AN APPLICATION OF THE BOTT SUSPENSION MAP TO THE TOPOLOGY OF $EIV$

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Consider the compact simply connected symmetric pair $(E_6, F_4)$. By a slight abuse of the notation of E. Cartan, the corresponding symmetric space is denoted by $EIV$. Let $W$ be the Cayley projective plane. The Bott suspension map $E: \Sigma(W) \to EIV$ (where $\Sigma$ denotes the nonreduced suspension) is defined by means of the set of minimal geodesic segments joining the two nontrivial points of the "center" of $EIV$. In this paper a map $q: S^{25} \to \Sigma(W)$ is constructed and $E$ is extended to a homeomorphism of $\Sigma(W) \cup qe_{26}$ onto $EIV$. Among other things, this gives canonical isomorphisms $\pi_j(EIV) \approx \pi_j(\Sigma(W))$, $0 \leq j \leq 24$. These groups are explicitly determined.

**Statement of results.** The maps $E$ and $q$ will be constructed in §2 and the following theorems will be proven.

**Theorem 1.1.** The map $E$ extends to a homeomorphism $E^*: \Sigma(W) \cup e_{26} \to EIV$. 

**Corollary 1.2.** $E_*: \pi_j(\Sigma(W)) \to \pi_j(EIV)$ is a bijection for $j \leq 24$, and a surjection for $j = 25$.

**Theorem 1.3.** $\text{Im}(q_*^*) = \text{Ker}(E_*)$ in homotopy in dimensions $\leq 32$, and

$$0 \longrightarrow \pi_{25}(S^{25}) \xrightarrow{q_*^*} \pi_{25}(\Sigma(W)) \xrightarrow{E_*} \pi_{25}(EIV) \longrightarrow 0$$

is exact and canonically split, with $\pi_{25}(EIV)$ a finite 2-primary group.

Having by (1.2) reduced the problem of computing $\pi_j(EIV)$, $j \leq 24$, to a somewhat easier problem, we devote the remaining sections of the paper to deducing the consequences listed below. We do not list $\pi_j(EIV)$ for $j \leq 15$, since isomorphisms $\pi_j(EIV) \approx \pi_j(S^8)$, together with the explicit values of these latter groups, are already well known for that range.

(1.4) $\pi_{10}(EIV) = 0$

(1.5) $\pi_{17}(EIV) = \mathbb{Z} + (\mathbb{Z}_2)^2$

(1.6) $\pi_{18}(EIV) = (\mathbb{Z}_2)^3$
\[ \pi_{19}(EIV) = \mathbb{Z}_g \]
\[ \pi_{20}(EIV) = \mathbb{Z}_{1512} + \mathbb{Z}_4 \]
\[ \pi_{21}(EIV) = 0 \]
\[ \pi_{22}(EIV) = \mathbb{Z}_3 \]
\[ \pi_{23}(EIV) = \mathbb{Z}_4 \]
\[ \pi_{24}(EIV) = \mathbb{Z}_{215} + (2\text{-primary group}) \]

\textbf{Remarks.} (1.4) was communicated to the author some time ago by Shôrô Araki who proved it by a somewhat different method (unpublished). The present paper actually resulted from attempts to verify this formula. (1.9) was proven in a different way in [8] and (1.5) and (1.10) remove the ambiguities from the partial determinations of these groups in that same paper. In (1.1) one gets a fully explicit cellular structure by recalling that

\[ \Sigma(W) = e_0 \bigcup_p e_2 \bigcup_p e_{17} \]

where \( p : S^8 \to e_0 \) is the only map possible and \( g : S^{16} \to e_0 \bigcup_p e_2 \approx S^8 \) is the suspension of the standard Hopf map \( f : S^{15} \to S^8 \).

In the course of this paper we will repeatedly (and without further reference) make use of the values of \( \pi_i(S^8) \) as found in [14].

2. The maps \( E \) and \( q \). Let \( e_6 \) be the Lie algebra of \( E_6 \) and \( \beta : e_6 \to e_6 \) the involution corresponding to \( EIV \). Let \( m \subset e_6 \) be the \(-1\) eigenspace of \( \beta \). Let \( t \subset m \) be a maximal abelian subalgebra (a two dimensional real vector space) and consider the root system of \( EIV \) relative to \( t \). This is a proper root system (in the sense of [2]) isomorphic to the root system of \( A_2 \), each root having multiplicity 8. Let \( \Delta \) be a fundamental simplex in \( t \).

The symmetric space \( EIV \) is canonically imbedded in \( E_6 \) as \( \exp(m) \). The adjoint action of \( F_4 \) on \( m \) passes over, under \( \exp \), to the adjoint action of \( F_4 \) on \( EIV \subset E_6 \).

\( \exp | \Delta \) is one-to-one (since \( EIV \) is simply connected) and \( \exp (\Delta) \) intersects each \( F_4 \)-orbit on \( EIV \) in one and only one point.

Let \( B \) denote the union in \( m \) of the \( F_4 \)-orbits of points of \( \Delta \). By the above remarks \( \exp : B \to EIV \) is onto. Let \( s(t), 0 \leq t \leq 1, \) describe the edge of \( \Delta \) opposite the vertex 0. Then \( x_0 = \exp(s(0)) \) and \( x_1 = \exp(s(1)) \) coincide with the nontrivial elements of the center \( Z_3 \) of \( E_6 \), while \( \exp s \) is a minimal geodesic joining \( x_0 \) and \( x_1 \). The following lemma and its corollary are completely straightforward.
LEMMA 2.1. B is homeomorphic to the standard closed cell $e_{26}$ and the boundary $\partial B \approx S^5$ is the union of the $F_4$-orbits of $s(t)$, $0 \leq t \leq 1$.

COROLLARY 2.2. Under the homeomorphism $B \approx e_{26}$, $\exp | B$ defines a surjection $e_{26} \to EIV$ which is a homeomorphism on the interior of $e_{26}$.

LEMMA 2.3. $\exp (\partial B) \approx \Sigma (W)$.

Proof. From [1] one knows that the centralizer in $F_4$ of $\exp(s(t))$, $0 < t < 1$, is the symmetric subgroup Spin (9) $\subset F_4$, while for $t = 0, 1$ the centralizer is clearly all of $F_4$. Since $W = F_4/\text{Spin}(9)$, the lemma follows.

COROLLARY 2.4. The inclusion $\exp (\partial B) \subset EIV$ is a Bott suspension $E: \Sigma (W) \to EIV$.

Proof. Let $\Omega = \Omega(EIV; x_0, x_1)$, the space of paths on $EIV$ joining $x_0$ and $x_1$. From the proof of (2.3) it is clear that the subspace of shortest geodesics in $\Omega$ is homeomorphic to $W$. The adjoint of the inclusion map $W \subset \Omega$ is precisely the Bott suspension [4], is one-to-one, and its image is $\exp (\partial B)$.

Of course, we define $q$ as $\exp | \partial B$ and immediately obtain (1.1) and (1.2).

