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SETS WITH ZERO-DIMENSIONAL KERNELS

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F. A. Valentine in his book ([1], p. 177, Problem 6.5) suggests that a sufficient condition for a nonempty compact and connected subset S of E^2 to have a kernel consisting of a single point is that each triple of points of S can see via S a unique point of S . The authors show that this condition is sufficient if S is any subset of a topological linear space which contains a noncollinear triple.

Let S be a subset of a linear space. A point p is said to be in the kernel of S if p can be joined to each point of S by a closed line segment that lies in S . F. A. Valentine ([1], p. 164, Problem 1.1) has posed the problem of determining conditions under which the kernel of S will be 0-dimensional, that is, will consist of a single point. He suggests ([1], p. 177, Problem 6.5) that if S is a compact and connected subset of the euclidean plane, then a sufficient condition for the kernel of S to be a single point is that each noncollinear triple of points of S be able to see a unique point of S via S , where this unique point depends on the triple chosen. In this paper we prove that this condition on the noncollinear triples of S implies that the kernel of S is 0-dimensional if S is any subset of a topological linear space containing a noncollinear triple. Thus we establish a stronger result than that suggested by Valentine. The result is secured by first establishing, in the form of lemmas, several properties of such a set S that culminate in the main result. The major property established, namely, that such a set S cannot contain a closed polygonal path made up of four line segments, is obtained in Lemmas 2 and 3. The notation and terminology is that used in [1].

Sufficient conditions for a set S to have a one point kernel. In the sequel S will denote a subset of a topological linear space L over the real field \mathcal{R} with the property that if x, y, z are noncollinear points of S , then there exists a unique point p of S such that the three segments xp, yp, zp all lie in S . (Here the point p may be one of the points x, y or z in which case one of the segments will be degenerate.) By the kernel K of S we mean the set of all points $p \in S$ with the property that if $x \in S$, then the segment $xp \subset S$.

LEMMA 1. *If $x, y \in S$, $x \neq y$, $xy \subset S$ and $z \in S$ such that x, y, z are noncollinear, then the unique point p of S with the property that xp, yp, zp all lie in S is on the line determined by x and y .*

Proof. Note first that S can not contain a triangle. (The vertices of a triangle form a noncollinear triple of points and if the triangle lies in S , then the point corresponding to the vertices in the assumed condition on noncollinear triples of S may be taken as any one of the vertices, contradicting the uniqueness of this point.) Thus if the point p of the lemma is not on the line determined by x and y , then S contains a triangle.

By the dimension of a subset of L we mean the dimension of a flat (also called linear variety) of least dimension that contains the set. A 1-dimensional flat (2-dimensional flat) will be called a line (a plane).

LEMMA 2. *The set S can not contain a closed polygon consisting of four line segments that lie in a plane.*

Proof. In order to show this, suppose the contrary and let a, b, c, d be the vertices of such a quadrilateral Q with sides ab, bc, cd, da all in S . If the quadrilateral Q is the boundary of a nonconvex subset of L , then one of the vertices, say a , is contained in the interior of the triangle formed by the remaining vertices. Let t be any point on side ab of Q that lies between a and b . Since t, d , and c are noncollinear points of S and since $cd \subset S$, the point t can be joined by a segment in S to a point r of the line of c and d . It follows that either the points a, d, r, t or the points t, b, c, r form the vertices of a quadrilateral that is the boundary of a convex subset of the linear space L . Thus we may assume that Q is the boundary of a convex subset of L .

Let $i(Q)$ denote the set bounded by Q relative to the plane containing Q . We now show that if $Q \subset S$, then $i(Q) \subset S$. This is clearly impossible since $i(Q)$ contains a triangle. Let t be any point of side ab of Q such that $a \neq t \neq b$. Since S contains no triangle and the points t, c, d are noncollinear, the unique point r_t of S to which each of the points t, c, d can be joined by a segment in S must be on side cd of Q between c and d . Let $p \in i(Q)$ and suppose $p \notin S$. Then for each $t \in ab$, either p is a point of that part of $i(Q)$ that is bounded by tb, bc, cr_t , and tr_t or p is in that part of $i(Q)$ bounded by $at, tr_t, r_t d$, and da . Let A be the set of all $t \in ab$ such that p is in that part of $i(Q)$ bounded by tb, bc, cr_t, tr_t ; and let $B = ab - A$. Then $a \in A$ and $b \in B$. If u and v are distinct points on ab , then the segments ur_u and vr_v do not intersect, where r_u and r_v denote the unique points on side cd to which u and v can be joined, respectively. Thus if $u \in A$ and $v \in ab$ such that the order, a, v, u holds on ab , then $v \in A$. It follows from this and the definition of A and B , that either A has a last point or B has a first point in the order from a to b . Suppose

