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**MODULAR PAIRS IN ORTHOMODULAR LATTICES**

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**Call an orthomodular lattice  $L$   $M$ -symmetric if  $M(e, f)$  implies  $M(f, e)$  for all  $e, f \in L$  and  $O$ -symmetric if  $M(e, f)$  implies  $M(f', e')$ . To check for these properties it is sufficient to consider only those modular pairs in which the two elements are complements. Every  $O$ -symmetric lattice is  $M$ -symmetric. In an atomic orthomodular lattice,  $M$ -symmetry is equivalent to the atomic exchange property.**

The orthomodular lattice  $L(H)$  of closed subspaces of a Hilbert space  $H$  satisfies both symmetries but apparently for separate reasons. G. W. Mackey has shown [4, Theorem III-6] that two closed subspaces of  $H$  form a dual modular pair in  $L(H)$  if and only if their vector sum is a closed subspace. Thus the natural symmetry,  $M$ -symmetry, depends on the topology of  $H$ .  $O$ -symmetry arises in  $L(H)$  as a consequence of properties of bounded linear operators on  $H$  with closed ranges. It is the purpose of this paper to investigate these symmetries in arbitrary orthomodular lattices. Recent results of A. Ramsay [8] and M. D. MacLaren [7] have shown  $M$ -symmetry to be of importance in the study of locally finite dimension lattices. The Baer  $*$ -semigroup coordinatization theory for orthomodular lattices, developed by D. J. Foulis [2], enables us to adapt the idea of an operator with a closed range and conveniently and naturally introduce  $O$ -symmetry into arbitrary orthomodular lattices.

Confining our attention to orthomodular lattices, we first establish certain general properties of modular pairs. Using these results we develop characterizations of  $M$ -symmetric lattices and  $O$ -symmetric lattices in Theorem 7 and Theorem 8. That  $O$ -symmetry implies  $M$ -symmetry is established in Theorem 9. In § 4, atomic orthomodular lattices satisfying the atomic exchange property introduced by MacLane [5] are considered. Using specializations of the approach and procedure used by Ramsay in [8] it is shown that  $M$ -symmetry is equivalent to the atomic exchange property.

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**2. Modular pairs.** In order to establish certain properties of modular pairs in orthomodular lattices, we shall make use of the Baer  $*$ -semigroup approach as defined and developed in [1, 2, 3]. We shall restrict our resume of definitions and results to a minimum of

chiefly notational conventions. An *orthomodular lattice* is a lattice  $L$  with 0 and 1 which is equipped with an *orthocomplementation*  $\prime: L \rightarrow L$  which satisfies the *orthomodular identity*  $e \leq f \Rightarrow f = e \vee (f \wedge e')$ . Recall that an orthocomplementation maps each element  $e \in L$  onto a complement  $e'$  of  $e$  in  $L$  in such a way that  $e'' = e$  and  $e \leq f \Rightarrow f' \leq e'$  for all  $e, f \in L$ . Examples and basic properties of orthomodular lattices may be found in [1, 2, 3]. While some of the following concepts make sense in arbitrary lattices we shall make the following restriction;  $L$  will always represent an orthomodular lattice.

If  $e, f \in L$ , the ordered pair  $(e, f)$  is called a *modular pair*, written  $M(e, f)$ , if  $g \in L$  with  $g \leq f$  implies  $g \vee (e \wedge f) = (g \vee e) \wedge f$ . For  $e, f \in L$ , we say that  $e$  *commutes* with  $f$ , written  $eCf$ , if  $e = (e \wedge f) \vee (e \wedge f')$ . If  $e \leq f$ , the *interval*  $L(e, f) = \{g \in L: e \leq g \leq f\}$  is an orthomodular lattice with orthocomplementation  $g \rightarrow g^* = e \vee (f \wedge g') = (e \vee g') \wedge f$ .