REMARK. The loop space $\Omega$ of $EIV$ is homology commutative, hence the theory of [5] can be applied to the Pontrjagin ring $H_*(\Omega)$. $W \subset \Omega$ proves to be a generating variety contributing generators $x_8, x_{16} \in H_*(\Omega) \approx Z[x_8, x_{16}], \dim (x_i) = i$. The diagram

$$
\begin{array}{ccc}
H_i(\Omega) & \xrightarrow{\sigma} & H_{i+1}(EIV) \\
\beta_* \downarrow & & \beta_* \\
H_i(\Omega) & \xrightarrow{\sigma} & H_{i+1}(EIV)
\end{array}
$$

is commutative, where $\sigma$ is homology suspension and the homomorphisms $\beta_*$ are induced by the involution $\beta$ of $E_6$. $\beta_*$ is $-1$ on $H_9(EIV) \approx Z$ [9] and $\sigma(x_8)$ generates this group. Thus $\beta_*(x_8) = -x_8$ and $\beta_*(x_{16}) = x_{16}$. $\beta_*$ is $-1$ on $H_{17}(EIV) \approx Z$ [9], so $\sigma(x_{16}) = 0$. $\sigma H_9(\Omega) = H_{17}(EIV)$, hence $\sigma(x_{16})$ generates that group. From the known homology of $EIV$ [9], it follows that $E_*: H_i(\Sigma (W)) \to H_i(EIV)$ is bijective, $i \leq 25$. (1.2) then follows by the Whitehead theorem. One can also deduce a map $q$ (defined
up to homotopy) and a weakened version of (1.1) in which $E'$ is only a homotopy equivalence. In point of fact, it was this somewhat roundabout line of thought that suggested (1.1).

We now take up the proof of (1.3). Consider the homomorphisms

$$q_\ast : \pi_j(S^{25}) \longrightarrow \pi_j(\Sigma(W))$$

and

$$\partial : \pi_{j+1}(EIV, \Sigma(W)) \longrightarrow \pi_j(\Sigma(W)).$$

**Lemma 2.5.** For $j \leq 32$ there is a natural bijection $h : \pi_j(S^{25}) \rightarrow \pi_{j+1}(EIV, \Sigma(W))$ such that $\partial \circ h = q_\ast$. 

*Proof.* $q$ defines a map $\bar{q} : (e_{26}, S^{25}) \rightarrow (EIV, \Sigma(W))$ and by [11, Chapter XI, Ex. B-3] (cf. the references given there to [10] and [16]), $\bar{q}_\ast$ is bijective in dimensions $\leq 33$. Let

$$\gamma : \pi_j(S^{25}) \longrightarrow \pi_{j+1}(e_{26}, S^{25}), \quad j \leq 32,$$

be the inverse of the boundary map. Then $h = \bar{q}_\ast \circ \gamma$ is as desired.

The first assertion of (1.3) follows immediately from (2.5). For the exactness of

$$0 \longrightarrow \pi_{25}(S^{25}) \xrightarrow{q_\ast} \pi_{25}(\Sigma(W)) \xrightarrow{E_\ast} \pi_{26}(EIV) \longrightarrow 0$$

we need only the following.

**Lemma 2.6.** $\partial : \pi_{26}(EIV, \Sigma(W)) \rightarrow \pi_{25}(\Sigma(W))$ is one-to-one.

*Proof.* From [8], $\pi_j(EIV, S^o) \approx \pi_{j-1}(S^o), \quad j \leq 31$. Thus, since $\pi_{26}(S^o)$ and $\pi_{25}(S^o)$ are finite groups, so is $\pi_{26}(EIV)$. Since $\pi_{26}(EIV, \Sigma(W)) \approx \mathbb{Z}$ by (2.5), the map $\pi_{26}(EIV) \rightarrow \pi_{26}(EIV, \Sigma(W))$ is zero. The lemma follows by exactness.

The fact that $\pi_{26}(EIV)$ is a finite 2-primary group also follows from the results in [8], so we are left with the task of proving that the above sequence splits. (If it splits at all, the splitting is canonical, since $\pi_{26}(EIV)$ will have to be identified with the torsion subgroup of $\pi_{26}(\Sigma(W))$.)

The imbedding $S^o \rightarrow EIV$ studied in [8] defines a generator of $\pi_9(EIV) \approx \mathbb{Z}$, hence $E$ can be assumed to define a map

$$i : (\Sigma(W), S^o) \rightarrow (EIV, S^o), \quad i \mid S^o = 1,$$

where $S^o \subset \Sigma(W)$ is given by our standard cellular decomposition of
\[ \Sigma(W) \]. Using \( \pi_{25}(EIV, S^9) \approx \pi_{24}(S^{16}) \approx (\mathbb{Z}_2)^{2} \) \[8\], we obtain a commutative diagram

\[
\begin{array}{ccccccc}
0 & \rightarrow & \pi_{25}(S^9) & \xrightarrow{r} & \pi_{25}(\Sigma(W)) & \xrightarrow{j} & \pi_{25}(\Sigma(W), S^9) \\
\downarrow & & \downarrow \pi_5^{EIV} & & \downarrow E_* & & \downarrow \iota_* \\
0 & \rightarrow & \pi_{25}(S^9) & \xrightarrow{r'} & \pi_{25}(EIV) & \xrightarrow{j'} & (\mathbb{Z}_2)^{2} \\
\end{array}
\]

where the second column and both rows are exact. Extending this diagram two more terms to the right, one easily establishes the surjective half of the five lemma.

**Lemma 2.7.** \( i_* : \pi_{25}(\Sigma(W), S^9) \rightarrow (\mathbb{Z}_2)^{2} \) is surjective and \( \text{Ker} (i_*) \subseteq \text{Im} (j) \).

**Lemma 2.8.** \( j^{-1}(\text{Ker} (i_*)) = \text{Ker} (E_*) \oplus \text{Im} (r) \).

**Proof.** \( j^{-1}(\text{Ker} (i_*)) = \text{Ker} (i_* \circ j) = \text{Ker} (J' \circ E_*) \). Now \( \text{Ker} (E_*) \) is infinite cyclic while \( \text{Im} (r) \) is a torsion group. Thus \( \text{Ker} (E_*) \cap \text{Im} (r) = 0 \). Furthermore, if \( J' (E_*(a)) = 0 \), then \( E_*(a) \in \text{Im} (r') \) and \( a = b + c \), \( b \in \text{Ker} (E_*), c \in \text{Im} (r) \).

**Corollary 2.9.** \( \text{Ker} (i_*) \) is the infinite cyclic group \( j(\text{Ker} (E_*)) \).

**Lemma 2.10.** \( (\mathbb{Z}_2)^{2} \subset \pi_{25}(\Sigma(W), S^9) \).

**Proof.** In \( \Sigma(W) = S^9 \cup_{g} e_{17} \), the attaching map \( g \) defines the characteristic map

\[ g : (e_{17}, S^{16}) \longrightarrow (\Sigma(W), S^9) \].

