CLOSED AND IMAGE-CLOSED RELATIONS

STANLEY PHILLIP FRANKLIN AND R. H. SORGENFREY
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S. P. FRANKLIN AND R. H. SORGENFREY

If $X$ and $Y$ are topological spaces, a relation $T \subseteq X \times Y$ is upper semi-continuous at the point $x$ of the domain $D(T)$ of $T$ if for each neighborhood $V$ of $T(x)$, there is a neighborhood $U$ of $x$ such that $T(U) \subseteq V$. Results so far published about such relations usually require that they be closed (as subsets of the product space) or image-closed ($T(x)$ is closed in $Y$ for each $x \in X$). Given any relation $T$, it seems natural to consider the associated relations $T'$ and $\bar{T}$, where $T'$ is defined by $T'(x) = T(x)$ and $\bar{T}$ is the closure of $T$ in the product space. In particular, it is pertinent to ask under what conditions the upper semi-continuity of $T$ implies that of $T'$ or $\bar{T}$, or that $T' = \bar{T}$. As might be expected, the answers to these questions take the form of restrictions on $Y$, and, indeed, serve to characterize regularity, normality, and compactness.

Other relation-theoretic characterizations have been given previously. In [6], Engelking characterizes regularity and compactness (in two ways), and in [10], Michael characterizes normality, collectionwise normality, perfect normality, and paracompactness. Ceder [1] characterizes $m$-compactness.

Terminology in this paper will follow Kelley [9]; in particular, regular and normal spaces need not be $T_1$. The following well known fact will be used: $T$ is upper semi-continuous (hereinafter abbreviated usc) on $D(T)$ if and only if the inverse under $T$ of each closed subset of $Y$ is closed in $D(T)$. A relation $T \subseteq X \times Y$ will be said to be on $X$ into $Y$ if and only if $D(T) = X$.

Statement of results. These are arranged so that for $n = 1, 2, 3, 4$, result $(2n)$ is in the nature of a converse of result $(2n-1)$, thus yielding the promised characterizations of regularity, normality, and various types of compactness.

(1) If $Y$ is regular and $T \subseteq X \times Y$ is usc at $x \in D(T)$, then $T'(x) = \bar{T}(x)$.

Regularity of $Y$ does not imply the upper semi-continuity of $T'$ or $\bar{T}$ for usc $T \subset X \times Y$ (see (6a) and (6b) below).

The statement of the next result, a converse of (1), and of several others will be expedited by a definition: Let $\mathcal{A}$ be a directed set and $p \notin \mathcal{A}$. Define a topology for $X = \mathcal{A} \cup \{p\}$ by letting each point of $\mathcal{A}$ be isolated and taking as a base at $p$ all sets of the form $S \cup \{p\}$ where.
S is a final segment in $\Delta$. When equipped with this topology, $X$ will be called the net-space of $\Delta$. It is clear that each net-space has at most one accumulation point and therefore a rather simple structure.

(2) If for each net-space $X$ and use $T$ on $X$ into $Y$, $T' = \overline{T}$, then $Y$ is regular.

(3) If $Y$ is regular and $T \subseteq X \times Y$ is usc and image-closed, then $T$ is closed in $D(T) \times Y$.

Under certain circumstances the hypothesis of regularity can be relaxed. A Fréchet space is one in which the closure of any subset $A$ is the set of all limits of sequences in $A$. Clearly, any first countable space is Fréchet, but the converse is not true (see [7]).

(3') Let $X$ and $Y$ be such that $X \times Y$ is a Fréchet space (e.g., $X$ and $Y$ first countable). If $Y$ is Hausdorff and $T \subseteq X \times Y$ is use and image-closed, then $T$ is closed in $D(T) \times Y$.

(4) If for each net-space $X$ and use relation $T$ on $X$ into $Y$, $T'$ is closed, then either (a) $Y$ is regular, or (b) every closed nonregular subspace of $Y$ fails to be $R_o$.

