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Let (X, T, Π) be a topological transformation group, where X is a nontrivial Hausdorff continuum, and T is a topological group which leaves an endpoint e of X fixed. Wallace showed that if X is locally connected and T is cyclic, T has another fixed point. In a later paper, Wallace asked the following question: if X is a peano continuum and T is compact or abelian, does T have another fixed point?

In 1952, Wang showed that if X is arcwise connected and T is compact, T has another fixed point; Chu has recently extended this result by showing T has infinitely many fixed points. Gray has shown that in the abelian case, the answer to Wallace's question is "no" (in general). However, if T is a generative group, and if X is arcwise connected, T has another fixed point. In this paper we will generalize the last result. In fact, we show that if X is arcwise connected or locally connected, and T is a group of the form AH, where H is a connected subgroup, and A is an abelian group generated by a compact subset, and A lies in the center of T, then T has another fixed point. We will generalize several known theorems by studying ordered spaces similar to those introduced by Wallace in 1945; in particular, we will obtain a generalized solution of the compact group problem (Theorem 2).

2. In this section, X will denote a compact Hausdorff space consisting of more than two points on which a reflexive, transitive, and antisymmetric order $\leqq$ is defined; if $z \in X$, let

$$L(z) = \{x; z < x\}, M(z) = \{x; x \leqq z\}, N(z) = \{x; z \leqq x\} .$$

We assume that $\leqq$ satisfies the following conditions:

(a) The set $M(z)$ is closed.

(b) The set $L(z)$ is open, and $N(z)$ is closed.

(c) X has a least element e under $\leqq$.

(d) Each set $M(z)$ is a chain, i.e. $M(z)$ is simply ordered by $\leqq$.

(e) X is directed by $\leqq$ in the following sense: if $x, y \in X$ and $x \neq e, y \neq e$, then there exists $z \neq e$ such that $z \leqq x$ and $z \leqq y$.

Wallace, [6], has proved:

(f) Each nonvoid closed subset of X contains a maximal element under $\leqq$.

We show:

(g) If C is a closed nonvoid subset of X with $e \notin C$, we have

$z \in X, z \neq e$, for which $z \leqq c$ for every $c \in C$.

Proof. If for some $z \in C$, $z \leqq c$ for every $c \in C$, we are finished. Otherwise, if $x, y \in C$ and $x \neq y$, choose $z_{xy} \neq e$ satisfying (e): $z_{xy} \leqq x$ and $z_{xy} \leqq y$. We show that the collection $\{L(z_{xy}); x, y \in C, x \neq y\}$ is an open cover of C. If $x \in C$, we have $y \in C$ for which $x \nleqq y$; it follows that $z_{xy} < x$, and hence $x \in L(z_{xy})$. Since X is compact and C is closed in X, there is a finite subset $\{z_i, \cdots, z_n\}$ of the set $\{z_{xy}; x, y \in C, x \neq y\}$ for which $C \subset \cup \{L(z_j), 1 \leqq j \leqq n\}$; since $z_j \neq e$ for every j, by (e) we have $z \in X$ for which $z \neq e$ and $z \leqq z_j$ for $j = 1, \cdots, n$. z is the desired element of X.

By an order isomorphism: $X \to X$, we mean a homeomorphism which preserves $\leqq$. If (X, T, Π) is a transformation group, we will assume that for each $t \in T$ the t-transition of (X, T, Π) is an order isomorphism.

If $A \subset X$ and $B \subset X$, we write $A \leqq B$ $[A < B]$ if, given $a \in A$ and $b \in B$, we have $a \leqq b$ $[a < b]$.

LEMMA 2.1. *Let (X, T, Π) be a topological transformation group. If there is a closed nonempty T-invariant subset $A \subset X$ such that $e \notin A$, then T has a fixed point other than e.*

Proof. Let $A \neq \varnothing$ be *any* closed subset of X such that $e \notin A$. Define

$$M(A) = \cap \{M(a); a \in A\} .$$

$M(A)$ is a closed chain, and $M(A) = \{z; z \leqq A\}$. By (g), $M(A)$ does not consist of e alone. By (f), $M(A)$ contains a maximal element $\mathscr{M}(A)$. Since $M(A)$ is a chain, $\mathscr{M}(A)$ is the largest element of $M(A)$. If $t: X \to X$ is an order isomorphism, then $t\mathscr{M}(A) = \mathscr{M}(tA)$. Then if A is T-invariant, $\mathscr{M}(A)$ is fixed under T. It is clear that $\mathscr{M}(A) \neq e$, so that the proof is complete.