Since suspension \( \Sigma : \pi_{24}(S^{16}) \rightarrow \pi_{25}(S^{17}) \) is one-to-one, it follows [11, p. 333] that

\[ g_* : \pi_{25}(e_{17}, S^{16}) \longrightarrow \pi_{25}(\Sigma(W), S^9) \]

is one-to-one. But \( \pi_{25}(e_{17}, S^{16}) \approx \pi_{24}(S^{16}) \approx (\mathbb{Z}_2)^{2} \).
Proposition 2.11. Ker $(E_\ast)$ is a direct summand of $\pi_{25}(\Sigma(W))$.

Proof. Write $\text{Ker} (E_\ast) \subset Z^1$, where $Z^1$ stands for a maximal infinite cyclic subgroup of $\pi_{25}(\Sigma(W))$. Im $(r) \cap Z^1 = 0$, so $j | Z^1$ is one-to-one. Thus $j(Z^1) \cap (Z^1)^2 = 0$, and, by (2.9), $\text{Im} (i_\ast) \supset (Z^1) \oplus j(Z^1)/j(\text{Ker} (E_\ast))$. Thus $j(\text{Ker} (E_\ast)) = j(Z^1)$, so $\text{Ker} (E_\ast) = Z^1$.

This completes the proof of (1.3). It also proves

\begin{equation}
\pi_{25}(\Sigma(W), S^9) \approx Z \oplus (Z^1)^2.
\end{equation}

3. The homotopy sequence of $(\Sigma(W), S^9)$. For the computation of $\pi_j(EIV), j \leq 24$, we are reduced to computing $\pi_j(\Sigma(W))$. We begin the attack on this latter problem by investigating the boundary operator $\delta$ in the homotopy sequence of $(\Sigma(W), S^9)$.

Recall that $\Sigma(W) = S^9 \bigcup_j e_{17}$ where $g$ is the suspension of the standard Hopf map $f: S^{15} \to S^9$. By [11, p. 334] one shows that $\bar{g}_\ast: \pi_j(e_{17}, S^9) \to \pi_j(\Sigma(W), S^9)$ is bijective for $j \leq 24$, $\bar{g}$ the characteristic map determined by $g$.

Let

\begin{equation}
F: \pi_j(\Sigma(W), S^9) \to \pi_{j-1}(S^{16}), j \leq 24,
\end{equation}

be the natural bijection obtained by composing $(\bar{g}_\ast)^{-1}$ with the natural isomorphism $\pi_j(e_{17}, S^9) \approx \pi_{j-1}(S^{16})$.

Lemma 3.1. $\delta: \pi_j(\Sigma(W), S^9) \to \pi_{j-1}(S^9)$ is given by $g_\ast \circ F$ if $j \leq 24$.

Next consider the commutative diagram ($n \leq 29$)

\begin{equation}
\begin{array}{ccc}
\pi_n(S^{16}) & \xrightarrow{g_\ast} & \pi_n(S^9) \\
\downarrow \cong & & \downarrow \Sigma \\
\pi_{n-1}(S^{15}) & \xrightarrow{f_\ast} & \pi_{n-1}(S^9)
\end{array}
\end{equation}

where the vertical maps are suspensions.

Lemma 3.2. $\text{Ker} \{\delta: \pi_j(\Sigma(W), S^9) \to \pi_{j-1}(S^9)\} \approx \text{Im} (f_\ast) \cap \text{Ker} (\Sigma)$ in $\pi_{j-1}(S^9), j \leq 24$.

Proof. By (3.1) we are reduced to finding $\text{Ker} (g_\ast)$. In the above diagram $f_\ast$ is injective (because it has Hopf invariant one [7, exposé 6, Proposition 5]). This immediately yields the assertion.
We study $\text{Im}(f^*) \cap \text{Ker}(\Sigma)$ by means of the exact suspension sequence [7, expose 6]:

$$\cdots \xrightarrow{\Sigma} \pi_{n+1}(S^9) \xrightarrow{H} \pi_n(\Omega(S^9), S^8) \xrightarrow{\partial} \pi_{n-1}(S^8) \xrightarrow{\Sigma} \pi_n(S^9) \xrightarrow{H} \cdots.$$ 

This gives $\text{Ker}(\Sigma) = \text{Im}(\partial)$. In order to study $\Delta$ we will consider the topology of $\Omega(S^9)$ in lower dimensions.

Let $i_s$ generate $\pi_8(S^8)$ and consider the Whitehead product $[i_s, i_s] \in \pi_{16}(S^9)$. Let $h: S^{15} \to S^8$ be in this homotopy class and set $X = S^8 \cup_i \epsilon_{16}$. It is known [7, exposé 5] that $\Omega(S^9)$ has the homotopy type of a CW complex obtained by attaching to $X$ cells of dimensions $\geq 24$. Thus the inclusion $(X, S^8) \subset (\Omega(S^9), S^9)$ is a homotopy equivalence in dimensions $\leq 22$, and in this range we can consider $\Delta$ as defined on $\pi_n(X, S^9)$. $h$ determines a characteristic map

$$\tilde{h}: (e_{16}, S^{15}) \to (X, S^8).$$

By [11, p. 334] we obtain

**Lemma 3.3.** $\tilde{h}_*: \pi_n(e_{16}, S^{15}) \to \pi_n(X, S^8)$ is bijective, $n \leq 22$.

**Corollary 3.4.** $\Delta = h_* \circ \partial \circ \tilde{h}_*^{-1}$ in $\dim \leq 22$, where

$$\partial: \pi_n(e_{16}, S^{15}) \approx \pi_{n-1}(S^8).$$

**Corollary 3.5.** $\text{Ker} \{\partial: \pi_j(\Sigma(W), S^9) \to \pi_{j-1}(S^9)\} \approx \text{Im} (f_*^*) \cap \text{Im} (h_*^*)$ in $\pi_{j-2}(S^8)$, $j \leq 23$.

4. $\pi_j(\Sigma(W))$, $j \leq 18$. For the simple proof of the following lemma I am indebted to S. Araki.

**Lemma 4.1.** Let $g$ be the suspension of the standard Hopf map $f: S^{15} \to S^8$. The class $[g]$ generates $\pi_{16}(S^9) \approx \mathbb{Z}_{\geq 40}$.

**Proof.** Let $\sigma \in \pi_7(SO(8))$ be the element defined by the natural action on $\mathbb{R}^8$ of the unit sphere of Cayley numbers. Let $\sigma' \in \pi_7(SO(9))$ be the image of $\sigma$ under the standard inclusion $SO(8) \subset SO(9)$. Then $\sigma'$ generates $\pi_7(SO(9)) \approx \mathbb{Z}$ [15]. The $J$-homomorphism $J: \pi_7(SO(9)) \to \pi_{16}(S^9) \approx \mathbb{Z}_{\geq 40}$ is surjective [12] and $J(\sigma') = [g]$.