The authors have been unable to remove the possibility (b) from the conclusion of this result. It is clear, however, that for $R_o$ (hence for $T_1$)-spaces, (3) and (4) characterize regularity.

(5) If $Y$ is normal and $T \subseteq X \times Y$ is use at $x \in D(T)$, then both $T'$ and $\overline{T}$ are use at $x$.

From (5) it is clear that if $Y$ is normal and $D(T)$ is closed, the upper semi-continuity of $T \subseteq X \times Y$ implies that of $\overline{T}$. That this need not be the case if $D(T)$ is not closed is shown by the following

**EXAMPLE.** Let $X$ and $Y$ be the reals with usual topology and $f: Y \to X$ be defined by $f(y) = y^{-1} \sin y$ for $y \neq 0$, $f(0) = 1$. Then $T = f^{-1}\{0\} \cup \{(0, y) \mid y \in Y\}$ is use on $D(T)$. However, $\overline{T}$ is not use at $0 \in D(\overline{T})$ since $V = \bigcup \{(n\pi - 1, n\pi + 1) \mid n \text{ an integer}\}$ is a neighborhood of $\overline{T}(0)$, but there is no neighborhood $U$ of 0 such that $T(U) \subseteq V$.

(6a) If for each net-space $X$ and use relation $T$ on $X$ into $Y$, $T'$ is use, then $Y$ is normal.

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1 A space is $R_o$ if and only if point closures partition it. (Davis [4].)
(6b) If \( Y \) is Hausdorff and for each net-space \( X \) and usc relation \( T \) on \( X \) into \( Y \), \( \bar{T} \) is usc, then \( Y \) is normal.

If \( Y \) is infinite and equipped with the co-finite topology,² then for every \( X \) and usc \( T \) on \( X \) into \( Y \), \( \bar{T} \) is usc; hence the Hausdorff hypothesis in (6b) cannot be weakened even to \( T \). Thus (5) and (6a) characterize normality, while (5) and (6b) characterize normality in the class of Hausdorff spaces.

Recall that for any infinite cardinal \( m \) (defined as an initial ordinal) a topological space \( Y \) is called \( m \)-compact if and only if each open cover of power \( \leq m \) has a finite subcover. Compact spaces are precisely those which are \( m \)-compact for each \( m \). \( \aleph_0 \)-compact spaces are the countably compact spaces. \( m \)-compact spaces have been characterized in terms of the behavior of usc relations on them by Ceder [1]. A space \( X \) is said to have local weight \( m \) if and only if \( m \) is the least cardinal such that each point of \( X \) has a basis of neighborhoods of power \( \leq m \). First countable spaces are those of local weight \( \leq \aleph_0 \).

(7) If \( Y \) is compact and \( T \subseteq X \times Y \) is closed, then \( T \) is usc on \( D(T) \).

This result is well known and was apparently first noticed by Choquet [3].

(7\(m\)) If \( X \) has local weight \( m \), \( Y \) is \( m \)-compact and \( T \subseteq X \times Y \) is closed, then \( T \) is usc on \( D(T) \).

(7\(\aleph_0\)) If \( X \) is first countable, \( Y \) is countably compact and \( T \subseteq X \times Y \) is closed. Then \( T \) is usc on \( D(T) \).

The corresponding results (7'), (7\(m\)') and (7\(\aleph_0\)') about functions, in which the hypotheses on \( X \) and \( Y \) are the same and the conclusion is that every function \( f: X \to Y \) with closed graph is continuous, are immediate corollaries. The net-space of an ordinal \( \alpha \) will be denoted by \( X_\alpha \).

(8) Let \( Y \) be \( T \). If for each net-space \( X \) every closed \( T \) on \( X \) into \( Y \) is usc, then \( Y \) is compact.

(8\(m\)) Let \( Y \) be \( T \). If for each ordinal \( \alpha \leq m \), every closed \( T \) on \( X_\alpha \) into \( Y \) is usc, then \( Y \) is \( m \)-compact.

² i.e., the topology generated by the complements of finite sets.
Let $Y$ be $T_1$. If every closed $T$ on the sequence space $X_{\aleph_0}$ into $Y$ is usc, then $Y$ is countably compact.