LEMMA 2.2. *Let (X, T, Π) be a transformation group. If there is a T-invariant chain $B \subset X$ which is not empty and does not consist of e alone, then T has a fixed point other then e.*

Proof. The collection of closed sets $\{N(b); b \in B\}$ has the finite intersection property since B is a chain. Hence the intersection, $N(B)$, of the $N(b)$ is not empty and is T-invariant since B is. Because $e \notin N(B)$, $N(B)$ satisfies the hypothesis of Lemma 2.1, and the proof is complete.

LEMMA 2.3. *Let $t_1, \cdots, t_n$ be commuting order isomorphisms: $X \to X$. Then the t_i have a fixed point other than e in common.*

Proof. Let z_0 be a maximal element of X. If $A = \{z_0, t_i^{-1}z_0; i = 1, \cdots, n\}$, then $e \notin A$. By (e) we have $z_1 \neq e, z_1 \leqq A$. For each i, $\{z_1, t_iz_1\} \subset M(z_0)$. We let T_i be the cyclic group generated by t_i and $T = T_1T_2 \cdots T_n$. Then for each i, T_iz_1 is a chain.

(1) If $s \in T$ and $t \in T$ such that sz_1 and tz_1 both compare to z_1, then sz_1 and tz_1 compare.

For if $sz_1 \leqq z_1$ and $tz_1 \leqq z_1$, the result follows from (d). If $sz_1 \geqq z_1$ and $tz_1 \geqq z_1$, apply the last case to $s^{-1}z_1$ and $t^{-1}z_1$ and use the fact that T is abelian. The final case follows by transitivity of $\leqq$.

(2) Each element of Tz_1 compares to z_1.

Let $t_1^{K_1} \cdots t_n^{K_n}z_1 \in Tz_1$, where the K_i are integers. Then $t_1^{K_1}z_1$ compares to z_1. We proceed by induction. If $t_1^{K_1} \cdots t_j^{K_j}z_1$ compares to z_1, where $1 \leqq j \leqq n-1$, then since $t_{j+1}^{K_{j+1}}z_1$ compares to z_1 also, $t_1^{K_1} \cdots t_{j+1}^{K_{j+1}}z_1$ compares to z_1; the desired result follows.

From (1) and (2) it follows that Tz_1 is a chain. Now $e \notin Tz_1$ so that Lemma 2.2 applies. The proof is complete.

A group T is *generative* if T is abelian and is generated by a compact neighborhood of the identity of T.

THEOREM 1. *Let (X, T, Π) be a transformation group, where T acts as a generative group of order isomorphisms on X. Then T has a fixed point other than e.*

Proof. Since T is generative, it is known that T has the form KZ^mR^n where Z and R denote the integers and reals, respectively, with the usual topology, and m and n are nonnegative integers. Thus T may be written in the form CA, where C is compact and A is a finitely generated abelian group. If x is a fixed point of X under A, with $x \neq e$, then $Tx = Cx$ is closed, T-invariant, and does not contain e. Hence Lemma 2.1 applies, and the proof is complete.

NOTE. Actually, in Theorem 1, we need only assume that the group T is abelian and is generated by a compact set. For if then C is a compact symmetric set which contains the identity of T and generates T, let x be a maximal element of X and let $z \leqq C^{-1}x$, where $e \neq z$. Then $Cz \subset M(x)$, hence Cz is a chain. Since T is abelian, we may argue as in the proof of Lemma 2.3 and show that C^nz is a chain for each positive integer n. Thus the set $\cup\{C^nz; n = 1, 2, \cdots\}$ is a T-invariant chain not consisting of e alone, and T has a fixed point other than e. This proves

THEOREM 1′. *If (X, T, Π) is a transformation group, where T is abelian and is generated by a compact subset, and if T acts as a group of order isomorphisms on X, then T has a fixed point other than e.*

We now consider a strengthened form of axiom (e):

(e_s) X is strongly directed by $\leqq$ in the following sense: if $x, y \in X$ and $x \neq e$, $y \neq e$, then there is a $z \in X$ with $z \neq e$ for which $z < \{x, y\}$.