In a Baer  $*$ -semigroup  $S$ ,  $P' = P'(S)$  denotes the set of all *closed* projections, i.e.,  $e \in P'$  if  $e = e^* = e^2$  and  $e = e''$ . In  $S$  one may generalize the concept of a bounded linear operator on a Hilbert space with closed range as follows: [1, p. 890]  $x \in S$  is *range-closed* if  $g \in P'(S)$ ,  $g \leq x''$  and  $(gx^*)'' = (x^*)''$  imply  $g = x''$ . This is equivalent to the condition  $[(ex^*)'x]'' = e' \wedge x''$  for all  $e \in P'$ . We shall use the following result of Foulis [3, Theorem 28].

**THEOREM 1.** *Let  $S$  be a Baer  $*$ -semigroup and  $e, f$  elements of the orthomodular lattice  $P'(S)$ . Then  $M(e, f)$  in  $P'(S)$  if and only if  $e'f$  is range closed in  $S$ .*

In  $P'(S)$  one has  $e \wedge f = (e'f)'f$  and  $eCf$  if and only if  $ef = fe$  in  $S$ .

**LEMMA 2.** *Let  $S$  be a Baer  $*$ -semigroup,  $x, y \in S$ . Then  $xy$  range-closed implies  $x''y$  is range-closed. Conversely, if  $x$  and  $x''y$  are range-closed, then  $xy$  is range-closed.*

*Proof.* Suppose  $xy$  is range-closed and  $g \in P'(S)$  with  $g \leq (x''y)''$  and  $(gy^*x'')'' = (y^*x'')''$ . Then  $g \leq (xy)''$  and  $(gy^*x^*)'' = ((gy^*x^*)'x^*)'' = (y^*x''x^*)'' = (y^*x^*)''$ . Hence  $g = (xy)'' = (x''y)''$  so  $x''y$  is range-closed.

Conversely, let  $x$  and  $x''y$  be range-closed. Then

$$\begin{aligned} [(ey^*x^*)'xy]'' &= \{[(ey^*)''x^*]'\}''y]'' = [(ey^*)' \wedge x'']y]'' \\ &= [(ey^*x'')'x''y]'' = e' \wedge (xy)'' \end{aligned}$$

for all  $e \in P'(S)$  so  $xy$  is range-closed.

Applying Theorem 1 and Lemma 2 one has for  $e \in P'(S)$  and a range-closed element  $x \in S$  that  $xe$  is range-closed if and only if  $M(x', e)$ .

Therefore, if  $x$  is range-closed and  $xe = ex$ ,  $xe$  must be range-closed, for  $xe = ex$  implies that  $x'e = ex'$ , so  $x'Ce$  and hence  $M(x', e)$ .

**THEOREM 3.** *Let  $e, f, g \in L = P'(S)$ . If  $M(e, f)$ ,  $gCe$  and  $gCf$ , then:*

- (i)  $M(e \vee g, f \vee g)$ .
- (ii)  $M(e \wedge g, f \wedge g)$ .
- (iii)  $M(e, f \wedge g)$ .
- (iv)  $M(e \vee g, f)$ .

*Proof.* By Theorem 1,  $e'f$  is range-closed. The hypothesis on  $g$  implies that  $g$  and  $g'$  permute with  $e'$  and  $f$  and thus with  $e'f$ . Now  $(e \vee g)'(f \vee g) = e'g'(f'g')' = e'(f \wedge g') = e'fg'$  which is range-closed by the preceding remark. Thus  $M(e \vee g, f \vee g)$ . The other three cases may be proved in a similar manner or by direct lattice theoretical arguments.

Of the four other possible alterations of the pair  $(e, f)$  using  $g$ , the pairs  $(e \vee g, f \wedge g)$  and  $(e \wedge g, f \wedge g)$  are automatically modular since the elements are comparable. The pairs  $(e, f \vee g)$  and  $(e \wedge g, f)$  are not necessarily modular.

One may derive another result as a consequence of Lemma 2 which amounts to an essential simplification of part of a theorem due to Foulis [1, Theorem 10].