These results are immediate consequences of the corresponding statements (8'), (8m') and (8K') in which it is hypothesized that each function $f$ from $X (X_m, X_{\aleph_0})$ into $Y$ with closed graph is continuous. If $Y$ is the set of natural numbers with the initial segments as a basis for the topology, then $Y$ is $T_0$, but not $T_1$, no function into $Y$ has closed graph, and $Y$ is not countably compact. Hence the $T_1$ hypothesis in (8'), (8m') and (8K') cannot be relaxed even to $T_0$. Clearly compactness (m-compactness, countable compactness) in $T_1$ spaces is characterized by (7) and (8) ((7m) and (8m), (7K) and (8K)) as well as by their corresponding function results.

The hypothesis of first countability on $X$ in (7, 8) can be relaxed if the hypothesis on $Y$ is strengthened.

If $X$ is a Hausdorff Fréchet space, $Y$ sequentially compact, and $T \subseteq X \times Y$ closed, then $T$ is usc on $D(T)$.

The corresponding function result (9') is again an immediate corollary. One might hope for a converse to (9) patterned after (8K), but the existence of compact, nonsequentially compact spaces (such as $\beta N$) makes the hope a vain one in view of (7).

Proofs of results. It will be convenient to give these in a somewhat different order from that of the statements.

Proof of (1). It is clear that for any relation, $T'(x) \subseteq \bar{T}(x)$. Suppose, therefore, that $y \in \bar{T}(x) \setminus T'(x)$. Since $Y$ is regular and $T'(x)$ is closed, there is a closed neighborhood $N$ of $T'(x)$ not containing $y$. Since $T$ is usc at $x$, there is an open neighborhood $U$ of $x$ such that $T(U) \subseteq N$. Then $U \times (Y \setminus N)$ is a neighborhood of $(x, y)$ not intersecting $T$, whence $(x, y) \notin \bar{T}$ or $y \notin T(x)$.

Proof of (3). For all $x \in D(T)$, $T(x) = \bar{T}(x) = T'(x)$ by hypothesis and $T'(x) = \bar{T}(x)$ by (1).

Proof of (3'). Suppose there exist $x \in D(T)$ and $y \in Y$ such that $(x, y) \notin \bar{T} \setminus T$. Since $X \times Y$ is Fréchet, there is a sequence $\{(x_n, y_n)\}$ in $T$ converging to $(x, y)$. Since $y \notin T(x)$, a closed set, and $\{y_n\} \to y$, there is an integer $k$ such that if $n > k$, $y_n \notin T(x)$. Thus $K = \{y_n | n > k\} \cup \{y\}$ and $T(x)$ are disjoint, and because $Y$ is Hausdorff, $K$ is closed. Since $T$ is usc, $T^{-1}(K)$ is closed in $D(T)$. But for $n > k$, $x_n \in T^{-1}(K)$ and $\{x_n\} \to x$, whence $x \in T^{-1}(K)$. Thus $T(x) \cap K \neq \emptyset$, a contradiction.
Proof of (5). Let $E$ be either $T'(x)$ or $T(x)$, and let $V$ be a neighborhood of $E$. Since $E$ is closed and $Y$ is normal, there is a closed neighborhood $N$ of $E$ contained in $V$. Since $T$ is use at $x$ and $N$ is a neighborhood of $T(x)$, there is an open neighborhood $U$ of $x$ such that $T(U) \subseteq N$. If $T(U) \not\subseteq N$, there are $z \in U$ and $y \in Y \setminus N$ such that $(z, y) \in T$. But $U \times (Y \setminus N)$ is a neighborhood of $(z, y)$ not intersecting $T$. Hence $T'(U) \subseteq T(U) \subseteq N \subseteq V$, and both $T'$ and $T$ are use at $x$.