If X is a space which satisfies (a)—(e) but does not satisfy (e_s), then it is easy to see that there is an $x \in X$ with $x \neq e$ such that $tx = x$ for every order isomorphism $t: X \to X$. If X satisfies (e_s), then we have

(g_s) If C is a closed nonempty subset of X with $e \notin C$, there is a $z \in X$ with $z \neq e$ for which $z < C$.

THEOREM 2. *Let (X, T, Π) be a transformation group, where X has an order $\leqq$ which satisfies* (b)—(d) *and* (e_s), *and T acts as a compact group of order isomorphisms on X. Let $x \in X$ with $x \neq e$. Let*

$$M(Tx) = \{y; y \leqq Tx\} = \cap\{M(y); y \in Tx\} .$$

Then T leaves each point of $M(Tx)$ fixed. Furthermore $M(Tx)$ is an infinite set.

Proof. The set $M(Tx)$ is a T-invariant chain by axiom (d). Let $z \in M(Tx)$. Then Tz is a compact subchain of A, and since (f) holds for $\leqq$ without assuming (a), Tz contains a maximal element m. Since Tz is a chain, m is the largest element of Tz, hence is fixed under T. Thus the orbit of z contains a fixed point under T, so that T leaves z fixed. Now (g_s) also holds for $\leqq$, so that the set $M(Tx)$ is infinite, and the proof is complete.

In what follows, let X be a nontrivial Hausdorff continuum. If $e \in X$, then e is an *end point* of X if, given an open set U with $e \in U$, there exists $y \in U$ such that $y \neq e$ and

$$X - y = V \cup W, e \in V \subset U, (\bar{V} \cap W) \cup (V \cap \bar{W}) = \varnothing .$$

If $x \in X$, let $E(e, x) = \{e, x\} \cup \{z; z \text{ separates } e \text{ and } x \text{ in } X\}$ Given two points $x, y \in X$, define $x \leqq y$ if and only if $x \in E(e, y)$. Then $\leqq$ satisfies (b)—(e) and (e_s). Furthermore, a homeomorphism: $X \to X$ which leaves e fixed is an order isomorphism. If in addition X is locally connected, $\leqq$ satisfies (a), and the results of this section apply to such a space. Hence if (X, T, Π) is a transformation group, where X is locally connected and $Te = e$, and if there is a closed nonempty T-invariant subset $A \subset X$ such that $e \notin A$, then T has a fixed point other than e.

From Theorem 2 we obtain

COROLLARY 2.1. *Let (X, T, Π) be a transformation group, where X is a nontrivial Hausdorff continuum and T is a compact group which leaves an end point e of X fixed. If $x \in X$ and $x \neq e$, let*

$$E(e, Tx) = \{y; y \text{ separates } e \text{ and } Tx \text{ in } X\} .$$

Then T leaves each point of $E(e, Tx)$ fixed.

We will call a metric continuum a *dendrite* if each two distinct points of the continuum is separated by a third point of the continuum. It is known [10] that each point of a dendrite is either a cut point or an end point.

COROLLARY 2.2. *Let X be a dendrite with a finite number, N, of end points. Then the only compact groups which can act effectively on X are the subgroups of S_N, the permutation group on N symbols.*

Proof. Let E be the set of end points of X and T be a compact group which acts effectively on X. Then for each $t \in T$, the restriction, $t \mid E$, of t to E is in S_N, and the mapping $t \rightarrow t \mid E$ is a homomorphic mapping of T onto a subgroup of S_N.