**Corollary 4.2.** $\pi_{16}(\Sigma(W)) = 0$

This establishes (1.4). For (1.5) and (1.6) we will need to make
For $h$ and $f$ as in § 3, the class $[\zeta] = [h] - 2[f]$ is a torsion element in $\pi_{10}(S^8)$, hence $\zeta: S^{15} \to S^8$ is the suspension of some map [7, exposé 6].

**Lemma 4.3.** Let $\beta \in \pi_{10}(S^{13}) \approx \mathbb{Z}_2$ be the generator. Then $h_*(\beta)$ is a suspension class.

**Proof.** Since $\beta$ is a suspension class, $h_*(\beta) = 2f_*(\beta) + \zeta_*(\beta) = \zeta_*(\beta)$ and this is a suspension class.

**Corollary 4.4.** $\text{Ker}\{\partial: \pi_{18}(\Sigma(W), S^9) \to \pi_{17}(S^9)\} = 0$.

**Proof.** By (4.3), $\text{Im}(h_*)$ in $\pi_{10}(S^8)$ is contained in the image of the suspension. Therefore $\text{Im}(f_*) \cap \text{Im}(h_*) = 0$ in $\pi_{10}(S^9)$. The conclusion follows by (3.5).

**Corollary 4.5.** $\pi_{17}(\Sigma(W)) \approx \mathbb{Z} + (\mathbb{Z}_2)^5$.

**Proof.** $\pi_{16}(\Sigma(W), S^9) \approx \pi_{17}(S^9) \approx \mathbb{Z}_2$ by (3.0), and $\pi_{15}(S^9) \approx (\mathbb{Z}_2)^3$. From the exact sequence of $(\Sigma(W), S^9)$ and (4.4) one obtains

$$0 \longrightarrow (\mathbb{Z}_2)^5 \longrightarrow \pi_{17}(\Sigma(W)) \longrightarrow \pi_{16}(\Sigma(W), S^9) \longrightarrow \pi_{15}(S^9).$$

Since $\pi_{16}(\Sigma(W), S^9) \approx \mathbb{Z}$ and $\pi_{15}(S^9)$ is finite, this gives an exact sequence

$$0 \longrightarrow (\mathbb{Z}_2)^5 \longrightarrow \pi_{17}(\Sigma(W)) \longrightarrow \mathbb{Z} \longrightarrow 0.$$

This completes the proof of (1.5).

Proceeding analogously as above, let $\beta \in \pi_{17}(S^{13}) \approx \mathbb{Z}_2$ be the generator and show that $h_*(\beta) \in \text{Im}(\Sigma)$. Then $\partial: \pi_{18}(\Sigma(W), S^9) \longrightarrow \pi_{18}(S^9)$ is one-to-one. Since, by (3.0), $\pi_{19}(\Sigma(W), S^9) \approx \mathbb{Z}_2$, and $\pi_{18}(S^9) \approx (\mathbb{Z}_2)^4$, one obtains

$$0 \longrightarrow (\mathbb{Z}_2)^5 \longrightarrow \pi_{18}(\Sigma(W)) \longrightarrow \pi_{18}(\Sigma(W), S^9) \overset{\partial}{\longrightarrow} \cdots$$

where $\partial$ is one-to-one by (4.4). This yields the following proposition and so proves (1.6).

**Proposition 4.6.** $\pi_{18}(\Sigma(W)) \approx (\mathbb{Z}_2)^5$.

5. Partial determinations of $\pi_j(\Sigma(W))$, $j = 19, 20$. The 3-primary components of these two groups present a special problem. The
ambiguities left by the partial determinations in this section will be removed in §7 by cohomological methods.

**Lemma 5.1.** \( \Delta : \pi_{17}(X, S^8) \to \pi_{19}(S^8) \) is one-to-one.

**Proof.** Consider the exact sequence

\[ \pi_{17}(X, S^8) \xrightarrow{\Delta} \pi_{19}(S^8) \xrightarrow{\Sigma} \pi_{17}(S^8) \xrightarrow{H} \cdots. \]

\( H \) is zero since \( \pi_{17}(S^8) \) is finite. Thus \( \Sigma \) is onto. Also \( \pi_{19}(S^8) \approx (\mathbb{Z}/4^4) \), \( \pi_{17}(S^8) \approx (\mathbb{Z}/3^3) \), so, by (3.3), \( \text{Im}(\Delta) \approx \mathbb{Z}_2 \approx \pi_{17}(X, S^8) \). It follows that \( \Delta \) is one-to-one.

**Corollary 5.2.** \( \Delta : \pi_{19}(X, S^8) \to \pi_{17}(S^8) \) is one-to-one.

**Proof.** By (5.1) the sequence

\[ \pi_{18}(X, S^8) \xrightarrow{\Delta} \pi_{19}(S^8) \xrightarrow{\Sigma} \pi_{18}(S^8) \xrightarrow{0} \]

is exact. Since \( \pi_{18}(S^8) \approx (\mathbb{Z}/2^2) \), \( \pi_{16}(S^8) \approx (\mathbb{Z}/2^3) \), we obtain \( \text{Im}(\Delta) = \text{Ker}(\Sigma) \approx \mathbb{Z}_2 \approx \pi_{18}(X, S^8) \).

**Corollary 5.3.** \( \Delta : \pi_{19}(X, S^8) \to \pi_{19}(S^8) \) is one-to-one.

**Proof.** By (5.2)

\[ \pi_{19}(X, S^8) \xrightarrow{\Delta} \pi_{19}(S^8) \xrightarrow{\Sigma} \pi_{19}(S^8) \xrightarrow{0} \]

is exact.

\( \pi_{18}(S^8) \approx (\mathbb{Z}/4^2) + \mathbb{Z}_2, \pi_{19}(S^8) \approx \mathbb{Z}_2^4 + \mathbb{Z}_2, \) and \( \pi_{19}(X, S^8) \approx \pi_{18}(S^1^5) \approx \mathbb{Z}_4 \).

The assertion follows.

By (5.3) and (3.4), \( h_* : \pi_{18}(S^1^5) \to \pi_{18}(S^8) \) is one-to-one. Let \( \beta \) generate \( \pi_{18}(S^1^5) \approx \mathbb{Z}_4 \). Then \( \beta \) is a suspension class and

\[ h_*(\beta) = 2f_*(\beta) + \zeta_*(\beta) \]

is of order 24. Since \( f_* \) is known to be one-to-one in all dimensions, \( f_*(\beta) \) is also of order 24. It follows that \( \zeta_*(\beta) \) is of order 24 or 8. This ambiguity affects the rest of this section.

**Lemma 5.4.** \( \partial : \pi_{20}(\Sigma W), S^5) \to \pi_{19}(S^8) \) has kernel 0 or \( \mathbb{Z}_3 \).

**Proof.** If \( \zeta_*(\beta) \) is order 24, then \( \text{Im}(f_*) \cap \text{Im}(h_*) \) is 0 in \( \pi_{18}(S^8) \). If \( \zeta_*(\beta) \) is of order 8, then \( \text{Im}(f_*) \cap \text{Im}(h_*) \approx \mathbb{Z}_3 \) in \( \pi_{19}(S^8) \). The lemma follows by (3.5).
Proposition 5.5. \( \pi_{19}(\Sigma(W)) \cong \mathbb{Z}_2 \) or \( \mathbb{Z}_6 \).