that s is the last point of A in the order from a to b . Then $s \neq b$ and p is inside the quadrilateral with sides sb, bc, cr_s , and sr_s . Let w be any point of the segment ab between s and b . Then $w \in B$ and p is inside the quadrilateral with sides aw, wr_w, dr_w , and da . Thus for each such w , p is inside the quadrilateral with sides $sw, wr_w, r_w r_s$, and sr_s . Now let the point w converge to the point s along the segment ab , then the point r_w converges to the point r_s on cd , for if this is not the case S contains a triangle. This implies that $p \in sr_s$, which is contrary to our assumption. If s is the first point of B in the order from a to b the argument is similar. Thus the lemma is proved.

LEMMA 3. *Let S contain a noncollinear triple. If $x, y \in S$ such that $xy \not\subset S$, then there is a unique point $p \in S$ such that xp and yp both lie in S .*

Proof. Since the set S is not linear there is at least one point $p \in S$ such that $xp, yp \subset S$. Let $q \in S$, $p \neq q$, and suppose that $xq, yq \subset S$. By Lemma 2 the four points x, y, p, q do not lie in a plane. Thus the situation is as follows: No one of the four points x, y, p, q is in the plane determined by the other three and exactly four of the six segments determined by these four points, namely, xp, yp, xq, yq , lie in S . Denote the plane determined by three noncollinear points a, b, c of L by $\pi(abc)$.

We now show that if $z \in S$ such that $zp \subset S$, then $z \in \pi(xyp)$. Suppose $zp \subset S$ and $z \notin \pi(xyp)$. If p, q, z are noncollinear, then the unique point $r \in S$ to which each can be joined by a segment in S must be on the line determined by z and p . Thus the points x, y and r are collinear since each can be joined by a segment in S to the distinct points p and q . This implies $r \in \pi(xyp)$ which is impossible since the line determined by z and p has only the point p in common with $\pi(xyp)$. Hence p, q, z are collinear. Since p, q , and z are collinear, the points y, q , and z are noncollinear. Thus the unique point $r \in S$ to which y, q, z can be joined by a segment in S is on the line determined by y and q . This implies that S contains a triangle or a plane quadrilateral since $z \in \pi(yq)$. Thus if $z \in S$ such that $zp \subset S$, then $z \in \pi(xyp)$. It follows that if $z \in S$ such that z can be joined by a segment in S to one of the points x, y, p, q then z is in the plane determined by the two segments in the set $\{xp, yp, xq, yq\}$ that have this point in common.

Consider now a point u on xq between x and q and a point v on yq between y and q such that the plane $\pi(uvp)$ does not intersect the line of x and y . Then there is a point $s \in \pi(xyp)$ to which each of

the points u, v, p can be joined by segments in S . Since $p \in S$ and $ps \subset S$, $p \in \pi(uvs)$. This is true since we can replace x, y, p in the argument in the preceding paragraph by u, v, s . Thus s is on the line of intersection of $\pi(xyp)$ and $\pi(uvp)$. The points p, s, q are noncollinear and hence the point q can be joined by a segment in S to a point t of the line of s and p . Since $qt \subset S$, $t \in \pi(xyq)$. This is impossible since $\pi(xyp)$ and $\pi(xyq)$ have only the line of x and y in common.

This completes the proof of the lemma.

THEOREM. *If S contains a noncollinear triple, then the kernel K of S consists of a single point.*

Proof. Since S contains a noncollinear triple it contains at least two noncollinear line segments with a common endpoint p . We will show that $K = \{p\}$. Let x and y denote the other two endpoints of this two segment path in S , and let q be any point of S not on this two segment path. If x, y, q are noncollinear, then the unique point r of S to which each can be joined by a segment in S must be the point p by Lemma 3. If the points x, y, q are collinear, then the points x, p, q are noncollinear and the points y, p, q are noncollinear. Thus the point q can be joined by a segment in S to a point r on the line of x and p . Also q can be joined by a segment in S to a point t of the line of y and p . If the points p, r, t are distinct, then we have a contradiction of Lemma 3. If exactly two of the points p, r, t are the same, then S contains a triangle. It follows that $p = r = t$ and $pq \subset S$. Thus the point p is in the kernel K of S . It is clear that K can not contain two distinct points. Thus the theorem is proved.

The assumed condition on triples of S is not a necessary condition for K to consist of a single point. This can be seen by considering a triangle together with its interior and with two of its sides extended through the base opposite their intersection.

REFERENCE

1. F. A. Valentine, *Convex sets*, McGraw-Hill Series in Higher Mathematics, McGraw-Hill, Inc., New York, 1964.

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