Let P be the set of all elements of T which leave each point of E fixed. P is a closed subgroup of T, and since $X = \cup \{E(x, y); x, y \in E\}$, it follows from Corollary 2.1 that P leaves each element of X fixed, and because T is effective, P is the identity alone. Thus if $t \mid E = s \mid E$, then $s^{-1}t \in P$, hence $s = t$. Thus the mapping $t \rightarrow t \mid E$, all $t \in T$, is an isomorphism.

3. In this section, X will denote a nontrivial locally connected Hausdorff continuum, and T is a group which leaves an end point e of X fixed. We remark that all the results of this section hold when X is arcwise connected but not necessarily locally connected (we replace the remark immediately preceding Corollary 2.1 by Wang's Lemma, [9]).

LEMMA 3.1. *Let (X, T, Π) be a transformation group, where T is connected. Then T has a fixed point other than e.*

Proof. Since X contains at least two noncut points, [8], let $x \neq e$ be another noncut point, and

$$X - z = U \cup V, e \in U, x \in V, (\bar{U} \cap V) \cap (U \cup \bar{V}) = \varnothing ;$$

now Tx contains only noncut points, and so $z \notin Tx$ since z is a cut point. Since Tx is connected, it follows that $Tx \subset V$. Because

$V \cup \{z\}$ is closed, we have $\overline{Tx} \subset V \cup \{z\}$. We have found a nonempty closed T-invariant set not containing e, so that the remark preceding Corollary 2.1 applies.

THEOREM 3. *Let (X, T, Π) satisfy the hypothesis of Lemma 3.1. Either e is the only noncut point in one of its neighborhoods, or else T has infinitely many fixed points.*

Proof. We use the order and notation of § 2. Let x_0 be a noncut point of X with $x_0 \neq e$. From the proof of Lemma 2.1, we see that $\mathscr{M}(\overline{Tx_0})$ is a fixed point different from e. Let $A_1 = \mathscr{M}(\overline{Tx_0}) \cup \overline{Tx_0}$. Since e does not belong to the closed set A_1, we may find $z \in X$ for which

$$X - z = U \cup V, e \in U, A_1 \subset V, (\bar{U} \cap V) \cup (U \cap \bar{V}) = \varnothing .$$

Suppose every neighborhood of e contains a cut point other than e, and let $x_1 \in U$ be such a point. Since z is a cut point, $z \notin Tx_1$ so that $\overline{Tx_1} \subset U \cup \{z\}$. Furthermore, a separation argument shows that if $x \in \overline{Tx_1}$, then $M(x) \subset U \cup \{z\}$ so that $\mathscr{M}(\overline{Tx_1}) \subset U \cup \{z\}$. Since $\mathscr{M}(Tx_0) \in V$, we have $\mathscr{M}(Tx_1) \neq \mathscr{M}(Tx_0)$. Set

$$A_2 = \overline{Tx_0} \cup \overline{Tx_1} \cup \mathscr{M}(\overline{Tx_0}) \cup \mathscr{M}(\overline{Tx_1}) ,$$

and complete the proof by induction.

THEOREM 4. *Let (X, T, Π) be a transformation group, with $T \approx AH$, where A is an abelian group which is generated by a compact subset and lies in the center of T, and H is a connected subgroup. Then T has a fixed point other than e.*

Proof. Let X be a fixed point under A, where $x \neq e$. Then $\overline{Tx} = \overline{Hx}$ is connected. If $e \notin \overline{Hx}$, we are finished (in view of previous results). If $e \in \overline{Hx}$, since $\overline{Hx}$ is a nontrivial Hausdorff continuum, $\overline{Hx}$ contains a noncut point $y \neq e$. Then for some $z \in X$,

$$X - z = U \cup V, e \in U, y \in V, (\bar{U} \cap V) \cup (U \cap \bar{V}) = \varnothing .$$

Because $\overline{Hx}$ is connected, z is a cut point of Hx. Since Hy contains only noncut points of $\overline{Hx}, z \notin Hy$, and $\overline{Hy} \subset (V \cap \overline{Hx}) \cup \{z\}$, for the last set is closed in X. Now A lies in the center of T, hence every point of Hx is fixed under A, so that $\overline{Hy}$ is a T-invariant set not containing e. By the remark at the end of § 2, the proof is complete.

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