Proof. Consider the exact sequence

\[
0 \longrightarrow \text{Ker} \ (\partial) \longrightarrow \pi_{20}(\Sigma(W), S^0) \xrightarrow{\partial} \pi_{19}(S^0) \longrightarrow \pi_{19}(\Sigma(W)) \longrightarrow 0
\]

where exactness holds on the right by the proof of (4.6).

\[
\pi_{20}(\Sigma(W), S^0) \cong \pi_{19}(S^{16}) \cong \mathbb{Z}_4 \quad \text{and} \quad \pi_{19}(S^0) \cong \mathbb{Z}_4 + \mathbb{Z}_2 .
\]

The proposition follows by (5.4).

Proposition 5.6. There is an exact sequence

\[
0 \longrightarrow \mathbb{Z}_{504} + \mathbb{Z}_2 \longrightarrow \pi_{20}(\Sigma(W)) \longrightarrow \pi_{19}(\Sigma(W)) \otimes \mathbb{Z}_2 \longrightarrow 0 .
\]

Proof. By (5.4) and (5.5) the kernel of \( \partial: \pi_{20}(\Sigma(W), S^0) \to \pi_{19}(S^0) \) is \( \pi_{19}(\Sigma(W)) \otimes \mathbb{Z}_2 \). This, together with \( \pi_{21}(\Sigma(W), S^0) \cong \pi_{20}(S^{16}) \cong 0 \) and \( \pi_{20}(S^0) \cong \mathbb{Z}_{504} + \mathbb{Z}_2 \), yields the proposition.

6. \( \pi_j(\Sigma(W)) \), \( 21 \leq j \leq 23 \). One has \( \pi_{21}(S^0) \approx 0 \) and \( \pi_{21}(\Sigma(W), S^0) \approx \pi_{20}(S^{16}) \approx 0 \), so the exact homotopy sequence of the pair yields the following proposition, completing the proof of (1.9).

Proposition 6.1. \( \pi_{21}(\Sigma(W)) \approx 0 \).

Now let \( \beta \) generate \( \pi_{21}(S^{15}) \cong \mathbb{Z}_2 \). As usual, \( h_*(\beta) = \zeta_*(\beta) \) so that \( \text{Im} (f_*) \cap \text{Im} (h_*) \) is \( 0 \) in \( \pi_{21}(S^0) \). Thus \( \delta: \pi_{23}(\Sigma(W), S^0) \to \pi_{22}(S^0) \) is one-to-one.

Proposition 6.2. \( \pi_{22}(\Sigma(W)) \approx \mathbb{Z}_2 \).

Proof. \( \pi_{23}(\Sigma(W), S^0) \cong \pi_{22}(S^{16}) \cong \mathbb{Z}_3 \), \( \pi_{22}(S^0) \cong \mathbb{Z}_6 \), and

\[
\pi_{22}(\Sigma(W), S^0) \cong \pi_{21}(S^{16}) \approx 0 .
\]

By the above remarks we obtain an exact sequence

\[
0 \longrightarrow \mathbb{Z}_2 \longrightarrow \pi_{22}(\Sigma(W)) \longrightarrow 0 .
\]

This also establishes (1.10). In order to prove (1.11) a slight change in approach is needed. The difficulty is that we are now out of the range of validity of (3.5).

There is an exact sequence

\[
\pi_{24}(\Sigma(W), S^0) \xrightarrow{\partial} \pi_{23}(S^0) \longrightarrow \pi_{22}(\Sigma(W)) \longrightarrow 0 .
\]
where exactness on the right follows from the fact that \( \partial \) is one-to-one on \( \pi_{23}(\Sigma(W), S^9) \). Substituting the known values of the first two groups (note that we are still in the range of validity for (3.0)) we obtain

\[
(6.3a) \quad Z_5 + Z_3 + Z_1 \xrightarrow{\partial} Z_{16} \xrightarrow{\partial} Z_{16} + Z_4 \longrightarrow \pi_{23}(\Sigma(W)) \longrightarrow 0 .
\]

Our problem will be to compute Ker(\( \partial \)) in (6.3a).

**Lemma 6.4.** Let \( \Delta: \pi_{23}(X, S^8) \rightarrow \pi_{21}(S^8) \) is one-to-one.

**Proof.** By (3.3), \( \pi_{23}(X, S^8) \approx \pi_{23}(S^15) \approx Z_6 \) and \( \pi_{21}(S^8) \approx Z_6 + Z_2 \), \( \pi_{23}(S^9) \approx Z_6 \). The suspension sequence of § 3 then yields

\[
Z_2 \xrightarrow{\Delta} Z_6 + Z_2 \xrightarrow{\Sigma} Z_4
\]

which necessitates \( \Delta \neq 0 \).

**Corollary 6.5.** Let \( \Sigma: \pi_{23}(S^8) \rightarrow \pi_{23}(S^9) \) is onto.

Recall that \( f_*: \pi_{23}(S^15) \rightarrow \pi_{23}(S^8) \) and \( \Sigma: \pi_{21}(S^9) \rightarrow \pi_{23}(S^8) \) are one-to-one and

\[
\pi_{23}(S^8) = \text{Im}(f_*) \oplus \text{Im}(\Sigma)
\]

Furthermore,

\[
\text{Im}(f_*) \approx Z_5 + Z_3 + Z_1
\]
\[
\text{Im}(\Sigma) \approx Z_3 + Z_8 + Z_4
\]
\[
\pi_{23}(S^9) \approx Z_{16} + Z_4
\]

It now follows from (6.5) that \( \Sigma: \pi_{23}(S^8) \rightarrow \pi_{23}(S^9) \) must vanish on \( Z_5 + Z_3 + Z_1 \subset \text{Im}(f_*) \) but must be one-to-one on \( Z_{16} \subset \text{Im}(f_*) \). The following lemma now holds by (3.2).

**Lemma 6.6.** Ker(\( \partial \)) in (6.3a) is \( Z_5 + Z_3 \).

**Proposition 6.7.** \( \pi_{23}(\Sigma(W)) \approx Z_4 \).

**Proof.** By (6.6), \( \text{Im}(\partial) \approx Z_{16} \) in (6.3a). Regardless of how the imbedding \( \text{Im}(\partial) \subset Z_{16} + Z_4 \) is realized, the quotient must be \( Z_4 \).

This completes the proof of (1.11).

7. The 3-primary components in \( \pi_j(EIV), j=19, 20 \). Our present aim is to complete the proofs of (1.7) and (1.8) which were begun in
§5. Let \( \Omega \) denote the space of loops on \( EIV \). From the spectral sequence one easily obtains:

**Lemma 7.1.** In dimensions \(< 32\), \( H^*(\Omega; \mathbb{Z}_3) \) has a basis \( \{1, x_8, x_{16}, x_8^2, x_8x_{16}, x_{24}\} \), \( \dim (x_i) = i \). Furthermore, \( x_8^2 = 0 \).