**THEOREM 4.** *The orthomodular lattice  $L = P'(S)$  is modular if and only if  $S_0$ , the sub-Baer \*-semigroup generated by the closed projections in  $S$  is range-closed.*

*Proof.* We point out that for an arbitrary orthomodular lattice  $L$  there are, unfortunately, many Baer \*-semigroups that coordinatize  $L$ . However, there is a unique (up to isomorphism) minimum one, namely  $S_0$ , coordinatizing  $L$ . If  $S_0$  is range-closed, that is, if every element of  $S_0$  is range-closed, then for all  $e, f \in L$ ,  $e'f$  is range-closed so  $M(e, f)$  and  $L$  is modular. Conversely, suppose  $L$  is modular. We use induction on the number of closed projection factors in the element  $x \in S_0$ . If  $x = e \in P'(S)$  then  $x$  is range-closed. Now assume that all products of  $n$  closed projections are range-closed and let  $x = e_1e_2 \cdots e_n e_{n+1}$ ,  $e_i \in P'(S)$  Since  $e_1e_2 \cdots e_n$  is range-closed and  $M((e_1e_2 \cdots e_n)', e_{n+1})$ , by the remark following Lemma 2,  $x$  is range-closed

The pair  $(e, f)$  is not modular if there is a  $g < f$  with  $g \vee (e \wedge f) < (g \vee e) \wedge f$ . Letting  $h = g \vee (e \wedge f)$  one has  $h \vee (e \wedge f) < (h \vee e) \wedge f$  where  $e \wedge f < h < f$ . As a result, the pair  $(e, f)$

is modular in  $L$  if and only if it is modular in  $L(e \wedge f, e \vee f)$ . More generally, if  $e, f \in L(a, b)$ , then  $(e, f)$  is modular in  $L$  if and only if  $(e, f)$  is modular in  $L(a, b)$ . This observation simplifies our considerations.

The ordered pair  $(e, f)$  is said to be a *dual modular pair*, written  $M^*(e, f)$ , if  $h \geq f$  implies  $h \wedge (e \vee f) = (h \wedge e) \vee f$ . As the final result of this section, we collect a number of facts about modular and dual modular pairs.

**THEOREM 5.** *Let  $e, f \in L$ . Then:*

- (i)  $M^*(e, f)$  if and only if  $M(e', f')$ .
- (ii) If  $e, f \in L(a, b)$ ,  $M^*(e, f)$  in  $L$  if and only if  $M^*(e, f)$  in  $L(a, b)$ .
- (iii)  $M(e, f)$  for all  $f \in L$  if and only if  $M^*(e, f)$  for all  $f \in L$ .
- (iv)  $M(e, f)$  for all  $f \in L$  if and only if  $M(e', f)$  for all  $f \in L$ .
- (v)  $M(e, f)$  for all  $e \in L$  if and only if  $M(e, f)$  for all complements  $e$  of  $f$  in  $L$ .
- (vi) If  $e \wedge f = 0$ ,  $e_1 \leq e$ ,  $f_1 \leq f$ , then  $M(e, f)$  implies  $M(e_1, f_1)$ .

**3. Symmetries of modular pairs.** In this section we consider some general properties of two symmetry conditions on modular pairs.

**DEFINITION.** An orthomodular lattice  $L$  is said to be *M-symmetric* if for all  $e, f \in L$ ,  $M(e, f)$  implies  $M(f, e)$ .

An *M-symmetric* lattice is sometimes said to be *semi-modular*. Referring to the remark following Theorem 3, if  $L$  is *M-symmetric*, we do have  $M(e, f)$ ,  $gCe$  and  $gCf$  imply  $M(e, f \vee g)$  and  $M(e \wedge g, f)$ . However, these implications do not imply that  $L$  is *M-symmetric*.

If  $P$  is any lattice property, then a lattice is *relatively P* if every interval  $L(a, b)$  satisfies  $P$ . Since we have for  $e, f \in L(a, b)$  that  $M(e, f)$  in  $L$  if and only if  $M(e, f)$  in  $L(a, b)$ , it follows that an *M-symmetric* lattice is *relatively M-symmetric*.

