

# Pacific Journal of Mathematics

**ON DUAL SERIES RELATIONS INVOLVING LAGUERRE  
POLYNOMIALS**

**K. N. SRIVASTAVA**

## ON DUAL SERIES RELATIONS INVOLVING LAGUERRE POLYNOMIALS

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**In this paper, we shall consider the problem determining the sequence  $\{An\}$ , such that**

$$\sum_{n=0}^{\infty} \{An/\Gamma(n + \alpha + 1)\} L_n^\alpha(x) = f_1(x), \quad 0 \leq x < y,$$

$$\sum_{n=0}^{\infty} \{An/\Gamma(n + \alpha + 1/2)\} L_n^\alpha(x) = f_2(x), \quad y < x \leq \infty, \alpha > -1/2,$$

**where  $L_n^\alpha(x)$  is a Laguerre polynomial, the functions  $f_1(x)$  and  $f_2(x)$  being prescribed. By expressing the sequence  $\{An\}$  in terms of a sequence of integrals involving an unknown function  $g(u)$  the problem is reduced to that of solving an Abel integral equation for  $g(u)$ .**

In recent years, dual series relations involving Fourier-Bessel, Dini series, trigonometric series and series of Jacobi polynomials have been investigated by various workers [1, 2, 5 to 12]. Here we shall apply the method developed by Sneddon and Srivastav for obtaining a solution of the dual series relations involving Laguerre polynomials.

As pointed out by Sneddon and Srivastav [6], with a view to simplify the calculations, we split the problem posed by the pair of dual equations given above into two parts: Problem (a). Determine the constants  $\{An\}$  satisfying the dual series relations

$$(1.1) \quad \sum_{n=0}^{\infty} \{An/\Gamma(n + \alpha + 1)\} L_n^\alpha(x) = f_1(x), \quad 0 \leq x < y,$$

$$(1.2) \quad \sum_{n=0}^{\infty} \{An/\Gamma(n + \alpha + 1/2)\} L_n^\alpha(x) = 0, \quad y < x \leq \infty, \alpha > -1/2.$$

Problem (b). Determine the constants  $\{An\}$  satisfying the dual series relations

$$(1.3) \quad \sum_{n=0}^{\infty} \{An/\Gamma(n + \alpha + 1)\} L_n^\alpha(x) = 0, \quad 0 \leq x < y,$$

$$(1.4) \quad \sum_{n=0}^{\infty} \{An/\Gamma(n + \alpha + 1/2)\} L_n^\alpha(x) = f_2(x), \quad y < x \leq \infty, \alpha > -1/2.$$

The solution of the general problem is obviously obtained merely by adding the solutions of problem (a) and (b). We suppose that functions  $f_1(x)$  and  $f_2(x)$  satisfy the following conditions:

(i)  $F_1(x) = x^\alpha f_1(x)$  is finite and continuously differentiable for  $0 \leq x < y$ ,

(ii)  $F_2(x) = \int_x^\infty e^{-x} f_2(x) dx$  is finite and continuously differentiable for  $y < x \leq \infty$ .

As we shall presently see the classes of functions  $f_1(x)$  and  $f_2(x)$  for which the problem under discussion is solvable, must satisfy the above conditions,

2. In this section we list some results for ready reference. By combining the results [3, p. 292 (2), (3)], we have

$$(2.1) \quad \int_0^\infty x^\alpha e^{-x} L_n^\alpha(x) L_m^\alpha(x) dx = (n+1)_\alpha \cdot \delta_{mn},$$

where  $\delta_{mn}$  is a Kronecker delta. From [4, p. 193 (27), (28)] we have

$$(2.2) \quad \frac{d}{dn} \{x^\alpha L_n^\alpha(x)\} = (n+\alpha) x^{\alpha-1} L_n^{\alpha-1}(x),$$

$$(2.3) \quad \int_x^\infty e^{-y} L_n^\alpha(y) dy = e^{-x} L_n^{\alpha-1}(x).$$

We shall also require the following results which are easily derived from the more general results given in [3, p. 293 (5), p. 405 (20)]. For  $\alpha > -1/2$

$$(2.4) \quad \int_x^\infty (y-x)^{-1/2} e^{-y} L_n^\alpha(y) dy = \Gamma(1/2) e^{-x} L_n^{\alpha-1/2}(x),$$

$$(2.5) \quad \int_0^x (x-y)^{-1/2} y^\alpha L_n^\alpha(y) dy = \frac{\Gamma(n+\alpha+1) \Gamma(1/2)}{\Gamma(n+\alpha+3/2)} x^{\alpha+1/2} L_n^{\alpha+1/2}(x).$$