In order to compute the 3-primary components of \( \pi_{18}(\Omega) \) and \( \pi_{19}(\Omega) \), we proceed by the method of killing cohomology classes in \( H^*(\Omega; \mathbb{Z}_3) \) via successive fibrations with appropriate Eilenberg-MacLane complexes as fibers. This yields the values of \( \pi_j(\Omega) \otimes \mathbb{Z}_3 \), \( j = 18, 19 \), and this information, together with § 5, will prove (1.7) and (1.8). In the computations of this section we will also set the stage for computation of \( \pi_{18}(\Omega) \otimes \mathbb{Z}_3 \) which will be completed in § 8.

A description the of \( \mathbb{Z}_3 \)-algebra \( H^*(\pi, n; \mathbb{Z}_3) \), \( \pi \) a finitely generated abelian group, will be essential. Since, in § 8, we will also need a description of \( H^*(\pi, n; \mathbb{Z}_6) \), we here discuss the general case of \( H^*(\pi, n; \mathbb{Z}_p) \), \( p \) an odd prime. For the proofs of our assertions cf. [6], especially exposés 9, 15, and 16.

Let \( I = (a_1, a_2, \cdots) \), a sequence of integers almost everywhere zero. \( I \) will be called admissible if
\[
\begin{align*}
    a_i &\equiv 0 \text{ or } 1 \mod (2p - 2) \\
    a_i &\geq pa_{i+1}.
\end{align*}
\]
The degree of \( I \) is defined as \( q(I) = \sum a_i \). \( I \) is said to be of the first kind if \( a_i \neq 1, \forall i \). Otherwise \( I \) is said to be of the second kind. If \( I = (a_1, \cdots, a_r, 0, 0, \cdots) \) is of the first kind, then one obtains an \( I' \) of the second kind by setting
\[
I' = (a, \cdots, a_r, 1, 0, \cdots).
\]
Define the numbers
\[
\begin{align*}
g(I) &= \left\lfloor \frac{pa_i}{(p - 1)} \right\rfloor - q(I) \\
n(I) &= \{pa_i/(p - 1)\} - q(I)
\end{align*}
\]
where \([b]\) denotes the greatest integer \( \leq b \) and \( \{b\} \) denotes the least integer \( \geq b \). Finally, let \( P^i, i = 0, 1, 2, \cdots \), denote the Steenrod reduced \( p \)-powers, \( \beta \) the mod \( p \) Bockstein, and define cohomology operations
\[
\begin{align*}
    St^a &= P^k, \quad b = 2k(p - 1) \\
    St^b &= \beta P^k, \quad b = 2k(p - 1) + 1 \\
    St^t &= St^{(a_1)} \circ St^{(a_2)} \circ \cdots, \quad I \text{ admissible}.
\end{align*}
\]

**Theorem 7.2.** (H. Cartan) If \( I \) is admissible of the first kind and if \( n(I') \leq n \), then...
$St^1: \mathbb{H}^{n+1}(\pi, n; \mathbb{Z}_p) \longrightarrow \mathbb{H}^{n+q(I')}(\pi, n; \mathbb{Z}_p)$

is a monomorphism. If also $n(I) \leq n$, then

$St^1: \mathbb{H}^n(\pi, n; \mathbb{Z}_p) \longrightarrow \mathbb{H}^{n+q(I)}(\pi, n; \mathbb{Z}_p)$

is a monomorphism. Let $A^*(\pi, n; \mathbb{Z}_p)$ be the direct sum of the images of all of the above monomorphisms, graded by $n + q(I')$ and $n + q(I)$ respectively. Then the operations $St^1$ define a graded homomorphism

$A^*(\pi, n; \mathbb{Z}_p) \longrightarrow H^*(\pi, n; \mathbb{Z}_p)$

which is an isomorphism onto the image of suspension

$\sigma: H^*(\pi, n + 1; \mathbb{Z}_p) \longrightarrow H^*(\pi, n; \mathbb{Z}_p)$.

Let $M_n \subset A^*(\pi, n; \mathbb{Z}_p)$ be the graded subspace consisting of the direct sum of the images of those of the above monomorphisms where $I'$ (respectively $I$) is required to satisfy the additional condition $g(I') < n$ (respectively $g(I) < \pi$). Then the algebra $H^*(\pi, n; \mathbb{Z}_p)$ is the free graded commutative $\mathbb{Z}_p$-algebra generated by $M_n$.

A further remark that is of use is that

$H^*(\pi, n; \mathbb{Z}_p) \cong \text{Hom}(\pi, \mathbb{Z}_p)$

$H^{n+1}(\pi, n; \mathbb{Z}_p) \cong \text{Hom}(\mathbb{Z}_p, \mathbb{Z}_p)$

where $\pi \subset \pi$ is the subgroup of elements of order $p$. One also notes that if $\pi = \pi$, then

$\beta: H^n(\pi, n; \mathbb{Z}_p) \longrightarrow H^{n+1}(\pi, n; \mathbb{Z}_p)$

is a bijection.

In the remainder of this section we understand $p$ to be 3. By the Adem relations [13] one has $P^2 = P^1 P^1, P^1, P^3$, and $\beta$ are trivial on $H^*(\Omega; \mathbb{Z}_3)$ since the nontrivial dimensions in this graded vector space are all of the form $8k$. Consequently $P^2$ is also trivial on $H^*(\Omega; \mathbb{Z}_3)$.

We kill the class $x_8 \in H^8(\Omega; \mathbb{Z}_3)$ by a fibration

$K(\mathbb{Z}, 7) \longrightarrow X_1 \longrightarrow \Omega$.

An application of (7.2) gives the following classes as a basis of $H^*(\mathbb{Z}, 7; \mathbb{Z}_3)$ in dimensions $\leq 25$ (where $\dim(y) = 7$): $1, y P^1(y), \beta P^1(y), P^3(y), \beta P^3(y), P^3 P^1(\beta), \beta P^4(\beta), \beta P^1(y), y \cdot P^1(y), y \cdot \beta P^1(y), y \cdot P^2(y), \beta P^4(y), P^1(y) \cdot \beta P^1(y), (\beta P^1(y))^2$. By straightforward computations using the spectral sequence of this fibration, one obtains
LEMMA 7.3. In \( \dim \leq 25 \), \( H^*(X_1; \mathbb{Z}_3) \) has basis \( \{1, u_{11}, \beta(u_{11}), P^1(u_{11}), \beta P^1(u_{11}), x_{16}, u_{19}, \beta(u_{19}), P^3(u_{11}), u_{11} \cdot \beta(u_{11}), u_{23}, \beta P^3(u_{11}), (\beta(u_{11}))^2, x_{24} \} \), where the dimension of an element is indicated by its subscript.