The second type of symmetry to be considered concerns the ordered pairs  $(e, f)$  and  $(f', e')$ . In terms of coordinatizing Baer \*-semigroups this arises in the following natural manner. In the semigroup of all bounded operators on a Hilbert space, it is known that an operator is range-closed if and only if its adjoint is range-closed. Thus if  $e'$  and  $f$  are projections, then  $e'f$  is range-closed if and only if  $(e'f)^* = fe'$  is range-closed. Using Theorem 1, this translates into  $M(e, f)$  if and only if  $M(f', e')$ . The orthocomplementation involved is reflected in the following terminology.

**DEFINITION.** An orthomodular lattice  $L$  is said to be *O-symmetric*

if for all  $e, f \in L$ ,  $M(e, f)$  implies  $M(f', e')$ .

The third possible symmetry, connecting  $(e, f)$  and  $(e', f')$ , will not be considered since there seems to be no natural way in which it occurs. Moreover, it is a composite of  $M$ -symmetry and  $O$ -symmetry.

Now, using the fact that the relative orthocomplement of  $g \in L(a, b)$  is  $g^\# = (g' \wedge b) \vee a$  and Theorem 3, we derive the following result.

**THEOREM 6.** *If  $e, f \in L(a, b)$  then  $M(f^\#, e^\#)$  if and only if  $M(f', e')$ . Thus an  $O$ -symmetric lattice is relatively  $O$ -symmetric.*

*Proof.* Assume  $M(f^\#, e^\#)$ . Since  $f^\#$  and  $e^\#$  both commute with  $a'$ ,  $M(f^\#, e^\#)$  implies  $M(f^\# \wedge a', e^\# \wedge a')$ . Since these two elements commute with  $b'$  we obtain  $M((f^\# \wedge a') \vee b', (e^\# \wedge a') \vee b')$ . But  $(f^\# \wedge a') \vee b' = ((f' \wedge b) \vee a) \wedge a' \vee b' = f'$  and  $(e^\# \wedge a') \vee b' = e'$  so we have  $M(f', e')$ . Conversely, since  $f'$  and  $e'$  commute with  $b$  and since  $f' \wedge b$  and  $e' \wedge b$  commute with  $a$ ,  $M(f', e')$  implies  $M((f' \wedge b) \vee a, (e' \wedge b) \vee a)$ , that is,  $M(f^\#, e^\#)$ . The second conclusion now follows immediately.

The mapping  $g \rightarrow g \wedge a'$  is an isomorphism from  $L(a, b)$  onto  $L(0, b \wedge a')$  that preserves relative orthocomplements. Denoting this mapping by  $\varphi$ , we have for  $e, f \in L(a, b)$ ,  $M(e, f)$  in  $L(a, b)$  if and only if  $M(e\varphi, f\varphi)$  in  $L(0, b \wedge a')$ . Since the first condition is equivalent to  $M(e, f)$  in  $L$  we see that to investigate  $L$  for  $M$ -symmetry it is enough to consider pairs  $(e, f)$  such that  $e \wedge f = 0$ . This establishes the equivalence of (i) and (ii) in the following characterization.

**THEOREM 7.** *Let  $L$  be an orthomodular lattice. The following conditions are mutually equivalent:*

- (i)  $L$  is  $M$ -symmetric.
- (ii) For  $e, f \in L$  with  $e \wedge f = 0$ ,  $M(e, f)$  implies  $M(f, e)$ .
- (iii) For  $e, f \in L$  with  $e$  a complement of  $f$ ,  $M(e, f)$  implies  $M(f, e)$ .

*Proof.* We need only establish that (iii)  $\Rightarrow$  (ii). Thus suppose that (iii) holds. Let  $e \wedge f = 0$  and  $M(e, f)$ . Then  $e \vee (e \vee f)'$  is a complement of  $f$  and since  $e$  and  $f$  commute with  $(e \vee f)'$ ,  $M(e, f)$  implies  $M(e \vee (e \vee f)', f)$ . By (iii) we obtain  $M(f, e \vee (e \vee f)')$ . Applying Theorem 5 (vi) we have  $M(f, e)$ .

For  $O$ -symmetry we have a similar result.