Proof of (6a). If $Y$ is not normal, there exist a closed $F \subset Y$ and a neighborhood $W$ of $F$ which contains no closed neighborhood of $F$. Direct the neighborhood system $\Delta$ of $F$ by $\subseteq$, and let $X = \Delta \cup \{p\}$ be the net-space of $\Delta$. Define $T$ on $X$ by $T(V) = V$ for all $V \in \Delta$, and $T(p) = F$. $T$ is use at $p$ (and hence on $X$) since for any neighborhood $V_\circ$ of $T(p) = F$, $U = \{V \in \Delta \mid V \subseteq V_\circ \} \cup \{p\}$ is a neighborhood of $p$, and $T(U) \subseteq V_\circ$. $T'$, however, is not use at $p$ since for each $V \in \Delta$, $T'(V) = \overline{V}$ is a closed neighborhood of $F$ and hence is not contained in the neighborhood $W$ of $T(p) = F$.

Proof of (6b). Suppose $Y$ is not normal. We will construct a net space $X$ and use $T$ on $X$ into $Y$ such that $T$ is not use.

Case 1. $Y$ is regular. By (6a) there is a net-space $X$ and use $T$ on $X$ into $Y$ such that $T'$ is not use. By (1), $T'' = T$, and the construction is accomplished.

Case 2. $Y$ is not regular. There exist a closed $F \subset Y$ and $p \in Y \setminus F$ such that the closure of every neighborhood of $p$ intersects $F$. Let $\Delta$ be the family of all neighborhoods of $p$ which do not intersect $F$, direct $\Delta$ by $\subseteq$, and let $X = \Delta \cup \{p\}$ be the net-space of $\Delta$. Then $T$ defined on $X$ by $T(x) = x$ is use.

We now show that $\overline{T(p)} = p$: Let $p \neq q \in Y$. Since $Y$ is Hausdorff, there exist $V_\circ \in \Delta$ and a neighborhood $W$ of $q$ such that $W \cap V_\circ = \emptyset$. Then $U = \{V \in \Delta \mid V \subseteq V_\circ \} \cup \{p\}$ is a neighborhood of $p$ in $X$, hence $U \times W$ is a neighborhood of $(p, q)$ in $X 	imes Y$. If $(V, y) \in U \times W$, then $y \in V = T(V)$ since $y \in W$ and $V \cap W \subseteq V_\circ \cap W = \emptyset$. Hence $(V, y) \in T$, i.e., $(U \times W) \cap T = \emptyset$, whence $(p, q) \not\in \overline{T}$, or $q \not\in \overline{T(p)}$.

$\overline{T}$ is not use at $p$ since if $V \in \Delta$, $\overline{T(V)} \supset \overline{V}$ and is therefore not contained in the neighborhood $Y \setminus F$ of $p = \overline{T(p)}$.

Proof of (4). Assuming the proposition not true, there is a closed nonregular subspace $Z$ of $Y$ which is $R_\circ$. The existence of a net space $X$ and a nonclosed, image-closed, use relation $T$ on $X$ into $Y$ will be
demonstrated. Since $Z$ is closed, $T$, regarded as a relation on $X$ into $Y$ will have the same properties and provide the desired contradiction.

There exist closed $F \subset Z$ and $q \in Z \setminus F$ which do not have disjoint neighborhoods. Direct $A = \{(V, W) | V \text{ is a neighborhood of } F \text{ and } W \text{ is a neighborhood of } q\}$ by $(V, W) > (V', W')$ if and only if $V \subseteq V'$ and $W \subseteq W'$, and let $X = A \cup \{p\}$ be the net-space of $A$. Define $T$ on $X$ into $Z$ by $T((V, W)) = \{p \in V \cap \{V, W\} \geq \{V, W\} \supset \{p\}$ if and only if $F \subseteq V$ and $W$ by $(V, W) > (V', W')$, and let $X = A \cup \{p\}$ be the net-space of $A$. Define $T$ on $X$ into $Z$ by $T((V, W)) = \{p \in V \cap (V, W) \setminus \{p\}$ is a neighborhood of $p$, and $(V, W) \in U$ implies $p_r, q \in V \cap W \subset V$, whence $T((V, W)) = \{p_r, w\} \supset \{p\}$. But the net $\{(V, W), p_r, w\} | (V, W) \in A$ in $T$ converges to $(p, q) \in T$, and $T$ is not closed.

