/p}$ is unbounded so that

$$F_1(xe^{i\varphi}) - (r-1)F_1\left(\frac{xe^{i\varphi}}{r-1}\right) = 0 \qquad (0 < x < \infty;\ 0 \leqq \varphi \leqq 2\pi) . \tag{43}$$

By Lemma 5, (43) can be written

$$\begin{aligned} &\left\{C_1(x) - (r-1)C_1\left(\frac{x}{r-1}\right)\right\} e^{i\varphi} \\ &\qquad + \left\{C_2(x) - (r-1)C_2\left(\frac{x}{r-1}\right)\right\} e^{-i\varphi} = 0 . \end{aligned} \tag{44}$$

Clearly this possible only if

$$C_j(x) = (r-1)C_j\left(\frac{x}{r-1}\right) \qquad (0 < x < \infty;\ j = 1, 2) . \tag{45}$$

Thus, if $r$ and $q$ are primes and $n$ is an integer,

$$C_j\left(\left(\frac{r-1}{q-1}\right)^n\right) = \left(\frac{r-1}{q-1}\right)^n C_j(1) \qquad (j = 1, 2) . \tag{46}$$

Now $\{(r - 1/q - 1)^n: r, q, \text{ primes}; n \text{ an integer}\}$ is dense in the positive real numbers. This is true since given $\varepsilon > 0$ there are infinitely many pairs of consecutive primes $q_n$, $q_{n+1}$ such that $q_{n+1} < (1 + \varepsilon) q_n$.

Since $C_j$ is continuous (46) then implies $C_j(x) = xC_j(1)$ for all $x$.

The proof of Theorem 1 follows from Lemmas 3, 5, and 6.

**4. The general case.** We remark here that Theorem 1 holds if we consider any compact Abelian group $G$. If $\Gamma$, the dual group of $G$, has elements of arbitrarily large order then it is possible to construct polynomials as in § 2 and the proof proceeds as in § 3. When, $\Gamma$, and hence $G$, has an exponent the construction of the polynomials is slightly different (it depends on the structure of $\Gamma$) but the remainder of the proof is similar.

## REFERENCES

1. H. Helson, J. P. Kahane, Y. Katznelson and W. Rudin, *The functions which operate on Fourier transforms*, Acta Math. **102** (1959), 139-157.
2. Walter Rudin, *Some theorems on Fourier coefficients*, Proc. Amer. Math. Soc. **10** (1959), 855-859.
3. H. S. Shapiro, *Extremal problems for polynomials and power series*, Thesis for S. M. Degree, Massachusetts Institute of Technology, 1951.

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