**THEOREM 8.** *Let  $L$  be an orthomodular lattice. The following conditions are mutually equivalent:*

- (i)  $L$  is  $O$ -symmetric.
- (ii) For  $e, f \in L$  with  $e \wedge f = 0$ ,  $M(e, f)$  implies  $M(f', e')$ .

(iii) For  $e, f \in L$  with  $e$  a complement of  $f$ ,  $M(e, f)$  implies  $M(f', e')$ .

*Proof.* (i)  $\Leftrightarrow$  (ii). By Theorem 6, when considering the pair  $(e, f)$  it is sufficient to consider the interval  $L(e \wedge f, e \vee f)$ . By the remark preceding Theorem 7, it is enough to consider only intervals of the type  $L(0, g)$ .

(iii)  $\Rightarrow$  (ii). Suppose  $M(e, f)$  where  $e \wedge f = 0$ . As  $e$  and  $f$  both commute with  $(e \vee f)'$ ,  $M(e \vee (e \vee f)', f)$  by Theorem 3. Since  $e \vee (e \vee f)'$  is a complement of  $f$ , we have  $M(f', e' \wedge (e \vee f))$ . Now  $f'$  and  $e' \wedge (e \vee f)$  both commute with  $(e \vee f)'$  so, again by Theorem 3, we obtain  $M(f' \vee (e \vee f)', [e' \wedge (e \vee f)] \vee (e \vee f)')$  which is precisely  $M(f', e')$ .

That (ii)  $\Rightarrow$  (iii) is clear.

We are now in a position to establish the following connection between the two symmetries.

**THEOREM 9.** *Every O-symmetric lattice is M-symmetric.*

*Proof.*<sup>1</sup> Let  $L$  be O-symmetric,  $e, f \in L$  with  $e \wedge f = 0$ . By Theorem 5,  $M(e, f)$  implies  $M(g, f)$  for all  $g \leq e$ . Thus, for all  $g \leq e$ ,  $M(f', g')$  which is equivalent to  $M^*(f, g)$ . This last statement implies  $M(f, e)$ .

Whether the converse of this theorem holds is still an open question. In addition to the similarities shown in Theorems 7 and 8, both properties are preserved under the taking of direct products.

**4. Atomic orthomodular lattices.** For  $a, b \in L$ , if  $a < b$  we say that  $b$  covers  $a$  if  $a < c \leq b$  implies  $c = b$ . An element that covers 0 is called an *atom*.  $L$  is called *atomic* if every nonzero element dominates an atom. In an atomic orthomodular lattice every nonzero element is the supremum of the atoms it dominates. An atomic orthomodular lattice is said to satisfy the *atomic exchange property* (AEP) if  $a$  and  $b$  atoms with  $a \leq b \vee e$  and  $a \not\leq e$  implies  $b \leq a \vee e$ .

Prime tools in the coordinatization theory of orthomodular lattices are the mappings  $\varphi_e: L \rightarrow L$  defined for each  $e \in L$  by  $a\varphi_e = (a \vee e') \wedge e$ .  $\varphi_e$  is called the *Sasaki projection* determined by  $e$ . Such mappings are monotone and preserve arbitrary suprema [2]. An atomic lattice is said to satisfy the *atomic projection property* (APP) if for all atoms  $a$  and for all  $e \in L$ ,  $a \not\leq e'$  implies  $a\varphi_e$  is an atom.

<sup>1</sup> The author is indebted to the referee for this simple proof.

LEMMA 10. *Let  $L$  be an atomic orthomodular lattice. The following conditions are mutually equivalent:*

- (i)  *$a$  an atom,  $a \wedge b = 0$  implies  $a \vee b$  covers  $b$ .*
- (ii)  *$L$  satisfies APP.*
- (iii)  *$L$  satisfies AEP.*
- (iv)  *$b$  an atom,  $x \leq b \vee e$  and  $x \not\leq e$  imply  $b \leq x \vee e$ .*
- (v) *For all atoms  $a \in L$ ,  $M(a, e)$  for all  $e \in L$ .*

The proof, which is routine, is omitted. If  $L$  satisfies AEP it also satisfies the condition  $a$  and  $b$  cover  $a \wedge b$  if and only if  $a \vee b$  covers  $a$  and  $b$ . In the sequel we make use of all these conditions.