We also note that if  $f(x)$  is continuously differentiable then Abel integral equation

$$(2.6) \quad f(x) = \int_0^x \frac{\phi(y)}{(x-y)^{1/2}} dy$$

has a continuous solution given by the equation

$$(2.7) \quad \phi(y) = \frac{1}{\Pi} \frac{d}{dy} \int_0^y \frac{f(x)}{(y-x)^{1/2}} dx.$$

Furthermore, if  $f(x)$  is continuously differentiable then the integral equation

$$(2.8) \quad f(x) = \int_x^\infty \frac{\phi(y)}{(y-x)^{1/2}} dy$$

has a continuous solution

$$(2.9) \quad \phi(y) = -\frac{1}{\Pi} \frac{d}{dy} \int_y^\infty \frac{f(x)}{(x-y)^{1/2}} dx .$$

This can be easily established by simple methods given in [13, p. 229]. The analysis given here is purely formal and no attempt is made to justify the interchange of various limiting processes.

**3. Solution of the problem (a).** Let us suppose that for  $0 \leq x < y$

$$(3.1) \quad \sum_{n=0}^\infty \{An/\Gamma(n + \alpha + 1/2)\} L_n^\alpha(x) = -e^x \frac{d}{dx} \int_x^y \frac{g_1(u)}{(u-x)^{1/2}} du .$$

Using the orthogonal property (2.1), it can be shown that

$$(3.2) \quad An = -\frac{\Gamma(n + \alpha + 1/2) \Gamma(1/2)}{\Gamma(n + \alpha + 1)} \int_0^y x^\alpha L_n^\alpha(x) \left( \frac{d}{dx} \int_x^y \frac{g_1(u)}{(u-x)^{1/2}} du \right) dx .$$

Since

$$(3.3) \quad -\frac{d}{dx} \int_x^y \frac{g_1(u)}{(u-x)^{1/2}} du = \frac{g_1(y)}{(y-x)^{1/2}} - \int_x^y \frac{d}{du} \left\{ \frac{g_1(u)}{(u-x)^{1/2}} \right\} du$$

we obtain with the help of (2.5), the equation

$$(3.4) \quad An = \Gamma(n + 1) \Gamma(1/2) \int_0^y g_1(u) u^{\alpha-1/2} L_n^{\alpha-1/2}(u) du , \quad n = 0, 1, 2, \dots$$

If in the equation (1.1), we substitute for the coefficients  $An$  from (3.4), on interchanging the order of summation and integration, we get

$$(3.5) \quad f_1(x) = \int_0^y g_1(u) u^{\alpha-1/2} K_1(u, x) du , \quad 0 \leq x < y ,$$

where

$$(3.6) \quad K_1(u, x) = \sum_{n=0}^\infty \frac{\Gamma(n + 1) \Gamma(1/2)}{\Gamma(n + \alpha + 1)} L_n^{\alpha-1/2}(u) L_n^\alpha(x)$$

with the help of equations (2.1) and (2.4) it can be shown that

$$(3.7) \quad K_1(u, x) = e^u x^{-\alpha} (x-u)^{-1/2} H(x-u)$$

where  $H(t)$  is Heaviside's unit function. (2.7) is easily proved. Let

$$K_1(u, x) = \sum_{n=0}^\infty a_n L_n^\alpha(x)$$

where the coefficients  $a_n$  are given by

$$\begin{aligned}
 a_n &= \frac{\Gamma(n+1)}{\Gamma(n+\alpha+1)} \int_0^\infty K_1(u, x) x^\alpha e^{-u} L_n^\alpha(x) \\
 &= \frac{\Gamma(n+1)}{\Gamma(n+\alpha+1)} e^x \int_x^\infty e^{-x}(x-u)^{-1/2} L_n^\alpha(x) dx \\
 &= \frac{\Gamma(n+1) \Gamma(1/2)}{\Gamma(n+\alpha+1)} L_n^{\alpha-1/2}(u).
 \end{aligned}$$

Thus the equation (3.5) is equivalent to

$$(3.8) \quad F_1(x) = x^\alpha f_1(x) = \int_0^x \frac{g_1(u) u^{\alpha-1/2} e^u}{(x-u)^{1/2}} du, \quad 0 \leq x < y.$$

This is Abel integral equation, since  $F_1(x)$  is finite and continuously differentiable, its solution is given by

$$(3.9) \quad u^{\alpha-1/2} e^u g_1(u) = \frac{1}{\Gamma} \frac{d}{du} \int_0^u \frac{x^\alpha f_1(x)}{(u-x)^{1/2}} dx.$$

The coefficients  $An$  may now be calculated with the help of the relations (3.4) and (3.9).