Proof of (2). Let $X$ be any net-space and $T$ be an image-closed use relation on $X$ into $Y$. Then $T = T'$ and, by hypothesis $T' = T$. Hence $T$ is closed and the hypothesis of (4) is satisfied. The present result will follow from (4) when it is shown that $Y$ (and hence every subspace of $Y$) is $R_v$. If this is not the case, there are points $q$ and $r$ of $Y$ such that $q \in (r)^{-}$ but $r \notin \{q\}$. Let $X$ be the net-space consisting of a sequence $\{x_n\}$ and its limit $p$, and define $T$ on $X$ into $Y$ by $T(x_n) = \{q, r\}; T(p) = \{q\}$. Since every neighborhood of $q$ contains $r$, $T$ is use at $p$. But $r \in T'(p) = \{q\}^{-}$, while $r \in T(p)$ since the sequence $\{(x_n, r)\}$ in $T$ converges to $(p, r)$. Hence $T' \neq T$.

Proof of (7m). If $F$ is a closed subset of $Y$, $\pi_X^{-1}(F) \cap T$ is closed in $X \times Y$. Since $Y$ is $m$-compact, $\pi_X$ is a closed mapping (Hanai [8]) and so $T^{-1}(F) = \pi_X(\pi_X^{-1}(F) \cap T)$ is closed in $X$ and therefore in $D(T)$.

Proof of (8m'). If $Y$ is not $m$-compact, it follows from a lemma attributed to Chittenden [2] (see Ceder [1]), that there is an $\alpha$-net $\{y_\beta\}_{\beta < \alpha \leq m}$ which has no cluster point. Define a function $f: X_a \rightarrow Y$ by $f(\beta) = y_\beta$ if $\beta < \alpha$ and $f(p) = y_0$, where $y_0$ is an arbitrarily chosen point of $Y$. $f$ is not continuous at $p$ since $\{\beta\}_{\beta < \alpha}$ converges to $p$ in $X_a$ but $\{f(\beta)\}_{\beta < \alpha} = \{y_\beta\}_{\beta < \alpha}$, having no cluster point, cannot converge to $f(p) = y_0$ in $Y$.

Suppose $(x, y) \in f$. If $x = \beta < \alpha$, then $y \neq y_\beta$. Let $W$ be any open neighborhood of $y$ not containing $y_\beta$. Then $(x, y) \in \{\beta\} \times W$, which is open and disjoint from $f$. If, on the other hand, $x = p$, then $y = y_0$. Since $y$ is not a cluster point of $\{y_\beta\}_{\beta < \alpha}$, there is an open neighborhood $U$ of $y$, not containing $y_\beta$, and a $\beta_0 < \alpha$ such that $\beta \geq \beta_0$ implies $y_\beta \notin U$. Let $N = \{\beta \mid \beta_0 \leq \beta < \alpha\} \cup \{p\}$. Then $(x, y) \in N \times U$ which is open and disjoint from $f$. #
(8') follows from (8m') since $Y$ is compact if and only if $Y$ is uncompact for all $m$ (Chittenden [2], Ceder [1]).

Proof of (9). Let $F$ be closed in $Y$. If $x_0 \in \text{cl}_{D(T)} T^{-1}(F)$, there is a sequence $\{x_n\} \subseteq T^{-1}(F)$ converging to $x_0$ (since subspaces of Fréchet spaces are Fréchet [7]). For each $n$ choose $y_n \in T(x_n) \cap F$ and let $\{y_{n_i}\}$ be a subsequence of $\{y_n\}$ converging to $y_0 \in Y$. But $y_0 \in F$ and $\{(x_{n_i}, y_{n_i})\}$ is contained in $T$ and converges to $(x_0, y_0)$. Thus, since $T$ is closed, $x_0 \in T^{-1}(F)$.

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