LEMMA 11. *Every  $M$ -symmetric, and hence every  $O$ -symmetric atomic lattice  $L$  satisfies AEP.*

*Proof.* For all atoms  $a \in L$  and all  $e \in L$   $M(e, a)$  and hence  $M(a, e)$ .

A nonempty subset  $A$  of  $L$  is an *ideal* if  $a, b \in A$  implies  $a \vee b \in A$  and  $a \in A, e \leq a$  implies  $e \in A$ . If  $A$  and  $B$  are ideals, then  $A \vee B = \{e: e \leq a \vee b \text{ for some } a \in A, b \in B\}$  is the smallest ideal containing  $A$  and  $B$ . For  $a, b \in L$ , say that  $a$  and  $b$  are *orthogonal*, written  $a \perp b$  if  $a \leq b'$ . For orthogonal elements write  $a \oplus b$  for  $a \vee b$ . If  $\{a_i: i = 1, 2, \dots, n\}$  is an orthogonal family, that is, if  $i \neq j$  implies  $a_i \perp a_j$  we write  $a_1 \oplus a_2 \oplus \dots \oplus a_n$  for  $a_1 \vee a_2 \vee \dots \vee a_n$ .

Henceforth,  $L$  will always denote an atomic orthomodular lattice. We shall call  $b \in L$  *finite* if  $b = 0$  or  $b$  is the join of a finite number of atoms. Let  $J$  denote the set of all finite elements in  $L$ . Minor modifications of the proof of [6, Theorem 4.1] enable us to show that if  $L$  satisfies AEP then  $J$  is an ideal of  $L$ . Moreover, if we define on  $J$  the integer valued function  $\text{deg}(b) = \min\{n: b \text{ is the join of } n \text{ atoms}\}$  then for  $b \in J$ ,  $\text{deg}(b) = n$  if and only if  $b = a_1 \oplus a_2 \oplus \dots \oplus a_n$  where each  $a_i$  is an atom.

LEMMA 12. *If  $L$  satisfies AEP and  $b \in J$  then  $b\varphi_e \in J$  for all  $e \in L$ . Moreover,  $\text{deg}(b\varphi_e) \leq \text{deg}(b)$ . If  $a \in L$  has a common complement with  $b$  then  $a \in J$  and  $\text{deg}(a) = \text{deg}(b)$ .*

*Proof.* Let  $b = a_1 \oplus \dots \oplus a_n, a_i$  atoms. Then  $b\varphi_e = a_1\varphi_e \vee \dots \vee a_n\varphi_e$  and each nonzero  $a_i\varphi_e$  is an atom by APP. Thus  $\text{deg}(b\varphi_e) \leq \text{deg}(b)$ . Now if  $e$  is a complement of both  $a$  and  $b$ , then  $b\varphi_{e'} = e'$  so  $e' \in J$  with  $\text{deg}(e') \leq \text{deg}(b)$ . Since  $e'\varphi_b = b$ ,  $\text{deg}(b) = \text{deg}(e')$ . Since  $e'\varphi_a = a$  and  $a\varphi_{e'} = e', a \in J$  and  $\text{deg}(a) = \text{deg}(e') = \text{deg}(b)$ .

THEOREM 13. *Let  $L$  satisfy AEP. Then  $b \in J$  implies  $M(c, b)$*



and  $M(b, c)$  for all  $c \in L$ . Therefore  $J$  is a modular sublattice of  $L$ .