In (7.3) the classes \( x_{16}, x_{24} \) are the pull-backs of the classes in the base \( \Omega \) that were denoted by the same symbols. \( u_{11} \) and \( u_{19} \) restrict respectively to \( P^1(y) \) and \( P^3(y) \) in the fiber. \( u_{23} \) corresponds to \( y \cdot x_8^2 \) in the \( E^2 \) term of the spectral sequence. Using these facts and the Adem relations [13] one verifies the following relations:

\[
\begin{align*}
\beta P^1 \beta(u_{11}) &= 0 \\
P^2(u_{11}) &= 0 \\
P^2 \beta(u_{11}) &= -\beta(u_{19}) \\
\beta P^3 \beta(u_{11}) &= 0 \\
P^3 \beta(u_{11}) &= \beta P^3(u_{11}) \\
\beta P^3 \beta(u_{11}) &= 0 .
\end{align*}
\]

Next kill \( u_{11} \) by a fibration

\[ K(\mathbb{Z}_3, 10) \longrightarrow X_2 \longrightarrow X_1 . \]

By (7.2), a basis for \( H^*(\mathbb{Z}_3, 10; \mathbb{Z}_3) \) in dimensions \( \leq 24 \) is given by the following classes (\( \dim(y) = 10 \)): \( 1, y, \beta(y), P^1(y), \beta P^1(y), P^1 \beta(y), \beta P^1 \beta(y), P^2(y), \beta P^2(y), P^3(y), \beta P^3(y), \beta P^3 \beta(y), y \cdot P^1(y) \).

LEMMA 7.4. Transgression

\[ t : H^{15}(\mathbb{Z}_3, 10; \mathbb{Z}_3) \longrightarrow H^{16}(X_1; \mathbb{Z}_3) \]

is bijective.

Proof. Otherwise the first nonvanishing \( H^i(X_2; \mathbb{Z}_3) \) for \( i > 0 \) occurs for \( i = 15 \), and this would give \( \pi_{15}(\Omega) \otimes \mathbb{Z}_3 \simeq \pi_{15}(X_2) \otimes \mathbb{Z}_3 \neq 0 \), contradicting (1.4).

Applying all of this information to the spectral sequence of the fiber space \( X_2 \) we obtain.

LEMMA 7.5. In \( \dim \leq 24 \), \( H^*(X_2; \mathbb{Z}_3) \) has a basis \( \{1, u_{16}, u_{19}, \beta(u_{18}), u_{19}, P^1(u_{19}), P^1 \beta(u_{18}), u_{23}, P^3(u_{19}), x_{24} \} \).

These classes satisfy the following relations:
\[ P^2(u_{16}) = -\beta P^1 \beta(u_{16}) \mod \pi_{24} \]
\[ \beta(x_{24}) = 0 \]
\[ \beta P^2(u_{16}) = 0 \] (a consequence of the above two)
\[ \beta(u_{19}) = 0 \]
\[ P^1(u_{19}) \equiv 0 \mod \pi_{23} \]
\[ \beta P^1(u_{19}) \equiv 0 \mod \pi_{24} \]

Note that, by (1.5), \( \pi_{18}(\Omega) \approx \mathbb{Z} + (\mathbb{Z}_2)^2 \), hence to kill \( u_{18} \) we need a fibration

\[ K(\mathbb{Z}, 15) \longrightarrow X_3 \longrightarrow X_2. \]

Using (7.2), (7.5), and the above relations, we obtain.

**Lemma 7.6.** In \( \dim \leq 24 \), \( H^*(X_3; \mathbb{Z}_3) \) has a basis \( \{1, u_{18}, \beta(u_{18}), u_{19}, u_{20}, P^1 \beta(u_{18}), u_{23}, P^1(u_{20}), x_{24}\} \) satisfying the relations: \( \beta P^1 \beta(u_{18}) \equiv 0 \mod \pi_{24}; \beta(u_{19}) = 0; P^1(u_{19}) \equiv 0 \mod \pi_{23}; \beta P^1(u_{19}) \equiv 0 \mod \pi_{24}. \)

**Corollary 7.7.** \( \pi_{18}(\Omega) \approx \mathbb{Z}_6. \)

**Proof.** By (7.6), \( \pi_{18}(\Omega) \otimes \mathbb{Z}_3 \approx \mathbb{Z}_5. \) By (5.5), \( \pi_{19}(\Omega) \approx \mathbb{Z}_3 \) or \( \mathbb{Z}_6. \)

This completes the proof of (1.7). Next we kill \( u_{19} \) by

\[ K(\mathbb{Z}, 17) \longrightarrow X_4 \longrightarrow X_3. \]

Using the spectral sequence and (7.6) one readily obtains:

**Lemma 7.8.** \( H^j(X_4; \mathbb{Z}_3) \approx 0, 0 < j < 19, \) and \( H^{19}(X_4; \mathbb{Z}_3) \approx \mathbb{Z}_3. \)

**Corollary 7.9.** \( \pi_{19}(\Omega) \approx \mathbb{Z}_{1512} + \mathbb{Z}_2. \)

**Proof.** By (5.6) and (7.7) there is an exact sequence

\[ 0 \longrightarrow \mathbb{Z}_9 + \mathbb{Z}_6 + \mathbb{Z}_7 + \mathbb{Z}_2 \longrightarrow \pi_{19}(\Omega) \longrightarrow \mathbb{Z}_3 \longrightarrow 0. \]

By (7.8), \( \pi_{19}(\Omega) \otimes \mathbb{Z}_3 \approx \mathbb{Z}_3. \) Hence \( \pi_{19}(\Omega) \approx \mathbb{Z}_{27} + \mathbb{Z}_8 + \mathbb{Z}_7 + \mathbb{Z}_2. \)

This completes the proof of (1.8). Evidently in the above lemmas we have obtained information on the cohomology of the spaces \( X_i \) in dimensions higher than necessary for the purposes of this section. This information will be used in the next section to help prove (1.12).

8. Partial determination of \( \pi_{34}(EIV) \). Notice that by the theory of [8] there is an exact sequence
\[ \pi_{24}(S^{18}) \longrightarrow \pi_{24}(S^9) \longrightarrow \pi_{24}(EIV) \longrightarrow \pi_{23}(S^{16}) \longrightarrow \pi_{23}(S^9) \]

which gives explicitly

\[(8.1) \quad (Z_2)^8 \longrightarrow Z_{240} + (Z_2)^3 \longrightarrow \pi_{24}(EIV) \longrightarrow Z_{240} \longrightarrow Z_{16} + Z_4.\]

Thus, to prove (1.12) we must compute \( \pi_{24}(EIV) \otimes Z_2 \) and \( \pi_{24}(EIV) \otimes Z_3 \).

Recall the fibration \( K(Z_3, 17) \to X_4 \to X_3 \). Recall also from (7.6) the relation \( \beta P^i \beta(u_{18}) \equiv 0 \mod x_{24} \). Replacing \( x_{24} \) with its negative if necessary, we obtain just two possibilities:

\[ \beta P^i \beta(u_{18}) = 0 \]

or

\[ \beta P^i \beta(u_{18}) = x_{24}. \]

In order to determine a basis for \( H^*(X_4; Z_3) \) it will be necessary to consider these two possibilities.