**4. Solution of the problem (b).** We start with the assumption that for  $y < x \leq \infty$

$$(4.1) \quad \sum_{n=0}^{\infty} \{An/\Gamma(n+\alpha+1)\} L_n^\alpha(x) = x^{-\alpha} \int_y^x \frac{g_2(u)}{(x-u)^{1/2}} du.$$

This is equivalent to assuming that

$$(4.2) \quad An = \Gamma(n+1) \Gamma(1/2) \int_y^\infty g_2(u) e^{-u} L_n^{\alpha-1/2}(u) du, \quad n = 0, 1, 2, \dots$$

If we multiply both sides of the equation (1.4) by  $\exp(-x)$  and integrate with respect to  $x$  from  $x$  to  $\infty$ ,  $y < x \leq \infty$ , we obtain

$$(4.3) \quad F_2(x) = \int_x^\infty e^{-x} f_2(x) dx = \sum_{n=0}^{\infty} \{An/\Gamma(n+\alpha+1)\} e^{-x} L_n^{\alpha-1}(x).$$

Substituting the values of the coefficients from (4.2) in the equation (4.3) we find on interchanging the order of summation and integration that

$$(4.4) \quad e^x F_2(x) = \int_y^\infty g_2(u) e^{-u} K_2(u, x) du, \quad y < x \leq \infty,$$

where

$$(4.5) \quad K_2(u, x) = \sum_{n=0}^{\infty} \frac{\Gamma(n+1) \Gamma(1/2)}{\Gamma(n+\alpha+1/2)} L_n^{\alpha-1/2}(u) L_n^{\alpha-1}(x).$$

From the relations (2.1) and (2.5) it easily follows that

$$(4.6) \quad K_2(u, x) - e^x u^{-\alpha+1/2} (u-x)^{-1/2} H(u-x).$$

Consequently the equation (4.4) reduces to the integral equation

$$(4.7) \quad F_2(x) = \int_y^\infty \frac{g_2(u) u^{1/2-\alpha} e^{-u}}{(u-x)^{1/2}} du, \quad y < x \leq \infty.$$

Since  $F_2(x)$  is finite and continuously differentiable, the solution of the above equation is given by

$$(4.8) \quad g_2(u) = -\frac{e^u u^{\alpha-1/2}}{H} \frac{d}{du} \int_0^u \frac{F_2(x)}{(x-u)^{1/2}} dx.$$

The coefficients  $A_n$  are given by the relations (4.2) and (4.8).

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S. J. Bernau, <i>The spectral theorem for unbounded normal operators</i> . . . . .	391
Lu-san Chen, <i>Asymptotic behavior of solutions of parabolic equations of higher order</i> . . . . .	407
Lawrence William Conlon, <i>An application of the Bott suspension map to the topology of EIV</i> . . . . .	411
Neal Eugene Foland and John M. Marr, <i>Sets with zero-dimensional kernels</i> . . . . .	429
Stanley Phillip Franklin and R. H. Sorgenfrey, <i>Closed and image-closed relations</i> . . . . .	433
William Jesse Gray, <i>A note on topological transformation groups with a fixed end point</i> . . . . .	441
Myron Goldstein, <i>K- and L-kernels on an arbitrary Riemann surface</i> . . . . .	449
George Joseph Kertz and Francis Regan, <i>The exponential analogue of a generalized Weierstrass series</i> . . . . .	461
Walter Leighton, <i>On Liapunov functions with a single critical point</i> . . . . .	467
Bernard Werner Levinger and Richard Steven Varga, <i>On a problem of O. Taussky</i> . . . . .	473
Lowell Duane Loveland, <i>Tame subsets of spheres in <math>E^3</math></i> . . . . .	489
Erik Andrew Schreiner, <i>Modular pairs in orthomodular lattices</i> . . . . .	519
K. N. Srivastava, <i>On dual series relations involving Laguerre polynomials</i> . . . . .	529
Arthur Steger, <i>Diagonability of idempotent matrices</i> . . . . .	535
Walter Strauss, <i>On continuity of functions with values in various Banach spaces</i> . . . . .	543
Robert Vermes, <i>On the zeros of a linear combination of polynomials</i> . . . . .	553
Elliot Carl Weinberg, <i>On the scarcity of lattice-ordered matrix rings</i> . . . . .	561
Harold Widom, <i>Toeplitz operators on <math>H_p</math></i> . . . . .	573
Neal Zierler, <i>On the lattice of closed subspaces of Hilbert space</i> . . . . .	583
Irving Leonard Glicksberg, <i>Correction to: "Maximal algebras and a theorem of Radó"</i> . . . . .	587
John Spurgeon Bradley, <i>Correction to: "Adjoint quasi-differential operators of Euler type"</i> . . . . .	587
William Branham Jones, <i>Erratum: "Duality and types of completeness in locally convex spaces"</i> . . . . .	588
Stanley P. Gudder, <i>Erratum: "Uniqueness and existence properties of bounded observables"</i> . . . . .	588