*Proof.* Since the modularity of the pair  $(b, c)$  depends only on the interval  $L(b \wedge c, b \vee c)$  we may assume  $b \wedge c = 0$ . Let  $\text{deg}(b) = n$  so  $b = a_1 \oplus \dots \oplus a_n$ ,  $a_i$  atoms. Let  $e \leq c$ . To show  $M(b, c)$  we must prove  $h = (e \vee b) \wedge c \wedge e' = 0$ . Assume  $h \neq 0$ . Since  $h \not\leq e$ ,  $h \leq a_j \vee e$  implies  $a_1 \leq h \vee e \leq c$  contradicting  $b \wedge c = 0$ . Thus  $h \not\leq a_1 \vee e$ . If  $h \not\leq a_1 \vee \dots \vee a_{j-1} \vee e$ , then  $h \leq a_1 \vee \dots \vee a_j \vee e$  implies  $a_j \leq a_1 \vee \dots \vee a_{j-1} \vee h \vee e \leq a_1 \vee \dots \vee a_{j-1} \vee c$ . Let

$$d = a_1 \oplus \dots \oplus a_{j-1} \oplus a_{j+1} \oplus \dots \oplus a_n$$

and join  $d \vee c \vee (b \vee c)'$  to both ends of the last inequality. This yields  $1 = d \vee c \vee (b \vee c)'$ . Since  $d \wedge [c \vee (b \vee c)'] = 0$ ,  $c \vee (b \vee c)'$  is a common complement of  $b$  and  $d$ . But then  $n - 1 = \text{deg}(d) = \text{deg}(b) = n$  by Lemma 12. This contradiction establishes that  $h \not\leq a_1 \vee \dots \vee a_j \vee e$ . By induction,  $h \neq 0$  implies  $h \not\leq b \vee e$ . However,  $h = (e \vee b) \wedge c \wedge e'$  so we must have  $h = 0$ . Thus  $M(b, c)$  for all  $c \in L$ .

By Theorem 5, to test  $M(c, b)$  for all  $c \in L$  it is sufficient to consider only complements of  $b$ . If  $c$  is a complement of  $b$  such that  $(c, b)$  is not modular, there exists  $e < b$  with  $e < (e \vee c) \wedge b$ . Since  $(e \vee c) \wedge b \in J$  and  $e$  have the common complement  $c \vee (e \vee c)'$ ,  $\text{deg}((e \vee c) \wedge b) = \text{deg}(e) < \text{deg}((e \vee c) \wedge b)$ . This contradiction establishes the theorem.

Note that by Theorems 5 and 13, if  $b \in J$  we also have  $M^*(c, b)$  for all  $c \in L$ .

A subset  $A$  of  $L$  is called *join-dense* in  $L$  if every element of  $L$  is the join of some subset of  $A$ . In an orthomodular lattices this is equivalent to the condition  $b \neq 0$  implies  $A \cap L(0, b)$  contains a nonzero element. We point out that completeness of  $L$  is not necessary for this equivalence. If  $L$  satisfies AEP,  $J$  is a join-dense modular ideal of  $L$ . For nonzero  $b \in L$ , let  $J(b) = J \cap L(0, b)$ . If  $L$  satisfies AEP, since  $L(0, b)$  and  $J$  are ideals of  $L$ ,  $J(b)$  is an ideal of  $L$  as well as an ideal of  $L(0, b)$ .

We now generalize the theorem of Mackey mentioned in the introduction. Note that for  $A, B \in L(H)$ ,  $A + B$  is closed and therefore equals  $A \vee B$  if and only if there are no one-dimensional subspaces contained in  $A \vee B$  but not in  $A + B$ . We first treat the analog of a finite dimensional subspace, in which case  $A + B = A \vee B$  always holds.

LEMMA 14. *Let  $L$  satisfy AEP. If  $a \in L$  and  $b \in J$ , then  $J(a \vee b) = J(a) \vee J(b)$ .*

*Proof.* Clearly  $J(a) \vee J(b) \subset J(a \vee b)$ . Now let  $e \in J(a \vee b)$ . Since  $e \vee b \in J$ ,  $(e \vee b) \wedge a \in J(a)$  and  $M(a, e \vee b)$  so  $((e \vee b) \wedge a) \vee b = e \vee b$ . Thus  $e \vee b \in J(a) \vee J(b)$  and since  $e \leq e \vee b$ ,  $e \in J(a) \vee J(b)$ .