**Lemma 8.2.** If \( \beta P^i \beta(u_{18}) = 0 \), then, in dim \( \leq 24 \), \( H^*(X_4; Z_3) \) has as a basis \( \{1, u_{19}, u_{20}, u_{21}, \beta(u_{21}), u_{23}, P^i(u_{20}), u_{23}, x_{24}\} \). The following relations are also satisfied: \( \beta(u_{19}) = 0; P^i(u_{19}) \equiv 0 \mod u_{23}; \beta P^i(u_{19}) \equiv 0 \mod x_{24} \).

**Lemma 8.3.** If \( \beta P^i \beta(u_{18}) = x_{24} \), then, in dim \( \leq 24 \), \( H^*(X_4; Z_3) \) has as a basis \( \{1, u_{19}, u_{20}, u_{21}, \beta(u_{21}), P^i(u_{20}), u_{23}\} \) with \( \beta(u_{19}) = 0, \beta P^i(u_{19}) = 0, P^i(u_{19}) \equiv 0 \mod u_{23} \).

We kill \( u_{19} \) by

\[ K(Z_{27}, 18) \longrightarrow X_5 \longrightarrow X_4. \]

The use of \( K(Z_{27}, 18) \) is dictated by (7.9). The 3-primary component of \( \pi_{19}(X_5) \) is 0.

Note that by (7.2) a basis of \( H^*(Z_{27}, 18; Z_3) \) is given by \( \{1, y_{18}, y_{19}, P^i(y_{18}), \beta P^i(y_{18}), P^i(y_{19}), \beta P(y_{19})\} \) in dim \( \leq 24 \). Here \( \beta(y_{18}) = 0 \).

**Lemma 8.4.** Transgression

\[ t: H^{19}(Z_{27}, 18; Z_3) \longrightarrow H^{20}(X_4; Z_3) \]

is bijective.

**Proof.** Otherwise, \( \pi_{19}(X_5) \otimes Z_3 \approx Z_3 \), contradicting the construction of \( X_5 \).

**Corollary 8.5.** \( H^i(X_5; Z_3) \approx 0, 0 < i < 21 \), while \( H^{21}(X_5; Z_3) \approx Z_3 \) and is generated by (the pull-back of) \( u_{21} \); \( \beta(u_{21}) \neq 0 \).
Lemma 8.6. \( t(P^i(y_{19})) = \pm u_{23}. \)

Proof. In either the hypothesis of (8.2) or of (8.3), \( t(P^i(y_{19})) = P^i(t_{19}) \equiv 0 \mod u_{23}. \) We must show \( P^i(u_{19}) \neq 0. \) Suppose the contrary. Then, killing \( u_{21} \) by \( K(Z_9, 20) \to X_5 \to X_6 \), one shows that \( H^i(X_6; Z_9) \approx 0, \) \( 0 < i < 22, \) and \( H^{22}(X_5; Z_9) \approx Z_3. \) Thus \( \pi_{22}(Q) \otimes Z_9 = \pi_{22}(X_5) \otimes Z_9 \approx Z_3, \) contradicting (1.11).

Lemma 8.7. In the hypothesis of (8.2), \( t(\beta P^i(y_{19})) = \pm x_{24}. \)

Proof. By (8.2), \( t(\beta P^i(y_{19})) = \beta P^i(u_{19}) \equiv 0 \mod x_{24}. \) We must show \( \beta P^i(u_{19}) \neq 0. \) Suppose the contrary. Kill \( u_{21} \in H^{21}(X_5; Z_9) \) by \( K(Z_9, 20) \to X_6 \to X_5. \) Using (8.2), (8.4), (8.5), and (8.6), one shows \( \pi_{23}(Q) \otimes Z_9 \approx \pi_{23}(X_5) \otimes Z_9 \approx Z_5 + Z_7. \) Here the two generators of \( H^{23}(X_5; Z_9) \) come from the \( w_{23} \) of (8.2) and from \( \beta P^i(y_{19}). \) This information, together with (8.1), implies that the 3-component of \( \pi_{23}(Q) \) is \( Z_3 + Z_7. \) Thus if \( w_{23}, v_{23} \in H^{23}(X_5; Z_9) \) are the two generators, \( \beta(w_{23}) \) and \( \beta(v_{23}) \) will be linearly independent. But \( \beta(w_{23}) \) and \( \beta(v_{23}) \) are \( \equiv 0 \mod x_{24}, \) so that we have reached a contradiction.

Lemma 8.8. In the hypothesis of (8.3), \( t(\beta P^i(y_{19})) = 0. \)

Proof. \( t(\beta P^i(y_{19})) = \beta P^i(u_{19}) = 0 \) by (8.3).

Putting all of this information together, one obtains.

\[ \text{Lemma 8.9. In either the hypothesis of (8.2) or of (8.3), } \]
\[ H^*(X_5; Z_9) \text{ has as a basis in dim } \leq 23 \text{ classes } 1, u_{21}, \beta(u_{21}), w_{23}. \]

Proposition 8.10. The 3-primary component of \( \pi_{23}(Q) \) is \( Z_9. \)

Proof. By (8.9) and the process of killing \( u_{21}, \) one finds \( \pi_{23}(Q) \otimes Z_9 \approx Z_9. \) The assertion now follows by (8.1).

There remains the task of finding the 5-primary component of \( \pi_{24}(EIV). \) Here we make use of (1.2) and of the mod 5 Steenrod algebra. Recall from [3, 19.6] that if \( x_i \) generates \( H^i(\Sigma(W); Z_5), \) \( i = 9, 17, \) then \( P^i(x_9) = \pm 2x_{17}. \)

Kill \( x_9 \) by \( K(Z, 8) \to X_1 \to \Sigma(W). \)

This gives the following lemma.
LEMMA 8.11. \( \text{In dim } \leq 25, H^*(X_1; \mathbb{Z}_6) \) has a basis \( \{1, u_{17}, u_{24}, \beta(u_{24}), u_{25}\} \) with relations \( \beta(u_{17}) = 0, P^1(u_{17}) \equiv \beta(u_{24}) \mod u_{25} \).

Since \( \pi_{17}(\Sigma(W)) \approx \mathbb{Z} + (\mathbb{Z}_2)^5 \), one needs
\[
K(\mathbb{Z}, 16) \longrightarrow X_2 \longrightarrow X_1
\]
to kill \( u_{17} \).

LEMMA 8.12. \( H^i(X_2; \mathbb{Z}_6) \approx 0, 0 < i < 24, \) and \( H^{24}(X_2; \mathbb{Z}_6) \approx \mathbb{Z}_6 \).

COROLLARY 8.13. The 5-primary component of \( \pi_{24}(\Sigma(W)) \) is \( \mathbb{Z}_{12} \).

Proof. By (8.12), \( \pi_{24}(\Sigma(W)) \otimes \mathbb{Z}_5 \approx \mathbb{Z}_5 \). The corollary now follows by (8.1).

Now by (8.1), (8.10), and (8.13) we can conclude (1.12).

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