**THEOREM 15.** *Let  $L$  be an atomic orthomodular lattice satisfying AEP. Then for  $a, b \in L$  the following conditions are mutually equivalent:*

- (i)  $J(a \vee b) = J(a) \vee J(b)$ .
- (ii)  $J(a) \vee J(b)$  is join-dense in  $L(0, a \vee b)$ .
- (iii)  $M^*(b, a)$ .

*Proof.* That (i)  $\Rightarrow$  (ii) is clear.

(ii)  $\Rightarrow$  (iii). Let  $a \leq c \leq a \vee b$ . To show  $c = a \vee (b \wedge c)$ , it is enough to show  $J(a) \vee J(b \wedge c)$  is join-dense in  $L(0, c)$ . Thus let  $e \in L(0, c)$  with  $e \neq 0$ . Since  $L(0, c) \subset L(0, a \vee b)$  there is an  $f \in J(a)$  and  $g \in J(b)$  with  $(f \vee g) \wedge e \neq 0$ . Set  $e_1 = (f \vee g) \wedge e$ . By Theorem 13,  $M(g, f \vee e_1)$  so  $f \vee (g \wedge (f \vee e_1)) = f \vee e_1$ . Since  $g \wedge (f \vee e_1) \in L(0, b \wedge c)$ ,  $f \vee e_1$  and hence  $e_1$  belongs to  $J(a) \vee J(b \wedge c)$ . Since  $e_1 \neq 0$ ,  $J(a) \vee J(b \wedge c)$  is join-dense in  $L(0, c)$ . Thus  $c = a \vee (b \wedge c)$ . That is,  $M^*(b, a)$ .

(iii)  $\Rightarrow$  (i). We need only show  $J(a \vee b) \subset J(a) \vee J(b)$ . Let  $e \in J(a \vee b)$ ,  $g = e\varphi_a$ , and  $f = (e \vee a) \wedge a'\varphi_b$ . Since  $M^*(b, a)$ ,  $f \vee g' = 1$ . As  $f \wedge g' = 0$ ,  $g'$  is a common complement of  $g$  and  $f$ . Thus, by Lemma 12,  $e \in J$  implies  $g \in J$  which in turn implies  $f \in J$ . Making use of the orthomodular identity and  $M^*(b, a)$  we have  $a \vee f = a \vee (a \wedge b) \vee f = a \vee (b \wedge (a \vee e)) = a \vee e$ . Since  $f \in J$ , by Lemma 14,  $J(a \vee e) = J(a \vee f) = J(a) \vee J(f)$ . Thus  $f \leq b$  implies

$$e \in J(a \vee e) \subset J(a) \vee J(b)$$

so (i) holds.

**COROLLARY 16.** *An atomic orthomodular lattice is  $M$ -symmetric if and only if it satisfies AEP.*

*Proof.* Necessity was established by Lemma 11. By Theorem 5,  $M$ -symmetry is equivalent to the symmetry of dual modular pairs. Sufficiency thus follows since condition (i) of Theorem 15 is symmetric in  $a$  and  $b$ .

#### REFERENCES

1. D. J. Foulis, *Conditions for the modularity of an orthomodular lattice*, Pacific J. Math. **11** (1961), 889-895.
2. ———, *A note on orthomodular lattices*, Portugal. Math. **21**, Fasc. 1 (1962), 65-72.

3. ———, *Relative inverses in Baer \*-semigroups*, Mich. Math. J. **10** (1963), 65-84.
4. G. Mackey, *On infinite-dimensional linear spaces*, Trans. Amer. Math. Soc. **57** (1945), 155-207.
5. S. MacLane, *A lattice formulation for transcendence degree and  $p$ -basis*, Duke Math. J. **4** (1938), 455-468.
6. M. D. MacLaren, *Atomic orthocomplemented lattices*, Pacific J. Math. **14** (1964), 597-612.
7. ———, *Nearly modular orthocomplemented lattices*, Trans. Amer. Math. Soc. **114** (1965), 401-416
8. A. Ramsay, *Dimension theory in complete orthocomplemented weakly modular lattices*, Trans. Amer. Math. Soc. **116** (1965), 9-31.

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